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On Characteristic p Zeta Functions

Dedicated to Bhaikaka on his 75th Birthday

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Abstract. We show that in contrast to the various known analogies, the orders of vanishing of the characteristic p zeta functions introduced by Goss sometimes follow interesting patterns involving base q-digits, providing a challenge to understand them in a general framework as in the classical case. We answer some questions raised by Goss about the connection of the zeta values and periods. We also generalize and simplify the proofs of some results.

1. Introduction

It is known in various contexts, that the order of vanishing and the leading term of zeta function have a lot of interesting arithmetic information encoded in them. The characteristic p zeta functions studied by Carlitz and Goss are no exception. Their values relate to the periods, the divisibilities of Bernoulli numbers analogues arising from them give information on class numbers of cyclotomic extensions in a way analogous to the classical case and they are related to the classical L-functions via congruence formulae. An important ingredient missing is that there is no functional equation in sight, even though there are analogues of gamma functions and Gauss sums. In the classical case, the gamma factors control the orders of vanishing of the zeta functions at the negative integers.

Using the congruence formula with the classical L-functions, Goss has provided [G4] lower bounds for orders of vanishing of these zeta functions at the negative integers and has raised a question whether they are exact. We show (Section 3) that there can be extra vanishing, even when these lower bounds are naive analogues of the exact orders of vanishing in the classical case. The patterns involving the q-digits which control these extra zeros are an interesting phenomenon giving us a challenge to put these zeros in a general framework, as is done classically.

In the other sections, we answer some questions about the connection of the zeta values at the positive integers with the periods and refine some results of Goss. We also give alternate simple proofs for some of the results in the literature.

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Let us briefly recall the classical case. If F is a number field, the Dedekind zeta function $\zeta_F(s) \in \mathbb{C}$ is defined, for $s \in \mathbb{C}$ with real part greater than 1, as $\sum 1/\mathrm{Norm}(I)^s = \prod(1 - 1/\mathrm{Norm}\,(P)^s)^{-1}$, where the sum is over all non-zero ideals I of the ring of integers of F and the product is over the primes P. (For $F = \mathbf{Q}$, this is the Riemann Zeta function). This function has meromorphic continuation to the whole complex plane with only pole at $s = 1$ and has a simple functional equation connecting $\zeta_F(s)$ to $\zeta_F(1 - s)$. Clearly, for $s > 1$, there are no zeros and hence analyzing the factors in the functional equation (essentially gamma functions), we see that at negative integer s, the zeta function vanishes to order $r_1 + r_2$ (r_2 respectively) if s is even (odd respectively). In addition, if F is totally real, then values at negative integers are rational or equivalently, by functional equation, for s a positive even integer, $(\zeta_F(s)/(2\pi i)^{r_1})^2 \in \mathbf{Q}$.

For a function field over a finite field $\mathbf{F}_q$, a complex valued zeta function was introduced and studied by Artin and Weil, by thinking of the norm as the number of residue classes and using similar definitions. It has functional equation and interesting theory of special values, but it turns out to be a rational function of q^{-s} and we do not have connection to $2\pi i$, for example.

If we want to consider the norm in the usual relative sense instead, as was done by Goss, (and in the simplest case by Carlitz), we have to deal with the fact that there are neither naturally given infinite places or rings of integers nor is there a naturally given base field (like $\mathbf{Q}$). Even the rational function field, which can be considered as a subfield, sits in the function field in various different ways. Even when you choose the base and the infinite places, the norm of an ideal is an ideal in the base and it may not be principal, if the base has class number greater than 1 and even if it is principal, there may be several units to consider, to get a well-defined generator out of it. Finally, once we get an element, we can raise it to an integral power only, not to any arbitrary complex power. This suggests either restricting to a class number 1 base with only one place at infinity (as is the case for $\mathbf{Q}$ or an imaginary quadratic field) so that the unit group is a finite group or more generally restricting to the extensions for which the norms are principal ideals or defining ideal exponentiation. This boils down to choosing representatives of the ideal classes of the base and dealing with vector (of length class number) valued zeta functions. We will discuss these in the next section.

The main reference is [G4] where the reader can find an exposition of various results and additional references. See also [T2]. We will assume some background, especially in the Section 4, on the relevant theory of Drinfeld modules of rank one. For this, see [H1], [H2] or the summary in [T3], whose notation we will follow. Otherwise, we have tried to give a self-contained exposition of the issues most relevant to this paper.

2. Zeta Values and Zeros

DEFINITIONS. Let K be a function field of one variable with field of constants $\mathbf{F}_q$ of characteristic p, ∞ be a place of K of degree δ, K_∞ be the completion of K at ∞ and A be the ring of elements of K having no pole outside ∞. We take Ω to be the completion of an algebraic closure of K_∞. Let us also denote by h the class number of K, so that the class number of A is $h\delta$. Fix a sign function sgn : $K_\infty^* \to \mathbf{F}_{q^\delta}^*$. (In examples, we usually take sgn such that x, y, t have sgn 1). Now let S be a set of representatives of $\mathbf{F}_{q^\delta}^*/\mathbf{F}_q^*$. Let $A_+ := \{a \in A : \mathrm{sgn}(a) \in S\}$. (This way of dealing with signs, when $\delta > 1$, was suggested in [T1] p. 46). Elements of sgn $= 1$ are called 'positive' or 'monic'. By deg we will denote the residue degree over $\mathbf{F}_q$. By convention $\deg(0) = -\infty$. Let $A_i := \{a \in A : \deg(a) = i\}$ and $A_{i+} := \{a \in A_+ : \deg(a) = i\}$. Let H be the Hilbert class field of A i.e., the maximal abelian unramified extension of K in which ∞ splits completely. Recall that $K = H$ if $h\delta = 1$. Let L be a finite separable extension of K and let $\mathcal{O}_L = \mathcal{O}$ denote the integral closure of A in L.

Now we define the relevant zeta functions.

For $s \in \mathbf{Z}$, define the 'absolute zeta function':

$$\zeta(s,X) := \zeta_A(s,X) := \sum_{i=0}^{\infty} X^i \sum_{a \in A_{i+}} \frac{1}{a^s} \in K[[X]],$$

$$\zeta(s) := \zeta_A(s) := \zeta(s,1) \in K_\infty.$$

A priori, the second definition makes sense only for s positive, as the terms then tend to zero (and also for $s = 0$ when $\zeta(s) = 1$), but we will show below that it works for for s a negative integer also and in fact, then $\zeta(s) \in A$. The results on the special values and the connection with Drinfeld modules mentioned in the paper later on, justifies the use of elements rather than ideals even when the class number is more than one. Also, we can look at a variant where a runs through elements of some ideal of A or we can sum over all ideals by letting s to be a multiple of class number and letting a^s to be the generator of I^s with appropriate sign. In the latter case, we have Euler product for such s, if $\delta = 1$.

If L contains H (note that this is no restriction if $h\delta = 1$, e.g. for $A = \mathbf{F}_q[t]$), then it is known that the norm of an ideal $\mathcal{I}$ of $\mathcal{O}$ is principal. Let $\mathbf{N}\mathcal{I}$ denote the generator whose sign is in S. Let us now define the relative zeta functions in this situation. Put, for $s \in Z$

$$\zeta_{\mathcal{O}}(s,X) := \zeta_{\mathcal{O}/A}(s,X) := \sum_{i=0}^{\infty} X^i \sum_{\deg \mathbf{N}\mathcal{I}=i} \frac{1}{\mathbf{N}\mathcal{I}^s} \in K[[X]],$$

$$\zeta_{\mathcal{O}}(s) := \zeta_{\mathcal{O}/A}(s) := \zeta_{\mathcal{O}}(s,1) \in K_\infty.$$

The same remark as above applies to the second definition.

Finally, we define the vector valued zeta function, which generalizes both definitions above and works without assuming that L contains H. We will leave the simple task of relating these definitions to the reader.

Let $\mathcal{C}_n$ ($1 \leqslant n \leqslant h\delta$) be the ideal classes of A and choose an ideal I_n in $\mathcal{C}_n^{-1}$. For $s \in \mathbf{Z}$, define the vector $Z_{\mathcal{O}}(s, X)$ by defining its nth component $Z_{\mathcal{O}}(s, X)_n$ via

$$Z_{\mathcal{O}}(s,X)_n := Z_{\mathcal{O}/A}(s,X)_n := \sum_{i=0}^{\infty} X^i \sum_{\substack{\deg \mathbf{N}\mathcal{I}=i \\ \mathbf{N}\mathcal{I}\in\mathcal{C}_n}} \frac{1}{(I_n\mathbf{N}\mathcal{I})^s} \in K[[X]],$$

$$Z_{\mathcal{O}}(s)_n := Z_{\mathcal{O}/A}(s)_n := Z_{\mathcal{O}}(s,1)_n \in K_\infty.$$

(Here $I_n\mathbf{N}I$ stands for the generator of this ideal with sign in S).

Again the same remark as above applies to the second definition.

Also note that Z depends very simply on the choice of I_n's, with components of Z, for different choices of I_n's, being non-zero rational multiples of each other. In particular, the questions of when it (i.e., all the components) vanishes, when it is rational or algebraic etc. are independent of such choice.

REMARKS. Goss [G4] defines the zeta functions for a bigger space using a bigger piece of the character space to define n^s. Since the results we are interested in deal only with s integral, for us X is an indeterminate and we deal only with integral s. Also note that Goss' function on the bigger space has many more zeros. We leave the task of relating our definitions to Goss' to the reader. □

We use X as a deformation parameter for otherwise discretely defined zeta values and hence we define the order of vanishing ord_s of ζ, $\zeta_{\mathcal{O}}$ or $Z_{\mathcal{O}}$ at s to be the corresponding order of vanishing of $\zeta(s,X)$, $\zeta_{\mathcal{O}}(s,X)$ or $Z_{\mathcal{O}}(s,X)$ at $X = 1$. This procedure is justified by the results described below. It is easy to see that the order of vanishing is the same for s and ps, as the characteristic is p.

Now we state and prove a theorem which is essentially due to Lee [L1]. (See [T2] for the relevant history). For a non-negative integer $k = \sum k_i q^i$, with $0 \leq k_i < q$, we let $l(k) := \sum k_i$, hence $l(k)$ is the sum of base q digits of k.

THEOREM 1. *Let W be a $\mathbf{F}_q$-vector space of dimension d inside a field (or ring) $\mathcal{F}$ over $\mathbf{F}_q$. Let $f \in \mathcal{F} - W$. If $d > l(k)/(q-1)$, then $\sum_{w\in W}(f+w)^k = 0$.*

Proof. (Compare [L1]). Let $w_1, \cdots w_d$ be a $\mathbf{F}_q$-basis of W. Then $(f+w)^k = (f + \theta_1 w_1 + \cdots + \theta_d w_d)^k$, $\theta_i \in \mathbf{F}_q$. When you multiply out the k brackets, terms involve at most k of θ_i's, hence if $d > k$ the sum in the theorem is zero, since we are summing over some θ_i a term not involving it and $q = 0$ in characteristic p. The next observation is that in characteristic p, $(a+b)^k = \prod (a^{q^i} + b^{q^i})^{k_i}$, hence the sum is zero, if $d > l(k)$, by the argument above. Finally, note that $\sum_{\theta\in\mathbf{F}_q} \theta^j = 0$

unless $q-1$ divides j. Expanding the sum above by the multinomial theorem, we are summing multiples of products $\theta_1^{j_1}\cdots\theta_d^{j_d}$ and hence the sum is zero; because the sum of the exponents being $l(k) < (q-1)d$, all the exponents can not be multiples of $q-1$. □

For the applications, note that $\{a \in A : \deg(a) < i\}$ form such $\mathbf{F}_q$ vector spaces whose dimensions are given by the Riemann-Roch theorem. Similarly, A_{i+} are made up of affine spaces as in the theorem. Hence

THEOREM 2 (Goss [G1]). *For a negative integer s, $\zeta(s) \in A$ and $\zeta((q-1)s) = 0$.*

Proof: Our remark above shows that by the Theorem 1, the sum over i is a finite sum, hence the first statement follows. Now if $k = -(q-1)s$, then

$$\zeta((q-1)s) = \sum_{i=0}^{\infty}\sum_{a\in A_{i+}} a^k = -\sum_{i=0}^{\infty}\sum_{a\in A_i} a^k = 0,$$

where the first equality holds since $\sum_{\theta\in\mathbf{F}_q}\theta^{q-1} = -1$ and the second equality is seen by using that the sum is finite and applying the Theorem 1 with $W := \{a \in A : \deg(a) < m\}$ for some large m. □

REMARKS. Goss' proof of the first part used a weaker version of the Theorem 1, with $d > k/(q-1)$, the simpler version $d > k$ mentioned in the proof would also suffice. Goss' proof of the second part involved an inductive argument. The proof given here was mentioned in [T2] in the simplest case dealt there. It is interesting to note (as detailed in [T2]) that though Carlitz and Lee almost proved this, nobody before Goss seems to have dealt with exactly the zeta sum at negative integer as such, though Carlitz [C1] obtained results for positive integers. (See below). Since the zeta sums turn out to be finite and since a^s interpolates by Fermat's little theorem, we have [G1] $\wp$-adic interpolation for zeta values at negative integers, once we remove the Euler factor at the prime $\wp$ of A. □

Let us now recall what we know of the values of the Riemann zeta function ζ at integers. There is a simple functional equation relating values at s and $1-s$. At $s=1$, we have a pole. At even positive s, Euler showed that $\zeta(s)$ is a rational multiple of π^s or rather $(2\pi i)^s$. For odd positive s, we know only that $\zeta(3)$ is irrational (Apery). At negative integer s, $\zeta(s)$ is a rational, which is zero, if s is even and nonzero if s is odd.

In our context, multiples of $q-1$ are considered even and the rest are considered odd, since the cardinalities of A^* and $\mathbf{Z}^*$ are $q-1$ and 2 respectively. There is an analogue $\tilde{\pi}$ of $2\pi i$ coming from the theory of Drinfeld modules. There is no reasonable functional equation known. (We do have $\zeta(sp^n) = \zeta(s)^{p^n}$ though trivially). At $s=1$, we have no pole (the zeta sum converges), but at even positive

s, modeling Euler's proof in the setting of the theory of Drinfeld modules, Carlitz for $A = \mathbf{F}_q[t]$ (where in fact he developed the relevant theory from scratch and also identified Bernoulli numbers and factorials occurring in the zeta value and proved many analogies for them) and Goss in general (for $\delta = 1$, but it is trivial to remove this restriction as noted on [T1] pa. 46) showed that $\zeta(s)/\tilde{\pi}^s \in H$. For odd positive values, there are results in [A1], [A-T], [Y1] etc. At negative integer s, $\zeta(s)$ is integer in A (by the Theorem 2), which is zero, if s is even. Hence the question arises whether it is non-zero if s is odd. That it is, was proved by Goss for $A = \mathbf{F}_q[t]$ using a recursion formula for the values. See [G1]. It is not known in general.

THEOREM 3. *Let $\delta = 1$ and s be a negative odd integer. Then $\zeta(s)$ is non-zero if A has a degree one rational prime $\wp$ (e.g. if $g = 0$ (this reproves $\mathbf{F}_q[t]$ case) or if q is large compared to the genus of K) or if $h = 1$. (Together they take care of the genus 1 case). In fact, in the first case, $\zeta(s)$ is congruent to 1 mod $\wp$.*

Proof. The first part follows by looking at the zeta sum modulo $\wp$ and noticing that $\sum_{\theta \in \mathbf{F}_q} \theta^{-s} = 0$, hence the only contribution is 1 from the $i = 0$ term. Apart from $g = 0$, there are only 4 other A's with $h\delta = 1$ (these have no degree 1 primes) and they are listed in [H1] p. 213. For $q = 2$, there are no odd integers. That leaves only two possibilities: (a) $A = \mathbf{F}_3[x, y]/y^2 = x^3 - x - 1$ and (b) $A = \mathbf{F}_4[x, y]/y^2 + y = x^3 + \zeta_3$. In both the cases, we look at contribution to the zeta sum for each i modulo $x : i = 0$ gives 1, $i = 1, 2, 3$ give 0 and for higher i's, with a modulo x we also get θa, $\theta \in \mathbf{F}_q$ and hence the contribution is zero as before. □

REMARKS. In many particular examples of A we can settle the non-vanishing by this method. If we ask, instead, for odd negative s's for which $\zeta(s)$ is always (i.e., for any A) non-zero, the Theorem 1, says that $\zeta(s) = 1$, if $l(-s) < q - 1$. Again, this can be pushed farther: For example, if $l(-s) < 2(q - 1)$ (and s odd negative as before), then by the Theorem 1 and the $\mathbf{F}_q[t]$ case of Theorem 3 (by taking t to be an element of A of the lowest positive degree), we see that $\zeta(s)$ is nonzero. □

It should be mentioned that there is a close connection known between divisibility properties of Bernoulli numbers arising from the Riemann zeta values and the class groups of cyclotomic fields. For similar results, due to Goss, Okada, Shu and Sinnott, coming from the values of ζ at both positive and negative integers, see [G4] and [T5].

Let us now turn to the relative zeta functions.

THEOREM 4 (Goss [G4]). *For a negative integer s, $\zeta_{\mathcal{O}}(s) \in A$. In fact, $Z_{\mathcal{O}}(s)_n \in A$ for every n.*

Proof. (Goss [G4]). The idea is to decompose the sum in $\mathbf{F}_q$-vector spaces and use the Theorem 1 to show that for large k, the k-th term of the zeta sum is zero. It is enough to show that for each ideal class $\mathcal{C}$ of $\mathcal{O}$, the contribution from $\mathcal{I} \in \mathcal{C}$ is zero. Let $\mathcal{J} \in \mathcal{C}$. Then $\mathcal{I}\mathcal{J}^{-1} = (i) \subset \mathcal{J}^{-1}$. Hence it is sufficient to show that

$$S_k := \sum_{\substack{i \in \mathcal{J}^{-1}\{0\}/\text{units} \\ \deg \mathrm{N}(i)=k}} \mathrm{N}(i)^{-s} = 0$$

Let $\infty_1, \ldots, \infty_n$ be the infinite places of L with relative residue degrees $f_1, \ldots, f_n$ and the corresponding embeddings $\sigma_1, \ldots, \sigma_n$ respectively. Then, writing the polar divisor as $-(i)_\infty = \sum m_l(\infty_l)$, we have $k = \deg \mathrm{N}(i) = \delta \sum m_l f_l$. We will show that parts of S_k corresponding to a fixed choice of $\{m_l\}$ also vanish. Namely, let $\overline{D} := \sum m_l(\infty_l)$, then we will show that if the $\deg(\overline{D})$ is large, then $\sum \mathrm{N}(i)^s = 0$, where the sum is over $S := \{i \in \mathcal{J}^{-1} - \{0\}/\text{units}, -(i)_\infty = \overline{D}\}$.

Once we fix a polar divisor, the ambiguity in i is only through a finite field. Hence, given $i \in S$, fix a representative $r(i)$ by demanding that the sign of $\sigma_1(r(i))$ is 1, for some choice of the sign in L_{∞_1}. $\mathcal{L} := \{j \in \mathcal{J}^{-1} : (j) + \sum(m_l - 1)(\infty_l) \geqslant 0\}$ acts on S by translation: $j \circ r(i) = r(i) + j$. This is an action because as k is assumed large, this translation does not change the top degree and hence polar divisor or a sign. Further, if $(j_1 + r(i))/(j_2 + r(i)$ is a unit, then since its polar divisor is zero, it has to be in a finite field, but then the choice of the sign makes it 1 and hence $j_1 = j_2$. In other words, there is no isotropy.

Now we show that the contribution of each orbit of S under this action is zero. In other words, we have to show that $\sum \mathrm{N}(j + r(i))^{-s} = 0$, where now the sum is over j running though $\mathcal{L}$, which is a $\mathbf{F}_q$-vector space. Now $\mathrm{N}(j + r(i))^{-s} = \mu \prod \sigma_l(j + r(i))^{-s}$, where μ is the root of unity to take care of the sign, and can be taken out of the sum, as it is the same for all j. Hence a slight modification of the Theorem 1, with $\prod(f_l + w)$ in the place of $f + w$ (see [G4]), we see that the sum is zero, when k is large. This finishes the proof. □

THEOREM 5. *For a negative integer s, $\zeta_\mathcal{O}((q-1)s) = 0$. In fact, $Z_\mathcal{O}((q-1)s)_n = 0$ for every n.*

Proof. By the previous theorem, the sum over i defining the zeta value is finite, so the proof follows exactly as in the Theorem 2, the fact that some norms get repeated being irrelevant. □

REMARKS. (a) Let H_1 be the class field of K corresponding to $K^* \times t^{\mathbf{Z}} \times U_+$, where t is an uniformizer at ∞ with $\mathrm{sgn}(t) = 1$ and U_+ is the subgroup of the idele group consisting of those ideles with unit components at the finite places and with component at ∞ of sgn $= 1$ (see [T3] for the alternate definition and summary of basic properties of H_1, we just mention that $H_1 = H$ when $\delta = 1$). In the case L contains H_1, this theorem follows from the results of Goss [G4] described in the next section, which use L-functions.

(b) The proof of the Theorem 4 can be used to get bounds, similar to those obtained by the Theorem 1, on i for which the ith term of $\zeta_\mathcal{O}$ can be nonzero. For the abelian extensions, using L-function factorization, much better bounds follow immediately from the Theorem 1.

3. Orders of Vanishing

Now we turn to the question of the order of vanishing. Goss [G4] gave a very nice method to find a lower bound for orders of vanishing, when L contains H_1 and mentioned as an open question whether they are exact. We will show that they are not and indeed that the patterns of extra vanishing are quite surprising in terms of the established analogies.

The main idea of Goss is to turn the similarity in the definition of our zeta function with the classical one into a double congruence formula, using the Teichmuller character, and to use the knowledge of the classical L function Euler factors to understand the order of vanishing. This is done as follows. (See the references mentioned in the introduction for the facts recalled from the theory of Drinfeld modules).

Let $\wp$ be a prime of A and let W be the Witt ring of $A/\wp$. The identification $W/pW \cong A/\wp$ provides us with the Teichmuller character $w : (A/\wp)^* \to W^*$ satisfying $w^k(a \bmod \wp) = (a^k \bmod \wp) \bmod p$.

Now let $\Lambda_\wp$ denote the $\wp$-torsion of the rank one, sgn-normalised Drinfeld module ρ of generic characteristic. Let L contain H_1 and let G be the Galois group of $L(\Lambda_\wp)$ over L. Then G can be thought of as a subgroup of $(A/\wp)^*$ and hence w can be thought of as a W-valued character of G. Let $L(w^{-s}, u) \in W(u)$ be the classical L-series of Artin and Weil in $u := q^{-sm}$, where m is the extension degree of the field of constants of L over $\mathbf{F}_q$.

Let $S_\infty := \{\infty_j\}$ denote the set of the infinite places of L and let G_j denote the Galois group of $L_{\infty_j}(\Lambda_\wp)$ over L_{∞_j}. Then $G_j \subset \mathbf{F}_q^*$. Given s, let $S_s \subset S_\infty$ be the subset of the infinite places at which w^{-s} is an unramified character of G. Then S_s does not depend on $\wp$. Put

$$\tilde{\zeta}_{\mathcal{O}}(s, X) := \zeta_{\mathcal{O}}(s, X) \prod_{\infty_j \in S_s} (1 - w^{-s}(\infty_j) X^{\deg(\infty_j)})^{-1}.$$

THEOREM 6 (Goss [G4]). *Let L contain H_1 and let s be a negative integer. Then $\tilde{\zeta}_{\mathcal{O}}(s, X) \in A[X]$.*

Sketch of the proof. Tracing through the definitions, the property of w mentioned above gives the double congruence formula $L(w^{-s}, X^m) \bmod p = \tilde{\zeta}_{\mathcal{O}}(s, X) \bmod \wp$ for infinitely many $\wp$. But as the L function is known to be a polynomial, the result follows. □

This immediately gives the following lower bound for the order of vanishing.

THEOREM 7 (Goss [G4]). *Let L contain H_1 and let s be a negative integer, then the order of vanishing of $\zeta_{\mathcal{O}}(s)$ is at least*

$$V_s := \mathrm{ord}_{X=1} \prod_{\infty_j \in S_s} (1 - w^{-s}(\infty_j) X^{\deg(\infty_j)}).$$

REMARKS. Note that V_s depends on s only through its value modulo $q-1$ and $V_s \leqslant [L:K]$. This is analogous to the properties of the exact orders of vanishing in the classical case, but not in our case as we will see below. □

EXAMPLES: (i) $L = K = H_1$: (This is possible, when $h\delta = 1$). Then $V_s = 0$ or 1, according as s is odd or even. Goss proved that for $A = \mathcal{O} = \mathbf{F}_q[t]$, the order of vanishing is V_s.

(ii) L is totally real extension of degree d of K containing H_1, i.e., ∞ splits completely in L: V_s is 0 or d according as whether s is odd or even. (Note that the bounds V_s in (i) and (ii) are in analogy with the orders of vanishing in the classical case mentioned in the introduction).

(iii) $L = K(\Lambda_\wp)$: $V_s = (q^{\deg(\wp)} - 1)/(q-1)$ for all s.

(iv) $L = \mathbf{F}_{q^n}(t)$ and $K = \mathbf{F}_q(t)$: $V_s = 0$ is s is odd and for s even $V_s = p^k$, if $n = p^k l$, $(l, p) = 1$.

(v) $L = \mathbf{F}_5(\sqrt{-t})$ and $K = \mathbf{F}_5(t)$: V_s is 1 or 0 according as whether 2 (not 4) does or does not divide s. □

First few calculations we made on situations (ii) to (v) showed equality of the bounds with the exact orders of vanishing in the examples. (We detail the data afterwards). But in the following situation of (i) (which can be thought of as special case of (ii) also), we found that the bound is not exact:

Let (a) $A = \mathbf{F}_2[x,y]/y^2+y = x^3+x+1$, (b) $A = \mathbf{F}_2[x,y]/y^2+y = x^5+x^3+1$. In both these situations, $h\delta = 1$ and hence $K = H_1$. For small s we computed $\zeta(s, X)$ by hand as follows: The genus of K is 1 and 2 for (a) and (b) respectively. Hence by the bounds obtained from the Theorem 2 and the Riemann-Roch theorem shows that for (a) and (b), $\zeta(-1, X) = 1 + 0 * X + (x + (x+1)) * X^2 = 1 + X^2 = (1+X)^2$. Hence the order of vanishing is 2 for $s = -2^n$. Similar simple hand calculation showed that the order of vanishing is 1 if $2^n \neq -s \leqslant 9$ for (a) and is 1 for $s = -7$ and 2 for other s with $-s \leqslant 9$ for (b).

As the pattern, if indeed there should be one, was not clear to us, more calculations (we list the results below) were performed, using the method and bounds described above, on macsyma by Jennifer Johnson for example (b), first for $-s \leqslant 43$ not divisible by 2 (without loss of generality). This showed that within the range of computation the order of vanishing at s is 2 or 1 depending on whether $l(-s) \leqslant 2$ or not. From this we could see the pattern and prove the following theorem.

THEOREM 8. *If $\delta = 1$, $q = 2$ and K is hyperelliptic, then the order of vanishing of $\zeta(s)$ at negative integer s is 2 if $l(-s) \leqslant g$, where g is the genus of K*

Proof. Let $x \in A$ be an element of degree 2 and let n be the first odd non-gap for A at ∞. Then the genus g of K is seen to be $(n-1)/2$. Let $S_A(i)$ denote the coefficient of X^i in $\zeta_A(s, X)$. A simple application of the Theorem 1 and the Riemann-Roch theorem shows that for $i > l(-s) + g$, $S_A(i) = 0$. Hence

$$\zeta_A(s, X) = \sum_{i=0}^{(n-1)/2} S_A(2i) X^{2i}$$

since $l(-s) \leqslant g = (n-1)/2$. On the other hand, as x has degree 1 in $\mathbf{F}_q[x]$, we have $S_A(2i) = S_{\mathbf{F}_q[x]}(i)$, for $0 \leqslant i \leqslant (n-1)/2$. Further, by the Theorem 1, $S_{\mathbf{F}_q[x]}(i) = 0$, if $i > (n-1)/2$, as $(n-1)/2 \geqslant l(-s)$. Hence, $\zeta_A(s, X) = \zeta_{\mathbf{F}_q[x]}(s, X^2)$. Hence by the example (i) above, the order of vanishing is $2 = 2 * 1$ as required. □

REMARKS: (i) We do not need a restriction on the class number of K, so apart from (a) and (b) this falls outside the scope of the theorem giving the lower bounds.

(ii) Let A be of genus g, with $\delta = 1$, and with the gap structure at ∞ so that $l(iq\infty) = i + 1$, for $1 \leqslant i \leqslant r$ and $l((g + r)\infty) = r + 1$, $rq \leqslant g + r$. Then a straightforward generalization of the proof of the theorem above gives the vanishing order q (the lower bound V_s is 1) at even s such that $l(-s)/(q-1) \leqslant r$. For $q > 2$, the simplest such example would mean $\delta = 1, q = 3, r = 1, g = 5$ and $l(3\infty) = 2$, $l(6\infty) = 3$. I am obliged to J. F. Voloch for providing the following concrete example of this situation: $x^2(x-1)^3 - y^2 + yx^3 + y^3 + y^5 = 0$ with ∞ being $(1, 0)$. In fact, for this example $l(7\infty) = 3$, so we can take $r = 2$. Other variants of this phenomenon are also possible.

(iii) Analogies between the number fields and the function fields are usually the strongest for $A = \mathbf{F}_q[y]$. But even in that case, here is an example of extra vanishing, when $A = \mathbf{F}_4[y]$ and $\mathcal{O} = \mathbf{F}_4[x, y]/y^2 + y = x^3 + \zeta_3$: Order of vanishing of $\zeta_{\mathcal{O}}$ at $s = -1$ is 2 (as can be verified by direct computation, for example, by using the remark (b) following the Theorem 5) whereas the lower bound of the Theorem 7 is 1. □

We refer to the last section for some issues raised by this theorem.

We do not know yet whether in these cases, the order of vanishing is V_s, except for the extra vanishing exceptional situation mentioned in the theorem. This is true in the computational range and we have the following partial result (which can again be pushed further by the proofs along the same line), which shows that situation in general can only be worse as far as the pattern of zeros is concerned.

PROPOSITION: *For the example (a) above, the order of vanishing of $\zeta(s)$ is 1 ($= V_s$), if $l(-s) = 2$ or if $-s \equiv 0, 3, 5$ or $6 \bmod 7$.*

Proof. We know that $\zeta(s) = 0$ by the Theorem 2. Hence we have

$$\begin{aligned}(\zeta(s,X)/(X-1))|_{X=1} &= \frac{\mathrm{d}}{\mathrm{dX}}(\zeta(s,X))|_{X=1} = \sum_{i=0}^{\infty} \sum_{a\in A_{2i+1}} a^{-s} \\ &= \sum_{i=0}^{\infty} \sum_{a\in A_{2i}} a^{-s}.\end{aligned}$$

Since the genus is 1, by the Riemann-Roch theorem and the Theorem 1, we see that in the first case, the terms of the last sum vanish if $i > 1$, and hence the sum is $1 + x^{-s} + (x+1)^{-s} \neq 0$ as claimed. For $i > 1$, the terms of the second sum are all zero modulo y (as they can be decomposed as sums over pairs b and $b+y$), hence the sum is congruent to $1 + x^{-s} + (x+1)^{-s}$ modulo y, which takes care of the second case. □

We end this section by listing the computational results mentioned above. Let O_s denote the order of vanishing of the relevant zeta function at s. Note that in (c), (d) and (e) below, it agrees with V_s.

(a) For the example (a) above, for $0 < -s \leqslant 32$, O_s is 2 if $l(-s) \leqslant 1$ and 1 otherwise.
(b) For the example (b) above, and for $A = \mathbf{F}_2[x,y]/y^2 + y = x^7 + x + 1$ and for $A = \mathbf{F}_2[x,y]/y^2 + y = x^9 + x + 1$ respectively, for $0 < -s < 65$, s not divisible by 2 (without loss of generality), O_s is 2 if $l(-s) \leqslant 2$ or 3 or 4 respectively and 1 otherwise.
(c) $A = \mathbf{F}_3[x,y]/y^2 = x^3 - x - 1$. O_s for $0 < -s < 28$ is 1 if s is even and 0 otherwise.
(d) $A = \mathbf{F}_3[t]$ with $\mathcal{O} = \mathbf{F}_3[t,y]/y^2 = t^3 - t - 1$ and $\mathcal{O} = \mathbf{F}_3[\sqrt{t}]$ respectively, for $0 < -s < 31$, O_s is 1.
(e) $A = \mathbf{F}_3[t]$ with $\mathcal{O} = \mathbf{F}_9[t]$, for $0 < -s < 179$, O_s is 1 if s is even and 0 otherwise.

4. Values at Positive Integers

We now look at the values of the zeta functions at the positive integers. The results obtained are analogues of the classical results with the base being Q (with $2\pi i$ as the relevant period) or an imaginary quadratic field (with the relevant period being the period of elliptic curve with complex multiplication by the field. In this second case, the similar results for the absolute and relative zeta values may be known, but we do not know a suitable reference). For the necessary background on Drinfeld modules and the explicit class field theory based on them see the references mentioned in the introduction. Notice that since there is no functional equation known linking the values at positive and negative integers, these seem to be two distinct theories. Let $\tilde{\pi}$ be a fundamental period of a sgn-normalized rank one Drinfeld A module ρ with the lattice corresponding to ρ being $\tilde{\pi}A$. (This determines $\tilde{\pi}$ up to multiplication by an element of $\mathbf{F}_q^*$). This is an analogue of $2\pi i$. As an analogue of Euler's theorem we have,

THEOREM 9. *Let s be a positive even (i.e., a multiple of $q-1$) integer, then* $\zeta_A(s)/\tilde{\pi}^s \in H_1$.

REMARK. Much more precise version of this theorem was proved by Carlitz for the case $A = \mathbf{F}_q[t]$, using (what is now understood as) $\mathbf{F}_q[t]$ Drinfeld module. Goss proved the Theorem 9 under the additional hypothesis $\delta = 1$ using the Drinfeld A modules in general, with essentially the same method. The full Theorem 9 was proved in [T1], with the only additional ingredient in the proof being the handling of the signs as in the Section 2. Also note that if we use a period of a Drinfeld module defined over H instead, the ratio belongs to H.

THEOREM 10. (Goss [G4]). *Let ρ be a sgn-normalized rank one Drinfeld module with corresponding lattice $\tilde{\pi}_\rho I_\rho$, where I_ρ is an ideal of A. Let J be an ideal of A, $\Lambda_{\rho,J}$ be the J-torsion of ρ and let $\alpha \in J^{-1}I_\rho$. Then for s a positive integer, we have*

$$M_s := \tilde{\pi}_\rho^{-s} \sum_{\substack{i \in I_\rho \\ \alpha + i \neq 0}} (\alpha + i)^{-s} \in H_1(\Lambda_{\rho,J}).$$

Proof. We will drop the subscripts ρ in the proof. Let $\mathrm{e}(z)$ be the exponential function corresponding to ρ. It has coefficients in H_1. Also, $\mathrm{e}(\alpha\tilde{\pi})$ is a J-torsion point of ρ and hence belongs to $H_1(\Lambda_J)$. (For a nontrivial J, this field is the same as $K(\Lambda_J)$, by the class field theoretic results of Hayes, see p. 245 of [T3]). Hence, if we write $1/\mathrm{e}(z + \alpha\tilde{\pi}) = \sum c_n z^n$, then $c_n \in H_1(\Lambda_J)$. Now $\mathrm{e}(z) = z\prod(1 - z/(\tilde{\pi}i))$, where the product runs over $i \in I - \{0\}$. Taking the logarithmic derivative, if $\alpha \notin I$, so that $\alpha + i \neq 0$, we get

$$\begin{aligned} \frac{1}{\mathrm{e}(z+\alpha\tilde{\pi})} &= \frac{1}{\mathrm{e}(z) + \mathrm{e}(\alpha\tilde{\pi})} = \sum_{i \in I} \frac{1}{z + \tilde{\pi}(\alpha + i)} \\ &= -\sum_{n=0}^{\infty} \sum_{i \in I} \frac{z^n}{(\tilde{\pi}(\alpha + i))^{n+1}}. \end{aligned}$$

We get a similar equality if we multiply both sides by z to get rid of the pole at 0, if $\alpha \in I$. Comparison of the coefficients now proves the theorem. □

Proof of the Theorem 9. The special case $\alpha = 0$ and $J = I = A$ of the previous theorem shows that $\tilde{\pi}^{-s}\sum a^{-s} \in H_1$, where a runs through the nonzero elements of A. By looking at the signs, we see that the sum is zero, if s is odd and is $-\zeta(s)$, if s is even. □

Now we turn to the relative zeta functions.

THEOREM 11. *Let L be an abelian totally real (i.e., split completely at ∞) extension of degree d of K containing H and let s be a positive even (i.e., a multiple of $q-1$) integer. Then $R_s := \zeta_{\mathcal{O}}(s)/\tilde{\pi}^{ds}$ is algebraic. Furthermore, $R_s \in K(\Lambda_J)$ in the notation introduced below, $R_s^2 \in H_1$ and $R_s^{2n} \in H$ if $q^\delta - 1$ divides $2dsn$.*

Proof. Let $G = \{\sigma_i : 1 \leqslant i \leqslant d\}$ be the Galois group of L over K. By the explicit class field theory, L is contained in the maximal totally real subfield $K(\Lambda_J)^+$ of $K(\Lambda_J)$, where J is the conductor of L and Λ_J denotes the J-torsion of ρ.

To make the main ideas clear, let us first consider the special case $h\delta = 1$. (e.g., $A = \mathbf{F}_q[t]$).

As G is abelian, the relative zeta function factors into L functions: $\zeta_{\mathcal{O}}(s) = \prod_{\chi\in\hat{G}} L(\chi, s)$, where $L(\chi, s) := \prod(1-\chi(P)P^{-s})^{-1} = \sum \chi(I)I^{-s}$, where I (P respectively) runs through (irreducible respectively) elements of A_+. Note that since s is even, I^s is independent of the choice of signs. (Here the subtlety, important when p divides d, (see also the remarks below or [G4] p. 345) is that we consider χ as coming from reductions to characteristic p of $\overline{\mathbf{Q}_p}$ valued characters, so that e.g., if $G = \mathbf{Z}/p$ we still consider p characters (all trivial), rather than just one). Since this decomposition is through the Euler factors matching and since each Euler factor of $\zeta_{\mathcal{O}}$ lies in K, it is enough to show the corresponding statements for $R_s' := \prod L(\chi, s)_J/\tilde{\pi}^{ds}$, where the subscript J indicates that Euler factors for J have been removed.

For $\sigma \in G$, let $\zeta_\sigma := \sum I^{-s}$, where the sum is over $I \in A_+$, prime to J and with the Artin symbol $\sigma_I = \sigma$.

Then we have $L(\chi, s)_J = \sum \chi(\sigma)\zeta_\sigma$. Furthermore, the ideal group corresponding to $K(\Lambda_J)^+$ consists of principal ideals generated by elements congruent to one modulo J. Hence ζ_σ is K-linear combination of sums in the Theorem 10, which then implies that $R_s \in K(\Lambda_J, \mu)$, where μ is the root of unity of the order equal to the exponent of G, so that $\chi \in \mathbf{F}_q(\mu)$. Hence R_s is algebraic. To get a better control on the field which contains it, we use the Dedekind determinant formula: $\prod L(\chi, s)_J = \det(\zeta_{\sigma_i^{-1}\sigma_j})_{d\times d}$.

Hence $R_s' = \det(\zeta_{\sigma_i^{-1}\sigma_j}/\tilde{\pi}^s)$, where the entries lie in $K(\Lambda_J)$ by above. The action of Gal$(K(\Lambda_J)/K)$ permutes the rows of the determinant, multiplying it by ± 1, hence $R_s^2 \in K$ and this completes the proof when $h\delta = 1$.

In the general case, in the definitions given above, $I \in A_+$ should be replaced by an ideal I. So if s is an even multiple of the class number $h\delta$ of A, we can replace I^s by an element in the obvious fashion and the proof above works just the same. But in general, to pass from ideals to elements, we proceed as follows.

Let $R := \{I_1, \ldots, I_{h\delta}\}$ be collection of the ideals of A prime to J representing the ideal classes of A. Since $H \subset L$ and since the Galois group of H over K

is naturally isomorphic to the ideal class group of A, we have a surjective map $\sigma \in G \mapsto r(\sigma) \in R$. Now $\prod r(\sigma)^2$ is the identity in the ideal class group, by a simple argument in group theory. (Pair elements with inverses and look at order 2 elements). Hence this product is a principal ideal (a) and since s is even, a^s is defined independent of the choice of the generator a of (a), just as $\zeta_{\mathcal{O}}(s)$ is then defined independent of the choice of S.

Next, we define I^s (for s an even integer) in such a way that $(IJ)^s = I^s J^s$ and for a principal ideal $I = (i)$, $I^s = i^s$, so that Dedekind determinant formula and factorization into L-series would get through.

The class group of A is a finite abelian group and hence a direct product of cyclic groups C_j of orders c_j, $1 \leqslant j \leqslant k$ say. We fix ideals $I_1, \ldots, I_k$ representing the classes given by some generators of these cyclic groups. Let $I_j^{c_j} = (a_j)$. Let I_i, $1 \leqslant i \leqslant h\delta$ be the ideals representing all the ideal classes of A obtained by taking the obvious monomials in the chosen ideals.

We first define for $1 \leqslant j \leqslant k$, $I_j^s := (a_j^s)^{1/c_j}$ where we fix the c_jth root arbitrarily. Then we extend this definition to I_i^s, $1 \leqslant i \leqslant h\delta$ by multiplicativity using the monomial defining I_i. Finally, given any I, for a unique I_l, we have $I = I_l(b_l)$ and we define $I^s := I_l^s b_l^s$. It is straightforward to check the consistency and the two properties mentioned above.

Notice that the ambiguity in passing from the principal ideal to its generator is irrelevant here, since the sth power that we take kills it. Hence, exactly as above, we finally get $\zeta_{\mathcal{O}}(s)_J/\tilde{\pi}^{ds} = \det(\zeta_{\sigma_i^{-1}\sigma_j} r(\sigma^{-1})^s r(\sigma_j)^s/\tilde{\pi}^s)a^{-2s}$, where the subscript J again denotes the restriction to prime to J ideals.

(Another way to see that we can extend the exponentiation multiplicatively from the principal ideals: $(a)^s = a^s$ to all ideals, is to note that since Ω^* is a divisible abelian group ($\mathbf{Z}$-module), it is injective and hence the exponentiation homomorphism from the group of principal nontrivial fractional ideals to Ω^* extends to the bigger group of all nontrivial fractional ideals. Notice also that when $\delta = 1$ or when the class of ideals is restricted to positively generated ideals (eg. norms from L if L contains H_1), then our approach defines the ideal exponentiation for any integral s satisfying the properties listed above, by choosing positive generators of the principal ideals involved. Similar remark holds when S can be chosen multiplicatively. Reader should also look at 3.3 of [G4], where the ideal exponentiation is defined on a much bigger space of exponents by more sophisticated technology. We do not need this for our limited purpose here).

Again, keeping in mind the natural correspondence between ideal classes and conjugate sgn-normalized Drinfeld modules, the entries in the determinant can be expressed in terms of the quantities of the Theorem 10 by the class field theory. The action of Gal$(H_1(\Lambda_J)/H_1)$ as before, implies $R_s^2 \in H_1$. The final statement follows since $\zeta_{\mathcal{O}}(s) \in K_\infty$ as well as $\tilde{\pi}^{q^\delta - 1} \in K_\infty$ by e.g., [T1] p. 19.

This finishes the proof of the theorem. □

REMARKS. (I) A version of the theorem, with $q^\delta - 1$ in place of $q-1$, is due to Goss [G4] and the proof here is also a minor variant of his proof. Classically, an analogous result for arbitrary (not necessarily abelian over $\mathbf{Q}$) totally real number field is known. For the history and references on this general as well as the abelian case using L-functions, as we have done here, see [K] and see [G4] for ideas about carrying over the proof in the general case. On the other hand, such an algebraicity result for the ratio of the relative zeta value with an appropriate power of the period $2\pi i$ is not expected for number fields which are not totally real. But in our case, we can have such a result even if L is not totally real, as was noted in [G3]: Let L be a Galois extension of degree p^k of K, then since all the characters of the Galois group are trivial, the L-series factorization shows that $\zeta_{\mathcal{O}}(s) = \zeta(s)^{p^k}$ and the result follows then from the Theorem 9. (More elementary way to see this when the degree is p and $\mathcal{O}$ is of class number one is to note that for $\alpha \in \mathcal{O} - A$, there are p conjugates with the same norms which then add up to zero, whereas for $\alpha \in A$, the norm is α^p). This can also be used to create examples of situations where the exact orders of vanishing are known.

(II) From the results stated in the introduction, values of the Dedekind zeta function at negative integers are all zero, if the number field is not totally real. By (I), similar result does not hold in our case. In fact, we do not even need degree to be a power of the characteristic, as can be seen from the examples (c) and (e) in the last section. The ramification possibilities for the infinite places are much more varied in the function field case.

(III) The following example shows that we can not replace H in the theorem, when $\delta = 1$, by K in general, answering a question raised by Goss [G4]. Take $A = \mathbf{F}_3[x,y]/y^2 = x^3 + x^2 - 1$, which has $h = 3$ and $\delta = 1$. Let L be H. Then L is a totally real abelian extension of K of degree 3 containing H and by the remark (I), $\zeta_{\mathcal{O}}(2)/\tilde{\pi}^6 = \zeta(2)^3/\tilde{\pi}^6 = (x - x^3)^3/x_1^3$ by (12) and (1) of [T4], where x_1 is the coefficient of F in ρ_x for ρ defined on pa. 214 of [H1] and hence generates H over K and hence x_1^6 can not be in K, otherwise H would not be a separable extension. □

5. Open Questions

Finally, we briefly discuss some open questions raised by these results. Clearly the main question is how to predict the exact orders of vanishing in general and whether there is any reasonable functional equation relating the values at positive integers to those at the negative integers.

The results of Goss and Sinnott, mentioned above, come via congruence with the classical L-function and hence relate the arithmetic of the class group to the V_sth term of the expansion of the zeta value at $X = 1$, rather than the leading term. It is not clear what information the leading term provides in the cases of the extra vanishing.

There are also simpler open questions such as whether $\zeta(s)$ can be zero at odd negative s if the hypotheses of the Theorem 3 are not met or whether there are any non-totally real extensions (which are not Galois of p-power degree) for which the values at positive even integers are related to the periods as in the Theorem 11 or whether anything special happens at the positive integers in the cases of extra vanishing. Another open question, already alluded to in the remark (I) of the Section 4, is whether the Theorem 11 generalizes to arbitrary totally-real extension (perhaps via Eisenstein series, see [G4]).

We hope that the reader will take the challenge and solve some of these questions.

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