M. RAPOPORT

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M. RAPOPORT[1] and M. RICHARTZ[2]
[1]*FB Mathematik, Bergische Universität, Gaussstrasse 20, 42097 Wuppertal, Germany*
[2]*Mathematisches Institut, Universität Bonn, Wegelerstr. 10, 53115 Bonn, Germany*

Abstract. We generalize Grothendieck's semicontinuity theorem for F-isocrystals over a base scheme of characteristic p to F-isocrystals with G-structure, where G is a connected reductive algebraic group over $\mathbf{Q}_p$.

Key words: F-isocrystal.

Contents

Introduction

Let k be a perfect field of characteristic $p > 0$ and let K be the fraction field of its Witt ring $W(k)$. Let σ be the Frobenius automorphism of K. An F-isocrystal over k is a finite-dimensional K-vector space V, together with a σ-linear bijection $\Phi: V \to V$.

The notion is due to Dieudonné who classified these objects in the 1950's in case k is algebraically closed. He showed that an F-isocrystal over an algebraically closed field is determined up to isomorphism by its Newton polygon or, equivalently, its slopes. In the 1960's Grothendieck introduced the notion of an F-isocrystal over a general base scheme S of characteristic p which makes precise the heuristic idea of a family of F-isocrystals over perfect fields parametrized by the points of S. Grothendieck ([G], appendix; comp. also [Ka]) proved the basic theorem that the Newton polygon rises under specialization $s \to s'$ and that its end point remains constant. Katz [Ka] subsequently investigated the question whether the

constancy of the Newton polygon under specialization implies the constancy of the F-isocrystal.

The theory took a new turn with the injection of algebraic groups into the theory by Kottwitz [K]. His starting point is the observation that the isomorphism classes of F-isocrystals (V, Φ) of height $h = \dim V$ are in bijective correspondence with the σ-conjugacy classes in $GL_h(K)$. He investigated the set $B(G)$ of σ-conjugacy classes in $G(K)$ where G is any connected reductive group over $\mathbf{Q}_p$, in case k is algebraically closed. (In fact, Kottwitz considers the case where $\mathbf{Q}_p$ is replaced by a finite extension, but in this introduction we will disregard this). Kottwitz introduces the subset $B(G)_{\text{basic}}$ of *basic elements* of $B(G)$, characterized by the fact that the *slope homomorphism* of any representative in $G(K)$ factors through the center of G, and gives a complete description of $B(G)_{\text{basic}}$. Furthermore, he is able to describe all of $B(G)$ in the case when G is quasi-split (thereby generalizing Dieudonné's results) by taking the basic sets of the various Levi subgroups as building blocks.

To explain our results consider the *Newton map*

$$B(G) \to X_*(T)_{\mathbf{Q}}/\Omega$$

which generalizes the Newton polygon associated to an F-isocrystal. Here T is a maximal torus of G with Weyl group Ω. The map associates to $\bar{b} \in B(G)$ the conjugacy class of the slope homomorphism of any representative in $G(K)$. The fibres of this map are principal homogeneous spaces under finite abelian groups of the form $H^1(\mathbf{Q}_p, J)$, where J is a Levi subgroup of a quasi-split inner form of G (depending on the image point). In the case of $G = GL_h$, these cohomology groups are trivial and we recover Dieudonné's results.

Our purpose in the present paper is to generalize Grothendieck's specialization theorems. To this end we introduce on the target space of the Newton map a partial ordering which generalizes the (reverse of the) usual partial ordering on Newton polygons with same end points. We define the notion of an F-isocrystal with G-structure over a base scheme S and associate to such an object a function $s \mapsto \bar{b}(s) \in B(G)$. The generalization of Grothendieck's theorem is that the Newton point of $\bar{b}(s)$ decreases under specialization. The proof is by reduction to Grothendieck's theorem. Furthermore, we prove that if S is connected and the Newton point of $\bar{b}(s)$ is constant, then so is $\bar{b}(s)$. For $G = GL_h$, this last statement is vacuous since in this case the Newton point determines the F-isocrystal up to isomorphism. For the proof we use the result of Katz mentioned above. The constancy of the end point of the Newton polygon in Grothendieck's theorem also has a counterpart in the general situation, but it has then a somewhat subtle cohomological meaning.

We now give a brief description of the various Sections. In Section 1 we give an account of most of the results of [K] with two noteworthy modifications. First, we use the algebraic fundamental group of Borovoi [B] instead of the center of the Langlands dual group used by Kottwitz. The gain is that the results are obviously

functorial. Second, as mentioned above, we reformulate some of his results in terms of the Newton map which enables us to say something even about the non-basic part of $B(G)$ for any connected reductive group and not only for quasisplit ones as in Kottwitz. This Section is largely expository and contains almost no proofs.

In Section 2 we introduce the partial order mentioned above. Our specialization result for Newton points is completely analogous to the specialization result for the Harder-Narasimhan polygons of vector bundles resp. G-bundles on a Riemann surface of Atiyah and Bott [AB]. However, perhaps surprisingly, their result is the exact opposite of ours. In this context the partial ordering had already been introduced in [AB]. We therefore content ourselves with quoting their results.

In Section 3 we prove the specialization results alluded to above. We also mention here the generalization (3.13) of Grothendieck's conjecture on the converse to his specialization theorem.

Section 4 is an afterthought to the proof in Section 3. In it we generalize Mazur's theorem that the Hodge polygon of an F-crystal over an algebraically closed field lies below the Newton polygon of the corresponding F-isocrystal and that both have the same end points.

1. The structure of $B(G)$

In this Section we give a presentation of some results of Kottwitz, [K], [K2].

1.1 – In this Section we will use the following notations, comp. [K].

k	–	an algebraically closed field of characteristic p.
K	–	the fraction field of the Witt ring $W(k)$.
$\bar{K}$	–	an algebraic closure of K.
F	–	a finite extension of $\mathbf{Q}_p$ in $\bar{K}$.
L	–	the compositum of K and F in $\bar{K}$.
σ	–	the Frobenius automorphism of L/F.
$W(\bar{K}/F)$	–	the Weil group, i.e. the group of continuous automorphisms of $\bar{K}$ which fix the elements of F and induce on the residue field k of $\bar{K}$ an integral power of the Frobenius automorphism.
Γ	–	the Galois group of $\bar{F}/F$.

1.2 – Let G be a connected reductive group over F. Let

$$B(G) = G(L)/\sim,$$

where the equivalence relation is σ-conjugacy, i.e.

$$x \sim y \Leftrightarrow x = gy\sigma(g)^{-1}, \quad g \in G(L).$$

The fact that k does not appear in this notation is justified by the following lemma (for $F = \mathbf{Q}_p$, this is proved in [RZ], (1.16); the general case is the same).

LEMMA 1.3 *Let $k' \subset k$ be an algebraically closed subfield and let $L', \sigma', B'(G)$ be the corresponding objects for k' instead of k. The obvious map*

$$B'(G) \to B(G)$$

is a bijection.

1.4 – There is an exact sequence of topological groups

$$1 \to \mathrm{Gal}(\bar{K}/L) \to W(\bar{K}/F) \to \langle \sigma \rangle \to 1,$$

where $\langle \sigma \rangle$ denotes the infinite cyclic (discrete) group generated by σ. By Steinberg's theorem the induced map

$$B(G) = H^1(\langle \sigma \rangle, G(L)) \to H^1(W(\bar{K}/F), G(\bar{K})),$$

is a bijection. On the other hand, the restriction homomorphism $W(\bar{K}/F) \to \mathrm{Gal}(\bar{F}/F)$ and the inclusion $G(\bar{F}) \subset G(\bar{K})$ define an injective map ([K], 1.8.3)

$$H^1(F, G) \to H^1(W(\bar{K}/F), G(\bar{K})) = B(G).$$

1.5 – Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of connected reductive groups over F. Then there is an exact sequence of pointed sets ([K], Sect. 1)

$$1 \to G_1(F) \to G_2(F) \to G_3(F) \to B(G_1) \to B(G_2) \to B(G_3) \to 1.$$

1.6 – Let F' be a finite extension of F contained in $\bar{K}$. Let G' be a connected reductive group over F' and let $B'(G')$ be the corresponding set for G', L', σ'. Then there is a Shapiro isomorphism ([K], Sect. 1).

$$B(\mathrm{Res}_{F'/F}(G')) \to B'(G').$$

1.7 – Let $\mathbf{D}$ be the pro-algebraic torus with character group $\mathbf{Q}$. For a connected reductive group G over F we put

$$\mathcal{N}(G) = (\mathrm{Int}\, G(L) \setminus \mathrm{Hom}_L(\mathbf{D}, G))^{\langle \sigma \rangle},$$

(set of σ-invariants in the set of conjugacy classes of homomorphisms $\mathbf{D}_L \to G_L$). For instance, if $G = T$ is a torus, then

$$\mathcal{N}(T) = X_*(T)^{\Gamma} \otimes \mathbf{Q}.$$

More generally, if $T \subset G$ is a maximal torus with Weyl group Ω, then

$$\mathcal{N}(G) = (X_*(T)_{\mathbf{Q}}/\Omega)^{\Gamma}.$$

THEOREM 1.8 ([K], Sect. 4) *Let $b \in G(L)$. Then there exists a unique element $\nu \in \mathrm{Hom}_L(\mathbf{D}, G)$ for which there exists an integer $s > 0$, an element $c \in G(L)$ and a uniformizing element π of F such that*

(i) $s\nu \in \mathrm{Hom}_L(\mathbf{G}_m, G)$.
   (Here $\mathrm{Hom}(\mathbf{G}_m, G) \subset \mathrm{Hom}(\mathbf{D}, G)$ *via the homomorphism* $\mathbf{D} \to \mathbf{G}_m$ *induced by the inclusion of character modules* $\mathbf{Z} \subset \mathbf{Q}$).
(ii) $\mathrm{Int}(c) \circ s\nu$ *is defined over the fixed field of* σ^s *in* L.
(iii) $c \cdot b \cdot \sigma(b) \cdot \ldots \cdot \sigma^s(b) \cdot \sigma^s(c)^{-1} = c \cdot (s\nu)(\pi) \cdot c^{-1}$.
   The element ν *is called the* slope *homomorphism associated to* b.

   Furthermore, the map $b \mapsto \nu = \nu_b = \nu_{G,b}$ *has the following properties.*

(a) $\sigma(b) \mapsto \sigma(\nu)$.
(b) $gb\sigma(g)^{-1} \mapsto \mathrm{Int}(g) \circ \nu$,$g \in G(L)$.
(c) $\nu_b = \mathrm{Int}(b) \circ \sigma(\nu_b)$.
(d) ν_b *is trivial if and only if* b *is in the image of the map (cf. (1.4))* $H^1(F, G) \to B(G)$.

1.9 – From (b) and (d) of the previous theorem it follows that the map $b \mapsto \nu_b$ induces a natural transformation of set-valued functors on the category of connected reductive algebraic groups

$$\bar{\nu} \colon B(.) \to \mathcal{N}(.) .$$

Here, denoting by a bar the σ-conjugacy class resp. the conjugacy class

$$\bar{\nu}_G(\bar{b}) = \bar{\nu}_b , \; b \in \bar{b}.$$

The map $\bar{\nu}_G$ is called the *Newton map of the group* G.

EXAMPLE 1.10 Let $G = GL(V)$, where V is a finite-dimensional F-vector space. Then $B(G)$ classifies the isomorphism classes of $\sigma - L$-spaces of height $h = \dim V$. To $b \in G(L)$ we associate the $\sigma - L$-space (i.e. a finite-dimensional L-vector space with a σ-linear bijective endomorphism)

$$(V_L, \Phi) = (V \otimes_F L, b(\mathrm{id}_V \otimes \sigma)).$$

There exist uniquely determined rational numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_r$$

and a uniquely determined decomposition

$$V_L = \bigoplus_{i=1}^{r} V_i,$$

into Φ-stable subspaces for which there exist O_L-lattices $M_i \subset V_i$ with

$$\Phi^{h_i} M_i = \pi^{d_i} M_i, \qquad h_i = \dim_L V_i,$$

where $d_i = \lambda_i \cdot h_i \in \mathbf{Z}$. The subspace V_i is called the *isotypical component of slope* λ_i. The associated homomorphism ν_b is equal to

$$\nu_b = \bigoplus_{i=1}^{r} \lambda_i \cdot \mathrm{id}_{V_i}.$$

Here $\lambda_i \cdot \mathrm{id}_{V_i}$ denotes the composition

$$\mathbf{D} \xrightarrow{\lambda_i} \mathbf{G}_m \subset GL(V_i).$$

In this case the map

$$\bar{\nu}_G \colon B(G) \to \mathcal{N}(G)$$

is injective, as follows from the Dieudonné classification of $\sigma - L$-spaces [K], Section 3. It is customary to use the slopes $(\lambda_1, \ldots, \lambda_r)$ and their multiplicities $(h_1, \ldots, h_r)$ to form the Newton polygon of the $\sigma - L$-space (V_L, Φ), which explains the name we have given to the map in general.

1.11 – Let $b \in G(L)$. We consider the following group-valued functor on the category of F-algebras. To an F-algebra R it associates the group

$$J_b(R) = \{g \in G(L \otimes_F R); \; g = b\sigma(g)b^{-1}\}.$$

Then ([RZ], (1.12)) this functor is representable by a connected reductive group J_b over F. Let $b' = h^{-1}b\sigma(h)$. Then Int h^{-1} induces an F-isomorphism

$$J_b \to J_{b'}. \tag{1}$$

Let $\bar{b} \in B(G)$ and let $b \in \bar{b}$ be an element such that $s\nu_b$ factors through $\mathbf{G}_m$ and is defined over the fixed field F_s of σ^s in L and such that

$$b\sigma(b) \ldots \sigma^{s-1}(b) = s\nu_b(\pi),$$

for a suitable integer $s > 0$ and a uniformizer π in F, cf. (1.8). Then ([RZ], (1.14)) $J_b \otimes_F F_s$ is the Levi subgroup of $G \otimes_F F_s$ which centralizes the 1-parameter subgroup $s\nu_b$,

$$J_b \otimes_F F_s \simeq G_{s\nu_b}.$$

Let h be such that $b = hb\sigma(h)^{-1}$. Then $h \in J_b(F)$. Therefore the F-isomorphism in (1) is unique up to inner automorphisms by elements in $J_b(F)$.

PROPOSITION 1.12 ([K], Sect. 5). *Let* $b \in G(L)$. *The following conditions are equivalent.*

(i) *The homomorphism* ν_b *factors through the center of* G.
(ii) *The* σ-*conjugacy class of* b *contains an element contained in an elliptic torus of* G.
(iii) *The group* J_b *is an inner form of* G.
(iv) (*in case* $G = GL(V)$, *as in example* (1.10)). *The slope decomposition has only one factor.*

In this case the element b resp. its σ-conjugacy class $\bar{b}$ is called *basic*. We denote by $B(G)_{\text{basic}}$ the set of basic σ-conjugacy classes.

1.13 – In order to state the next results we introduce the algebraic fundamental group of a connected reductive group G over F ([B], comp. also [M]). Let $T \subset B \subset G_{\bar{F}}$ be a maximal torus and a Borel subgroup *defined over* $\bar{F}$. We have an action of Γ on $X_*(T)$ defined by

$$\tau \bullet \nu = \text{Int}(g) \circ \tau(\nu), \qquad \tau \in \Gamma,$$

where $g \in G(\bar{F})$ satisfies $g(\tau(T, B))g^{-1} = (T, B)$. We obtain an induced action of Γ on

$$\pi_1(G, T) = X_*(T)/\Sigma_{\alpha \in \Phi(G,T)} \mathbf{Z} \alpha^{\vee},$$

which is independent of the choice of B. Here $\Phi(G, T)$ denotes the set of roots of T and for $\alpha \in \Phi(G, T)$ we denote by $\alpha^{\vee}$ the corresponding coroot.

If $T' = gTg^{-1}$, $g \in G(\bar{F})$, then $\text{Int}(g)$ induces a Γ-equivariant isomorphism

$$\pi_1(G, T) \rightarrow \pi_1(G, T')$$

which is independent of the choice of g. We therefore may define $\pi_1(G)$ as the common value of these Γ-modules. It is called the *algebraic fundamental group of* G. The functor π_1 is an exact functor from the category of connected reductive groups over F to the category of finitely generated discrete Γ-modules, ([B], (1.5)). If G' is an inner form of G, there is a canonical isomorphism

$$\pi_1(G) = \pi_1(G').$$

1.14 – Kottwitz [K] formulates his results in terms of the center $Z(\hat{G})$ of the Langlands dual group, which a priori is functorial only for morphisms with image a normal subgroup. We prefer to formulate his results in terms of the algebraic fundamental group since this is functorial for all morphisms. To make

the connection we point out that there is a canonical isomorphism of Γ-modules ([B], (1.10))

$$X^*(Z(\hat{G})) = \pi_1(G).$$

In particular

$$X^*(Z(\hat{G})^{\Gamma}) = \pi_1(G)_{\Gamma},$$
$$\text{Hom}(\pi_0(Z(\hat{G})^{\Gamma}), \mathbf{C}^{\times}) = (\pi_1(G)_{\Gamma})_{\text{tors}}.$$

(coinvariants resp. torsion subgroup in coinvariants).

We also point out that

– for a torus T,

$$\pi_1(T) = X_*(T),$$

– for a semi-simple group G, with simply connected covering $\varrho\colon G_{sc} \to G$,

$$\pi_1(G) = \text{Ker}\,\varrho(-1) = \varinjlim_n \text{Hom}(\boldsymbol{\mu}_n(\bar{F}),\ \text{Ker}\,\varrho),$$

– if the derived group of the connected reductive group G is simply connected,

$$\pi_1(G) \xrightarrow{\ \sim\ } \pi_1(G_{ab}).$$

Here, as in the rest of the paper, G_{ab} denotes the factor group of G by its derived group.

THEOREM 1.15 ([K2], Sect. 6), [K]) (i) *There exists a unique natural transformation*

$$\gamma\colon B(\cdot) \to \pi_1(\cdot)_{\Gamma}$$

of set-valued functors on the category of connected reductive groups over F such that the following diagram is commutative.

$$
\begin{array}{ccc}
L^{\times} & \xrightarrow{\ \text{ord}_L\ } & \mathbf{Z} \\[2pt]
\big\downarrow & & \big\| \big\| \\[2pt]
B(\mathbf{G}_m) & \xrightarrow{\ \gamma_{\mathbf{G}_m}\ } & \pi_1(\mathbf{G}_m)_{\Gamma}.
\end{array}
$$

Here the valuation on L is normalized by $\text{ord}_L(\pi_L) = 1$ for a uniformizer π_L.

Furthermore, the induced maps

$$B(G)_{\text{basic}} \to \pi_1(G)_\Gamma,$$

and

$$H^1(F, G) \to (\pi_1(G)_\Gamma)_{\text{tors}} \tag{$cf.$ (1.4)}$$

are bijections for all G. This puts the structure of abelian groups on $B(G)_{\text{basic}}$ and on $H^1(F, G)$ in a functorial way. The action of the subgroup $H^1(F, G)$ on $B(G)_{\text{basic}}$ by translations preserves the fibres of (the restriction to $B(G)_{\text{basic}}$ of) the Newton map and is simply transitive on each fibre of $\bar{\nu}_G|B(G)_{\text{basic}}$.

(ii) *Let $G = T$ be a torus, in which case $B(T) = B(T)_{\text{basic}}$ and $\pi_1(T) = X_*(T)$. Then the structure of abelian group on $B(T)$ defined in* (i) *is the natural structure. Let $\tilde{\gamma}_T$ be the composition*

$$X_*(T) \to X_*(T)_\Gamma \xrightarrow{\gamma_T^{-1}} B(T).$$

Let E be a finite extension of F contained in $\bar{K}$ such that T splits over E and let $E_0 = E \cap L$ be the maximal subfield of E unramified over F. Let π_E be a prime element in E. Then

$$\tilde{\gamma}_T(\mu) = \textit{the σ-conjugacy class of } N_{E/E_0}(\mu(\pi_E)), \qquad \mu \in X_*(T).$$

The following diagram is commutative, if E is a finite Galois extension of F.

$$
\begin{array}{ccc}
\hat{H}^{-1}(E/F, X_*(T)) & \xrightarrow{\text{TN}} & H^1(E/F, T(E)) = H^1(F, T) \\
\downarrow{\scriptstyle\text{can}} & & \downarrow{\scriptstyle\gamma_T} \\
X_*(T)_\Gamma & \longleftarrow & (X_*(T)_\Gamma)_{\text{tors}}
\end{array}
$$

Here TN denotes the Tate-Nakayama isomorphism (cup product with the fundamental class in $H^2(E/F, E^\times)$).

(iii) *For any connected reductive group G over F, the natural homomorphism $G \to G_{ab}$ induces an isomorphism of vector spaces*

$$\pi_1(G)^\Gamma \otimes \mathbf{Q} = \pi_1(G_{ab})^\Gamma \otimes \mathbf{Q}.$$

The functor $\pi_1(\,.\,)^\Gamma \otimes \mathbf{Q}$ is an exact functor from the category of connected reductive groups over F to the category of finite-dimensional $\mathbf{Q}$-vector spaces.

There is a unique natural transformation of functors on the category of connected reductive groups

$$\delta : \mathcal{N}(\,.\,) \to \pi_1(\,.\,)^\Gamma \otimes \mathbf{Q}$$

such that for a torus T this is the natural identification (cf. (1.7))

$$\mathcal{N}(T) = X_*(T)^\Gamma \otimes \mathbf{Q} = \pi_1(T)^\Gamma \otimes \mathbf{Q}.$$

The following diagram is functorial with exact rows (in the sense of pointed sets).

$$
\begin{array}{ccccc}
H^1(F, G) & \longrightarrow & B(G) & \xrightarrow{\ \bar{\nu}_G\ } & \mathcal{N}(G) \\
\gamma \downarrow & & \gamma \downarrow & & \downarrow \delta \\
(\pi_1(G)_\Gamma)_{\text{tors}} & \longrightarrow & \pi_1(G)_\Gamma & \longrightarrow & \pi_1(G)^\Gamma \otimes \mathbf{Q}
\end{array}
\tag{2}
$$

Here the arrow in the right lower corner is given as

$$\bar{\mu} \mapsto |\Gamma \cdot \mu|^{-1} \sum_{\mu' \in \Gamma \cdot \mu} \mu'.$$

We often write $\delta(\bar{b}) = \delta_G(\bar{b})$ instead of $\delta \circ \bar{\nu}(\bar{b})$, for $\bar{b} \in B(G)$.

1.16 – The statement (i) in the previous theorem gives a complete description of the *basic subset* of $B(G)$. We now want to describe the fibres of the Newton map through an arbitrary element $\bar{b} \in B(G)$ which is not necessarily basic.

PROPOSITION 1.17 *Let $\bar{b} \in B(G)$ and let $\mathcal{F}$ be the fibre of the Newton map through $\bar{b}$,*

$$\mathcal{F} = \{\bar{b}' \in B(G); \ \bar{\nu}_{G,\bar{b}'} = \bar{\nu}_{G,\bar{b}}\}.$$

Let $b \in G(L)$ be a representative of $\bar{b}$, with associated group J_b, cf. (1.11). Then there is a natural identification

$$\mathcal{F} = H^1(F, J_b).$$

This identification is induced from the map which associates to a representative b' of $\bar{b}' \in \mathcal{F}$ the J_b-torsor whose values in a F-algebra R are given by

$$J_{b',b}(R) = \{g \in G(L \otimes R); \ g \cdot b \cdot \sigma(g)^{-1} = b'\}.$$

Furthermore, the resulting action of the finite abelian group $H^1(F, J_b)$ on $\mathcal{F}$ (cf. (1.15), (i)) is independent of the choice of $\bar{b}$ and b in the following sense. If b' is a representative of another element of $\mathcal{F}$, then $J_{b'}$ is an inner form of J_b and hence $H^1(F, J_{b'})$ is canonically isomorphic to $H^1(F, J_b)$, cf. (1.13).

Proof. We first prove that $J_{b',b}$ is a torsor, i.e. is non-empty. Replacing b by a σ-conjugate we may assume that $\nu_b = \nu_{b'}$ and that we have identities for a suitable integer $s > 0$,

$$b \cdot \sigma(b) \ldots \cdot \sigma^{s-1}(b) = s\nu(\pi) = b'\sigma(b') \ldots \sigma^{s-1}(b').$$

Then $s\nu$ is defined over the fixed field F_s of σ^s in L and $b, b' \in G(F_s)$ (comp. [RZ], (1.9)). The above identity implies that we have equality of norms, $Nm_{F_s/F}(b) = Nm_{F_s/F}(b')$. But then it follows ([K5], 5.2) that there exists $g \in G(F_s \otimes \bar{F})$ with $b' = gb\sigma(g)^{-1}$. The image of g in $G(L \otimes \bar{F})$ is a point in $J_{b',b}(\bar{F})$.

We next prove the surjectivity of the map. Let $c \in H^1(F, J_b)$. By the theorem of Steinberg there exists a finite unramified extension F' of F contained in L trivializing c. We may represent c by a cocycle also denoted by c of Gal (F'/F) with values in $J_b(F')$. However, it is obvious that under the natural injection

$$J_b(F') \to G(L \otimes_F F') = \prod_{\tau \in \mathrm{Gal}(F'/F)} G(L)$$

there exists $g \in G(L \otimes_F F')$ such that

$$c_\tau = g^{-1} \cdot g^\tau, \qquad \tau \in \mathrm{Gal}(F'/F).$$

Putting $b' = gb\sigma(g)^{-1}$, we have $b' \in G(L)$ and $[J_{b',b}] = c \in H^1(F, J_b)$, hence b' maps to c.

To prove the injectivity of the map we remark that $J_{b',b}$ is a $(J_{b'}, J_b)$-bitorsor (with J_b acting from the right and $J_{b'}$ acting from the left). Using the customary notation for contraction we have for any b, b', b'' an identification of $(J_b, J_{b''})$-bitorsors,

$$J_{b'',b'} \times^{J_{b'}} J_{b',b} = J_{b'',b}. \tag{3}$$

Hence if b' and b'' define the same cohomology class in $H^1(F, J_b)$ it follows that the $J_{b''}$-torsor $J_{b'',b'}$ is trivial. Any element in $J_{b'',b'}(F)$ σ-conjugates b' into b'', which proves the injectivity.

It is obvious that $J_{b'}$ is an inner form of J_b. By what we have proved already, the last assertion is equivalent to the statement that the map

$$J_{b'',b} \mapsto J_{b'',b'}$$

induces a translation on $H^1(F, J_b) = H^1(F, J_{b'})$. To prove this we may assume that the derived group of G is simply connected which implies the same fact about J_b and $J_{b'}$. But then

$$H^1(F, J_b) = H^1(F, J_{b,ab}) = H^1(F, J_{b',ab}) = H^1(F, J_{b'}).$$

Since the identity (3) above shows that the map induced on $H^1(F, J_{b,ab})$ is a translation, this concludes the proof. $\square$

Remark 1.18 In order to have a complete description of $B(G)$ by this method one would need to describe the image of the Newton map $\bar{\nu}_G : B(G) \to \mathcal{N}(G)$. We do not know how to do this in the most general case. In the case when G is quasi-split the following description of the image follows from [K], Section 6. Let M be a Levi subgroup of G and denote by φ_M the composition of the following obvious maps

$$\pi_1(M)_\Gamma \dashrightarrow \pi_1(M_{ab})_{\mathbf{Q}}^\Gamma = \pi_1(Z_M)_{\mathbf{Q}}^\Gamma \to \mathcal{N}(G).$$

Here Z_M denotes the center of M. The image of $\bar{\nu}_G$ is equal to

$$\mathrm{Im}(\bar{\nu}_G) = \bigcup_M \mathrm{Im}(\varphi_M),$$

where M ranges through all Levi subgroups of G.

EXAMPLE 1.19 (= example (1.10) continued). In case $G = GL(V)$, an element $\bar{\nu} \in \mathcal{N}(G)$ is given by a sequence of rational numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_r$$

and multiplicities (positive integers) $h_1, \ldots, h_r$ such that

$$\sum_{i=1}^r h_i = h = \dim V.$$

The condition that $\bar{\nu}$ be the Newton point of an isocrystal is that the break points of the Newton polygon associated to $\bar{\nu}$ occur at integer points, i.e.

$$\lambda_i \cdot h_i \in \mathbf{Z}, \qquad i = 1, \ldots, r.$$

This is equivalent to the condition appearing in remark (1.19). Indeed, if this condition is satisfied put

$$M = \prod_{i=1}^r GL_{h_i}.$$

Then

$$\pi_1(M)_\Gamma = \bigoplus_{i=1}^r \pi_1(\mathbf{G}_m)_\Gamma = \bigoplus_{i=1}^r \mathbf{Z}.$$

The element $\bar{\nu}$ is the image of

$$(\lambda_1 h_1, \ldots, \lambda_r h_r).$$

The converse is also easy to see.

2. A partial ordering on the set of Newton points

2.1 – Let G be a connected reductive group over an algebraically closed field of characteristic zero, with associated root datum

$$(X^*(T), \Phi, X_*(T), \Phi^\vee).$$

We fix a basis Δ of the set of roots and denote by $\Delta^\vee$ the corresponding basis of the set of coroots. Let

$$\bar{C} = \{x \in X_*(T)_{\mathbf{R}}; \ \langle x, \alpha \rangle \geqslant 0, \ \alpha \in \Delta\}$$

$$C^\vee = \left\{x \in X_*(T)_{\mathbf{R}}; \ x = \sum_{\alpha^\vee \in \Delta^\vee} n_{\alpha^\vee} \cdot \alpha^\vee, \ n_{\alpha^\vee} \in \mathbf{R}_{\geqslant 0}\right\}$$

be the corresponding closed *Weyl chamber* resp. *obtuse Weyl chamber*. Hence $\bar{C}$ is a fundamental domain for the action of the Weyl group Ω on $X_*(T)_{\mathbf{R}}$. On the other hand, $C^\vee \subset X_*(T_{\text{der}})_{\mathbf{R}}$, where $T_{\text{der}} = T \cap G_{\text{der}}$. For the following lemma we refer to [AB], Section 12, and the references quoted there (in loc.cit. this lemma is stated in the context of compact groups but the proofs carry over to the present set-up.)

LEMMA 2.2 *Let* $x, x' \in X_*(T)_{\mathbf{R}}$. *The following conditions are equivalent.*

(i) *x lies in the convex hull of the finite set*
$$\{wx'; \ w \in \Omega\}.$$

(ii) *Let $\tilde{x}$ resp. $\tilde{x}'$ be the representatives in $\bar{C}$ of x resp. x' for the action of Ω. Then*
$$\tilde{x}' - \tilde{x} \in C^\vee.$$

(iii) *Let $\tilde{x}'$ be the representative of x' in $\bar{C}$. Then*
$$\tilde{x}' - wx \in C^\vee, \ w \in \Omega.$$

Let us write $x \prec x'$ if these equivalent conditions are satisfied. Then this condition only depends on the orbits under Ω of x resp. x'.

(iv) *For any representation $\varrho \colon G \to GL(V)$, denoting by $T' \subset GL(V)$ a maximal torus containing $\varrho(T)$ we have*
$$\varrho(x) \prec \varrho(x').$$

Let our algebraically closed field be the algebraic closure of a subfield F and assume that G is defined over F. Let Γ be the Galois group of F and assume that $x, x' \in (X_*(T)_{\mathbf{R}}/\Omega)^\Gamma$. Then it suffices to check condition (iv) on F-rational representations $\varrho \colon G \to GL(V)$. Indeed, this follows from the proof of loc. cit. by the following two observations. First, since $x' - x$ is Γ-invariant it suffices to check $\langle x' - x, \lambda \rangle \geqslant 0$ for any Γ-invariant dominant integral weight λ. Second, a

positive multiple of a Γ-invariant dominant integral weight is the highest weight of an F-rational representation.

2.3 – We now return to the set-up of Section 1. A point $\bar{\nu} \in \mathcal{N}(G)$ gives a well-defined orbit $\Omega \cdot \bar{\nu}$ under the Weyl group in $X_*(T)_{\mathbf{R}}$, where T is a maximal torus in G over $\bar{F}$. The Definition (2.2) therefore defines a partial order on $\mathcal{N}(G)$ resp. $B(G)$

$$\bar{\nu} \prec \bar{\nu}' \Leftrightarrow \Omega \cdot \bar{\nu} \ \prec \Omega \cdot \bar{\nu}',$$

$$\bar{b} \prec \bar{b}' \ \Leftrightarrow \Omega \cdot \bar{\nu}_b \prec \Omega \cdot \bar{\nu}_{b'}.$$

PROPOSITION 2.4 *Recall the map* $\delta \colon B(G) \to \pi_1(G)^{\Gamma} \otimes \mathbf{Q}$.

  (i) *For any* $\bar{b}, \bar{b}' \in B(G)$, *if* $\bar{b} \prec \bar{b}'$ *then* $\delta(\bar{b}) = \delta(\bar{b}')$.
 (ii) *Let* $\bar{b}, \bar{b}' \in B(G)$ *with* $\delta(\bar{b}) = \delta(\bar{b}')$. *Then* $\bar{b} \prec \bar{b}'$ *if* $\bar{b} \in B(G)_{\text{basic}}$.
(iii) *For all* $\bar{b} \in B(G)$ *the set* $X_{\bar{b}} = \{\bar{b}' \in B(G); \ \bar{b}' \prec \bar{b}\}$ *is finite.*
(iv) *Let* $G = GL(V)$, *as in* (1.10). *Let* $\bar{b}, \bar{b}' \in B(G)$. *Then* $\bar{b} \prec \bar{b}' \Leftrightarrow$ *the Newton polygon of* $\bar{b}$ *lies* above *the Newton polygon of* $\bar{b}'$ *and has the same end point.*

Proof. (i) holds because $C^{\vee} \subset X_*(T_{\text{der}})_{\mathbf{R}}$. We next prove (iv). By (i), if $\bar{b} \prec \bar{b}'$ we have $\delta(\bar{b}) = \delta(\bar{b}')$, i.e. (1.15 (iii)) the Newton polygons of $\bar{b}$ and $\bar{b}'$ have the same end points. Let

$$\lambda_1 \leqslant \cdots \leqslant \lambda_h \quad \text{resp.} \quad \lambda_1' \leqslant \cdots \leqslant \lambda_h'$$

be the slopes with multiplicities of $\bar{b}$ resp. $\bar{b}'$..By the above remark we may assume that

$$\sum_1^h \lambda_i = \sum_1^h \lambda_i'.$$

Using the form of the simple coroots for PGL_h (i.e. $\alpha_i^{\vee} = e_i^{\vee} - e_{i+1}^{\vee}$, $1 \leqslant i \leqslant h - 1$, in terms of the standard cocharacters of the diagonal torus), we see that

$$\bar{b} \prec \bar{b}' \Longleftrightarrow \sum_{j=1}^i \lambda_j' \leqslant \sum_{j=1}^i \lambda_j, \quad i = 1, \ldots, h.$$

The condition on the right is precisely the condition on the Newton polygons of $\bar{b}$ and $\bar{b}'$ appearing in the statement of (iv).

We now prove (iii). The fibres of the Newton map are finite, hence it suffices to see that the image of $X_{\bar{b}}$ in $\mathcal{N}(G)$ is finite. Choose a faithful representation ϱ of G,

$$\varrho \colon G \to G' = GL(V).$$

Then, as is easily seen, the induced map $\mathcal{N}(G) \to \mathcal{N}(G')$ has finite fibres. Therefore in fact the fibres of the induced map

$$B(G) \to B(G')$$

are finite which reduces us to the case of $G' = GL(V)$. Using (iii) the statement follows from the fact that the number of Newton polygons with a fixed end point above a fixed one is finite (since they have integral break points).

The statement (ii) follows from the fact ([Bou], VI, Sect. 1,6) that

$$C \cap X_*(T_{\mathrm{der}})_{\mathbf{R}} \subset C^{\vee}. \qquad \square$$

3. Variation of F-isocrystals with additional structure

In this Section we prove the specialization theorems. Throughout this Section we will denote by S a connected scheme of characteristic p.

3.1 – We first recall the following definitions. There is the concept of a *locally free crystal on S* ([Ka]). If $S = \mathrm{Spec}\, R$ is the spectrum of a perfect ring, a locally free crystal on S is simply a module over the Witt ring $W(R)$ which is free of finite rank, locally on S. The locally free crystals on S form a category. A morphism $f\colon M \to M'$ in this category is called an *isogeny* if there exists a morphism $g\colon M' \to M$ such that $fg = p^n$ and $gf = p^n$ for some $n \geqslant 0$.

An *isocrystal on S* is an object of the category of locally free crystals on S, up to isogeny. If $S = \mathrm{Spec}\, R$ is the spectrum of a perfect ring, an isocrystal on S is a module over $W(R)_{\mathbf{Q}} = W(R) \otimes_{\mathbf{Z}} \mathbf{Q}$ which is free of finite rank, locally on S.

An *F-isocrystal on S* is an isomorphism of isocrystals on S,

$$F\colon M^{(p)} \to M.$$

Here $M^{(p)} = \mathrm{Frob}^*(M)$ is the pullback under the Frobenius morphism $\mathrm{Frob}\colon S \to S$.

The category $F\text{-}\mathrm{Isoc}(S)$ of F-isocrystals over S is a tannakian category over $\mathbf{Q}_p$ with neutral object $\mathbf{1}$. If $S = \mathrm{Spec}\, R$ is the spectrum of a perfect ring, the neutral object is given by

$$\sigma\colon W(R)_{\mathbf{Q}} \to W(R)_{\mathbf{Q}}.$$

To have the required isomorphism $\mathbf{Q}_p \xrightarrow{\sim} \mathrm{End}(\mathbf{1})$ it is necessary to assume that S is connected.

3.2 – Let G be a linear algebraic group over $\mathbf{Q}_p$. We follow a suggestion of J. de Jong in adapting a definition from the theory of vector bundles resp. Higgs bundles (cf. [S], Sect. 6)

DEFINITION 3.3 An *F-isocrystal on S with G-structure* is an exact faithful tensor functor

$$M : \mathrm{Rep}_{\mathbf{Q}_p} G \to F\text{-Isoc}(S).$$

REMARKS 3.4 (i) Assume that $S = \mathrm{Spec}\, k$ is the spectrum of an algebraically closed field. Then the set $B(G)$ of Section 1 classifies the F-isocrystals with G-structure up to isomorphism. More precisely, let K be the fraction field of the Witt ring $W(k)$. Then any element $b \in G(K)$ defines an exact faithful functor $M = M_b$ via

$$M((V, \varrho)) = (V \otimes K, \varrho(b) \cdot (\mathrm{id}_V \otimes \sigma)),$$

and conversely by Steinberg's Theorem any such functor is defined by a unique $b \in G(K)$, comp. (3.5) below.

   (ii) Let $G = GL_n$. Then an F-isocrystal on S with G-structure is simply an F-isocrystal X on S of height n, i.e. such that the pullback to any geometric point $\bar{s} \colon \mathrm{Spec}\, k \to S$ gives a K-vector space of dimension n. Equivalently, the dimension of the F-isocrystal X on S in the sense of tannakian categories ([D], 7.1) is equal to n, i.e., the composition

$$1 \xrightarrow{\ \delta\ } X \otimes \overset{\vee}{X} \xrightarrow{\ \mathrm{ev}\ } 1$$

is equal to $n \in \mathbf{Q}_p = \mathrm{End}\,(\mathbf{1})$. Indeed, if M is an exact faithful tensor functor as in definition (3.3), we put

$$X = M(V_{\mathrm{nat}}),$$

where V_{nat} is the natural representation of GL_n. Conversely, let X be an F-isocrystal of height n over S. The above identity forces on us the value of M on V_{nat}. Every irreducible representation of GL_n is isomorphic to the image $\mathbf{S}_\lambda(V_{\mathrm{nat}})$ of V_{nat} of a uniquely determined Schur functor (comp. [FH]). Here λ is a partition of an integer $d \geqslant 0$ and $\mathbf{S}_\lambda(V_{\mathrm{nat}})$ is a subrepresentation of $V_{\mathrm{nat}}^{\otimes d}$. Let V be an arbitrary representation of GL_n. Then V is isomorphic to a representation of the form

$$\bigoplus_\lambda \mathbf{S}_\lambda(V_{\mathrm{nat}})^{m_\lambda}, \quad m_\lambda \geqslant 0.$$

We consider the pair

$$\left(\left(\bigoplus_\lambda \mathbf{S}_\lambda(X)^{m_\lambda} \right)_\alpha, \varphi_{\beta\alpha} \right),$$

where α ranges over $\mathrm{Isom}(\bigoplus_\lambda \mathbf{S}_\lambda(V_{\mathrm{nat}})^{m_\lambda}, V)$, and where

$$\varphi_{\beta\alpha} \colon \left(\bigoplus \mathbf{S}_\lambda(X)^{m_\lambda} \right)_\alpha \to \left(\bigoplus \mathbf{S}_\lambda(X)^{m_\lambda} \right)_\beta$$

is defined as $\beta^{-1} \circ \alpha \in \mathrm{Aut}(\oplus \mathbf{S}_\lambda(X)^{m_\lambda})$. We define a set-valued functor $M(V)$ on F-Isoc(S) by putting

$$\mathrm{Hom}(T, M(V)) = \left\{\left(\psi_\alpha \in \mathrm{Hom}\left(T, \left(\oplus \mathbf{S}_\lambda(X)^{m_\lambda}\right)_\alpha\right)\right)_\alpha; \right.$$

$$\left. \varphi_{\beta\alpha} \circ \psi_\alpha = \psi_\beta\right\}. \tag{4}$$

It is obvious that this functor is representable by an object $M(V) \in F$-Isoc(S) isomorphic to $\oplus \mathbf{S}_\lambda(X)^{m_\lambda}$. This defines the tensor functor M associated to X and it is obvious that these two constructions are inverse to one another. We note that the definition of $\mathbf{S}_\lambda(X)$ makes sense through the usual formulas (*loc. cit.*) since F-Isoc(S) is a tannakian category over a field of characteristic zero.

(iii) Similarly, let G be the symplectic group Sp_{2n}. As in example (ii) above using Schur functors one sees that an F-isocrystal over S with G-structure is the same as an F-isocrystal X of height $2n$ over S together with a non-degenerate alternating pairing

$$X \otimes X \to \mathbf{1}.$$

Similarly, let G be the group of symplectic similitudes. There is an exact sequence

$$1 \to Sp_{2n} \to G \xrightarrow{c} \mathbf{G}_m \to 1.$$

Here c denotes the multiplier homomorphism. As above one sees that an F-isocrystal over S with G-structure is the same as an F-isocrystal X of height $2n$ over S with a non-degenerate alternating pairing

$$X \otimes X \to C,$$

where C is an F-isocrystal on S of height 1.

(iv) Let G be a linear algebraic group over $\mathbf{Q}_p$ and let $H \subset G$ be a connected closed subgroup. Let G' be the centralizer of H in G. Then an F-isocrystal on S with G'-structure is the same as an F-isocrystal M on S with G-structure, equipped with an action of $H(\mathbf{Q}_p)$, i.e.

$$\alpha\colon H(\mathbf{Q}_p) \to \mathrm{Aut}(M),$$

such that

$$\mathrm{tr}(h; V) = \mathrm{tr}(\alpha(h); M(V)), \qquad V \in \mathrm{Rep}_{\mathbf{Q}_p} G.$$

Here again the trace is to be understood in the sense of tannakian categories. To see this one uses the fact that $H(\mathbf{Q}_p)$ is a Zariski-dense in H to conclude

that $\mathrm{Rep}_{\mathbf{Q}_p} G'$ is the category of representations of G, equipped with an action of $H(\mathbf{Q}_p)$.

(v) As a concrete example let B be a semi-simple algebra of finite dimension over $\mathbf{Q}_p$ equipped with an involution $b \mapsto b^*$ and let V be a finite B-module, equipped with a non-degenerate alternating bilinear form

$$(\, , \,) \colon V \otimes V \to \mathbf{Q}_p$$

such that

$$(bv, v') = (v, b^* v'), \quad v, v' \in V, \quad b \in B.$$

Let G be the linear algebraic group defined over $\mathbf{Q}_p$ with points in a $\mathbf{Q}_p$-algebra R

$$G(R) = \{ g \in GL_B(V \otimes R); \ (gv, gv') = c(g) \cdot (v, v'), \ c(g) \in R^\times \}.$$

Let $\mathbf{X}$ be a p-divisible group over S which is equipped with an action of an order O_B of B stable under the involution and with a non-degenerate alternating form

$$\mathbf{X} \times \mathbf{X} \to \hat{\mathbf{G}}_m,$$

satisfying an identity similar to the one above for $b \in O_B$ and such that

$$\mathrm{tr}(b; V) = \mathrm{tr}(\iota(b); X), \quad b \in O_B.$$

Here X denotes the F-isocrystal over S associated to the p-divisible group by the crystalline Dieudonné theory and the trace on the right-hand side is again in the sense of tannakian categories. Combining the examples (ii–iv) we see that X, equipped with the induced action by B and the induced alternating pairing

$$X \otimes X \to \mathbf{1}(-1)$$

is an F-isocrystal with G-structure. Here $\mathbf{1}(-1)$, the F-isocrystal associated to $\hat{\mathbf{G}}_m$, denotes the dual of the Tate object $\mathbf{1}(1)$. If $S = \mathrm{Spec}\, R$ is the spectrum of a perfect ring, $\mathbf{1}(1)$ is given by

$$F = p \cdot \sigma \colon W(R)_{\mathbf{Q}} \to W(R)_{\mathbf{Q}}.$$

3.5 – Let $\bar{s} \to S$ be a geometric point of S. Let $K(\bar{s}) = K(\kappa(\bar{s})) = \mathrm{Fract}\, W(\kappa(\bar{s}))$ be the fraction field of the Witt ring of the residue field of $\bar{s}$. There is an obvious exact faithful tensor functor (inverse image)

$$F\text{-Isoc}(S) \to F\text{-Isoc}(\bar{s}).$$

Let M be an F-isocrystal on S with G-structure. We obtain by pull-back an F-isocrystal $M_{\bar{s}}$ on $\bar{s}$ with G-structure. Composing with the obvious fibre functor $F\text{-Isoc}(\bar{s}) \to \text{Vect}_{K(\bar{s})}$ we obtain a fibre functor over $K(\bar{s})$,

$$\omega_{\bar{s}} \colon \text{Rep}_{\mathbf{Q}_p} G \to \text{Vect}_{K(\bar{s})}.$$

From now on we assume again that G is a connected reductive group over $\mathbf{Q}_p$. By Steinberg's Theorem we may choose an isomorphism of fibre functors on $\text{Rep}_{\mathbf{Q}_p} G$,

$$\omega_{\bar{s}} \simeq \omega^{\text{st}} \otimes_{\mathbf{Q}_p} K(\bar{s}),$$

where ω^{st} denotes the standard fibre functor. It then follows that for every object $(V, \varrho) \in \text{Rep}_{\mathbf{Q}_p} G$ the F-isocrystal structure on $\omega_{\bar{s}}(V, \varrho)$ is given by $\varrho(b(\bar{s})) \cdot (\text{id}_V \otimes \sigma)$, for a uniquely determined element $b(\bar{s}) \in G(K(\bar{s}))$. Any other choice of an isomorphism of fibre functors changes $b(\bar{s})$ into a σ-conjugate element. Therefore we obtain a well-determined element $\overline{b(\bar{s})} \in B(G)$.

This element $\overline{b(\bar{s})}$ only depends on the point $s \in S$ underlying the geometric point $\bar{s}$ in the sense of Lemma (1.3). We shall use the symbol $\bar{b}(s) = \bar{b}_M(s) \in B(G)$ for the element thus defined.

Summarizing, we have associated to an F-isocrystal on S with G-structure a function on S,

$$S \to B(G) \colon s \mapsto \bar{b}(s).$$

THEOREM 3.6 *Let S be a connected scheme of characteristic p and let M be an F-isocrystal with G-structure on S. Then the following statements hold.*

(i) *The function on S,*

$$s \mapsto \delta(\bar{b}(s)) \in \pi_1(G)^{\Gamma} \otimes \mathbf{Q}.$$

is constant.

(ii) *Fix $\bar{b}_0 \in B(G)$. The subset*

$$\{s \in S; \ \bar{b}(s) \prec \bar{b}_0\}.$$

is Zariski-closed and locally on S the zero set of a finitely generated ideal.

(iii) *The subset*

$$\{s \in S; \ \bar{b}(s) \in B(G)_{\text{basic}}\}.$$

is Zariski-closed and locally on S the zero set of a finitely generated ideal.

In the proof of this theorem we are going to use the following result of Grothendieck (cf. [Ka], 2.3.1).

Let $F : M^{(p)} \to M$ be an F-isocrystal over a connected scheme S of characteristic p. Then the set of points $s \in S$ where the Newton polygon of its

fiber $F_s \colon M_s^{(p)} \to M_s$ *at* s *lies above a given continuous* **R***-valued function on* $[0, \mathrm{height}(M)]$ *which is linear between successive integers is Zariski-closed in* S *and is locally the zero set of a finitely generated ideal. Furthermore, the integer valued function*

$$s \mapsto \mathrm{ord} \det F_s$$

is constant.

Since the last statement is not contained in *loc. cit.* we indicate a proof. There exists an isomorphism of isocrystals

$$V \colon M \to M^{(p)},$$

such that $VF = FV = p^n$, some $n \geqslant 0$. Clearly

$$\mathrm{ord} \ \det(F_s) + \mathrm{ord} \ \det(V_s) = \mathrm{ord} \ \det p^n.$$

Both summands on the left are upper semi-continuous functions on S, whereas the right-hand side is a locally constant function on S. The result follows. For a different proof, see [C], 1.7. $\Box$

PROOF OF THEOREM (3.6): (i) For this statement, replacing G by its maximal abelian quotient, we may assume that $G = T$ is a torus. Let $\chi \colon T \to \mathbf{G}_m$ be a $\mathbf{Q}_p$-rational character and denote by the same symbol the maps induced by functoriality,

$$\chi \colon B(T) \to B(\mathbf{G}_m), \qquad \chi \colon \pi_1(T)^\Gamma \otimes \mathbf{Q} \to \pi_1(\mathbf{G}_m)^\Gamma \otimes \mathbf{Q}.$$

Denoting by $\chi(M)$ the induced F-isocrystal with $\mathbf{G}_m$-structure we have

$$\chi(\delta_T(\bar{b}_M(s))) = \delta_{\mathbf{G}_m}(\bar{b}_{\chi(M)}(s)),$$

which is a constant function in s by the second part of Grothendieck's result quoted above. The being true for all $\mathbf{Q}_p$-rational characters χ the assertion (i) follows.

(ii) By (2.2), (iv) a point s lies in this subset if and only if for any representation $\varrho \colon G \to GL_n$ the induced F-isocrystal $\varrho(M)$ of height n over S (cf. (3.4), (iii)) satisfies

$$\bar{b}_{\varrho(M)}(s) \prec \varrho(\bar{b}_0).$$

In fact, it suffices to check this on a finite number of representations ϱ (this is a minor complement to the statement of (2.2), (iv)). We may therefore assume that $G = GL_n$. In this case this subset is empty unless the Newton polygons of $\bar{b}(s)$ and $\bar{b}_0$ have the same end points, in which case the assertion follows from Grothendieck's result via (2.4), (iv).

(iii) By (i) the function $s \mapsto \delta(\bar{b}(s))$ is constant. Therefore the assertion (iii) follows from (ii) since by (2.4), (ii) the subset appearing in (iii) if it is non-empty can be described as

$$\bigcap_{s_0 \in S} \{s \in S; \ \bar{b}(s) \prec \bar{b}(s_0)\},$$

and since by (2.4), (i), this is a finite intersection.   □

3.7 – In the case $G = GL_n$ Theorem (3.6) contains all there can be said about the specialization of F-isocrystals since in this case the Newton map $\bar{\nu}_G$ is injective (this follows by (1.17) from the vanishing of $H^1(\mathbf{Q}_p, J_b)$ for $b \in G(K)$ in this case).

In the general case Theorem (3.6) is complemented by the following result.

THEOREM 3.8 *Let S be a connected locally noetherian scheme of characteristic p and let M be an F-isocrystal with G-structure on S. Let $\bar{b}_0 \in B(G)$ and assume (constancy of the Newton point)*

$$\bar{\nu}(\bar{b}(s)) = \bar{\nu}(\bar{b}_0), \qquad s \in S.$$

Then the subset

$$\{s \in S; \ \bar{b}(s) = \bar{b}_0\}$$

is either empty or all of S.

Proof. We have to show that the map $s \mapsto \bar{b}(s)$ is locally constant on S provided that $\bar{\nu}(\bar{b}(s))$ is so. It suffices to prove that $s \mapsto \bar{b}(s)$ is unchanged under specialization. Since S is locally noetherian we may assume, after base changing the whole situation, that S is the spectrum of a discrete valuation ring which we may assume complete with algebraically closed residue field k, i.e.

$$S = \operatorname{Spec} k[[t]].$$

Let s_1 resp. s_0 be the generic resp. special point of S. Let $R = k[[t]]^{\mathrm{perf}}$ be the perfect closure of $k[[t]]$. We are going to use the following result of Katz [Ka], 2.7.4.

Let k be an algebraically closed field of characteristic p. Let M be an F-isocrystal on $\operatorname{Spec} k[[t]]$ and assume that the Newton polygons at the two points s_1 and s_0 of $\operatorname{Spec} k[[t]]$ coincide. Let M_{s_0} resp. M_{s_1} resp. M_R be the inverse images of M on s_0 resp. s_1 resp. $\operatorname{Spec} R$.

Let a be the composition of morphisms

$$\operatorname{Spec} R \to \operatorname{Spec} k \xrightarrow{s_0} S.$$

Then there is a unique isomorphism

$$M_R \to a^*(M),$$

which induces the identity on M_{s_0}.

In *loc. cit.* it is only stated that the F-isocrystal M_R on Spec R is constant, i.e., that there exists an isomorphism as above. The uniqueness statement follows from the following lemma.

LEMMA 3.9 *Let R be a perfect k-algebra where k is a perfect field. Let M and M' be F-isocrystals over k and denote by M_R and M'_R the F-isocrystals obtained by base change. Then the map*

$$\operatorname{Hom}(M, M') \to \operatorname{Hom}(M_R, M'_R)$$

is injective. If k is algebraically closed this map is bijective.

Proof. By definition we have $\operatorname{Hom}(M, M') = \underline{\operatorname{Hom}}(M, M')^F$, where on the right-hand side we have the invariants of F in the internal Hom. Hence the injectivity amounts to saying that for an F-isocrystal N we have an injection $N^F \to N_R^F$. This is obvious since $N \subset N \otimes_{W(k)_{\mathbf{Q}}} W(R)_{\mathbf{Q}}$.

Now we assume that k is algebraically closed. We may also assume that N is isoclinic. If the slope of F is positive, F is topologically nilpotent and

$$N^F = N_R^F = (0).$$

Something similar holds if the slope of F is negative, i.e. F^{-1} is topologically nilpotent. If the slope is zero, we may assume since k is algebraically closed that $N = W(k)_{\mathbf{Q}}$ and that $F = \sigma$ is the Frobenius automorphism σ. In this case the assertion reduces to $W(R)_{\mathbf{Q}}^{\sigma} = W(k)_{\mathbf{Q}}^{\sigma} = \mathbf{Q}_p$. □

We return to the notations introduced in the proof of Theorem (3.8). Let $\bar{s}_1$ be a geometric point above s_1 which factors through Spec R. The result of Katz quoted above yields a commutative diagram of exact faithful tensor functors

Here F-Isoc$(S)_{\text{const}}$ denotes the full tannakian subcategory of F-isocrystals on S with constant Newton polygons. The functor $a_{\bar{s}_1}^*$ is an equivalence of categories.

Starting now with the F-isocrystal with G-structure M on S, our assumption on M implies that the functor M factors as

$$M : \mathrm{Rep}_{\mathbf{Q}_p} G \to F\text{-Isoc}(S)_{\mathrm{const}}.$$

The above diagram therefore yields an isomorphism of tensor functors

$$M_{s_0} \simeq M_{\bar{s}_1},$$

which had to be shown. $\qquad\square$

COROLLARY 3.10 *Let S be a locally noetherian scheme of characteristic p and let M be an F-isocrystal with G-structure on S. Let $\bar{b}_0 \in B(G)_{\mathrm{basic}}$. Then the set*

$$\{s \in S;\ \bar{b}(s) = \bar{b}_0\}$$

is closed in S.

Proof. Indeed, by Theorem (3.8) the set above is open and closed in the set appearing in (3.6), (ii). This follows from the two following trivial observations:

If $\bar{b} \in B(G)$ and $\bar{b}_0 \in B(G)_{\mathrm{basic}}$ with $\bar{b} \prec \bar{b}_0$, then $\bar{b} \in B(G)_{\mathrm{basic}}$.

If $\bar{b}, \bar{b}' \in B(G)_{\mathrm{basic}}$ with $\delta(\bar{b}) = \delta(\bar{b}')$, then $\bar{\nu}(\bar{b}) = \bar{\nu}(\bar{b}')$. $\qquad\square$

As B. Totaro pointed out to us, Theorem (3.8) allows us to strengthen the statement of the assertion (i) of Theorem (3.6). The following proof was worked out jointly with B. Totaro. Recall from (1.15) the map $\gamma \colon B(G) \to \pi_1(G)_\Gamma$.

COROLLARY 3.11 *Let S be a connected locally noetherian scheme of characteristic p and let M be an F-isocrystal with G-structure on S. The function on S,*

$$s \mapsto \gamma(\bar{b}(s)) \in \pi_1(G)_\Gamma$$

is constant, provided that at least one of the following conditions is satisfied:

— the derived group of G is simply connected, or
— S is locally of finite type over a field.

Proof. Let us first assume that the derived group of G is simply connected. Using the bijection $\pi_1(G)_\Gamma = \pi_1(G_{ab})_\Gamma$ we are then immediately reduced to the case where $G = T$ is a torus. In this case the map γ yields a bijection

$$B(T) = B(T)_{\mathrm{basic}} = \pi_1(T)_\Gamma.$$

By Theorem (3.6), (i) the function in question is constant up to torsion, hence can assume only finitely many values. Therefore Corollary (3.10) implies the assertion in this case.

The case of a general group will be reduced to the previous one using a z-extension $\alpha\colon \tilde{G} \to G$ ([K5], Sect. 1). Since we are assuming now that S is of finite type over a field we may, in order to prove that $s \mapsto \gamma(\bar{b}(s))$ is unchanged under specialization, base change the whole situation to Spec $k\{t\}$ where k is an algebraically closed field and where $k\{t\}$ is the henselization at (t) of $k[t]$. The advantage over the power series ring is that the facts we are going to need are in the literature, although they should also hold true in the other case. The result follows from the following lemma.

LEMMA 3.12 *Let*

$$1 \to T \to \tilde{G} \to G \to 1$$

be a z-extension. Let M be an F-isocrystal with G-structure on Spec $k\{t\}$, where k is an algebraically closed field. Then its inverse image M_R on $R = k\{t\}^{\mathrm{perf}}$ comes from an F-isocrystal with $\tilde{G}$-structure (reduction of structure from G to $\tilde{G}$).

Proof. We need to use the description of the category of F-isocrystals over Spec $k\{t\}$ given in [Ka], (2.4). (The reader checks readily that $k\{t\}$ satisfies the conditions of *loc. cit.*). Let B be the p-adic completion of the henselization of $W(k)[t]$ in (p, t) and let $\Sigma\colon B \to B$ be the lifting of the Frobenius endomorphism σ of $k\{t\}$ which sends t to t^p and induces the Frobenius automorphism on $W(k)$. To give an F-isocrystal over Spec $k\{t\}$ is equivalent to giving a free module M of finite rank over $B[1/p]$, an integrable $W(k)_{\mathbf{Q}}$-connection ∇ and a horizontal isomorphism

$$F\colon \Sigma^*(M) \to M \tag{1}$$

such that (M, ∇) is induced by a pair $(M_\circ, \nabla_\circ)$ where $M_\circ$ is a free B-module and $\nabla_\circ$ is a nilpotent $W(k)$-connection. The choice of Σ defines injections $B \to W(R)$ and $B[1/p] \to W(R)_{\mathbf{Q}}$ and the inverse image of the F-isocrystal on R corresponds under this equivalence to tensoring the isomorphism (1) up to $W(R)_{\mathbf{Q}}$ and forgetting about ∇.

Let now M be our F-isocrystal with G-structure on Spec $k\{t\}$. It therefore defines a fibre functor

$$\omega_M\colon \mathrm{Rep}_{\mathbf{Q}_p} G \to (\text{free } B[1/p]\text{-modules})$$

and an isomorphism of fibre functors

$$F\colon \Sigma^*(\omega_M) \to \omega_M.$$

By our above remarks it suffices to show that ω_M is induced by a fibre functor $\omega_{\tilde{M}}$ on $\mathrm{Rep}_{\mathbf{Q}_p} \tilde{G}$ and that the isomorphism F lifts to an isomorphism $\tilde{F}\colon \Sigma^*(\omega_{\tilde{M}}) \to \omega_{\tilde{M}}$.

But ω_M differs from the obvious fibre functor by an element on $H^1(\mathrm{Spec}\,B[1/p],$ $G)$ (étale or fppf topology, it comes to the same).

Consider the long exact cohomology sequence where we have set $X = \mathrm{Spec}\,B$ and $X_\eta = \mathrm{Spec}\,B[1/p]$

$$\to H^1(X_\eta, T) \to H^1(X_\eta, \tilde{G}) \to H^1(X_\eta, G) \to H^2(X_\eta, T).$$

The image of ω_M in $H^2(X_\eta, T)$ is a torsion element which is fixed under the induced action Σ^*. For the existence of $\omega_{\tilde{M}}$ it suffices to show that this element is trivial. Similarly, the obstruction to lifting the isomorphism F into $\tilde{F}$ lies in $H^1(X_\eta, T)$ and it therefore suffices to show that this last group is trivial. Now T is a product of tori of the form $\mathrm{Res}_{E/\mathbf{Q}_p}\,\mathbf{G}_m$. Therefore, using Shapiro's Lemma we are reduced to proving the following statements. Let A be a complete discrete valuation ring of unequal characteristic $(0, p)$ with algebraically closed residue field k and with uniformizer π. Let B be the p-adic completion of the henselization of $A[t]$ in (π, t). Put $X = \mathrm{Spec}\,B$ and $X_\eta = \mathrm{Spec}\,B[1/\pi]$. Let Σ be the endomorphism of B lifting the Frobenius σ on $B/\pi B$ which sends t to t^p and induces on A an extension of the Frobenius automorphism of $W(k)$. Then

(i) $H^1(X_\eta, \mathbf{G}_m) = (0)$.

(ii) $H^2(X_\eta, \mu_\ell)^{\Sigma^*} = (0)$ for any prime number ℓ.

Here the first statement follows immediately from the fact that $B[1/\pi]$ is factorial. For the second statement we first reduce ourselves to a geometric situation. Let B° be the henselization of $A[t]$ in (π, t) and put $X^\circ = \mathrm{Spec}\,B^\circ$ and $X_\eta^\circ = \mathrm{Spec}\,B^\circ[1/\pi]$. The homomorphism $B^\circ \to B$ ($= p$-adic completion) induces a natural map

$$H^2(X_\eta^\circ, \mu_\ell) \to H^2(X_\eta, \mu_\ell).$$

By a general theorem of Fujiwara [F], (6.6.4), and of Huber [H], (3.2.11), this map is an isomorphism. We are therefore reduced to proving $H^2(X_\eta^\circ, \mu_\ell)^{\Sigma^*} = (0)$.

If $\ell \neq p$ we have $H^2(X_\eta^\circ, \mu_\ell) = (0)$ by the classical purity theorem (this special case is in fact an easy consequence of the smooth base change theorem).

Consider now the case $\ell = p$. An easy reduction allows us to assume that A contains a primitive p-th root of unity (form $A' = A[\zeta_p]$ and extend Σ by $\Sigma(\zeta_p) = \zeta_p$ so that it commutes with the action of the Galois group $\mathbf{Z}/(p-1)\mathbf{Z}$; then observe that taking invariants under $\mathbf{Z}/(p-1)\mathbf{Z}$ in an $\mathbf{F}_p$-vector space is an exact functor). We may therefore apply the theorem of Bloch and Kato [BK], $\star$ Corollary (1.4.1) which defines a canonical filtration of $H^2(X_\eta^\circ, \mu_p)$ and identifies the associated graded. Their result shows that the associated graded is isomorphic to a direct sum of groups of one of the following forms

$$\Omega_{Y,\log}^q, \ \Omega_Y^q \ (q \geqslant 1), \ \Omega_Y^q/\mathrm{Ker}\,d \ (q \geqslant 0),$$

$\star$ We thank L. Illusie for pointing out this reference to us.

where we have put $Y = \operatorname{Spec} B/\pi B$.

The action of Σ^* respects this filtration and induces on the associated graded the action induced by the Frobenius σ on Y, which evidently is zero. It follows that Σ^* is a nilpotent endomorphism and hence can have no invariants.   □

3.13 – Theorem (3.6) and Corollary (3.11) above suggest the following conjecture. Let $\bar{b}_0, \bar{b}_1 \in B(G)$ with $\bar{b}_0 \prec \bar{b}_1$ and $\gamma(\bar{b}_0) = \gamma(\bar{b}_1)$. Then there exists a scheme S of characteristic p and an F-isocrystal M on S with G-structure such that

$$\bar{b}_0 = \bar{b}_M(s_0), \qquad \bar{b}_1 = \bar{b}_M(s_1),$$

where s_0 and s_1 are points of S such that s_0 is a specialization of s_1.

A weakened version of the conjecture above would be that there exists an F-isocrystal M on S with G-structure such that $\bar{\nu}(\bar{b}_M(s_i)) = \bar{\nu}(\bar{b}_i)$, $i = 1, 2$. However, the stronger version seems reasonable to us.

In the case where $G = GL_n$ and where $\bar{b}_0$ and $\bar{b}_1$ are given by the F-isocrystals of p-divisible groups this conjecture is essentially due to Grothendieck [G] who asked for a *p-divisible group over S* with fibres at s_1 resp. its specialization s_0 to give the elements $\bar{b}_1$ resp. $\bar{b}_0$. A similar conjecture can be posed in the context of p-divisible groups with additional structure, as in (v) of Remark (3.4), assuming that the group G defined there is connected. Results announced by Oort ([O], Cor. 2.8) confirm this conjecture in the case of the group of symplectic similitudes. F. Oort informs us that he can also handle the case of GL_n, i.e. Grothendieck's original conjecture.

4. Hodge points

In this Section we give a generalization of Mazur's Theorem.

4.1 – In this Section we use the notations introduced in (1.1). In addition we assume that the group G is quasi-split and split over an unramified extension of F. We fix a hyperspecial point Λ_0 in the building of G over F. Let $\mathcal{V}$ be the orbit of Λ_0 under $G(L)$ in the building of G over L. There is an action on $\mathcal{V}$ of the semi-direct product of $G(L)$ and $\langle \sigma \rangle$, the infinite cyclic group generated by σ.

Let $K \subset G(L)$ denote the stabilizer of $\Lambda_0 \in \mathcal{V}$. Then there are canonical bijections (comp. [K3], (1.3))

$$G(L) \setminus (\mathcal{V} \times \mathcal{V}) = K \setminus G(L)/K = X_*(T)/\Omega.$$

Here T denotes a maximal torus contained in a Borel subgroup in G, with Weyl group Ω. We denote by

$$\operatorname{inv}: \mathcal{V} \times \mathcal{V} \to X_*(T)/\Omega$$

the map induced by composition of the above identifications. It depends only on the $G(F)$-orbit of Λ_0. For $\Lambda \in \mathcal{V}$ and $b \in G(L)$ we put

$$\mu_\Lambda(b) = \mathrm{inv}(\Lambda, b\sigma(\Lambda)).$$

The element $\mu_\Lambda(b) \in X_*(T)/\Omega$ is called *the Hodge point associated to* Λ *and* b. We have the following identities.

$$\mu_{g\Lambda}(gb\sigma(g)^{-1}) = \mu_\Lambda(b), \quad g \in G(L),$$
$$\mu_{\sigma(\Lambda)}(\sigma(b)) = \sigma(\mu_\Lambda(b)).$$

An element $\mu \in X_*(T)/\Omega$ defines in the obvious way an element in $\pi_1(G)$. Its image in $\pi_1(G)_\Gamma$ will be denoted by $\mu^\natural$. On the other hand, μ may not be invariant under σ. We define $\bar{\mu} \in (X_*(T)_\mathbf{Q}/\Omega)^{\langle \sigma \rangle}$ to be the class of the average of the representative of μ in the closed Weyl chamber corresponding to a Borel subgroup in G containing T over its orbit under σ.

THEOREM 4.2 *Let* $b \in G(L)$ *with corresponding σ-conjugacy class* $\bar{b} \in B(G)$. *Let* $\Lambda \in \mathcal{V}$.

(i) *The elements* $\gamma(\bar{b})$ *and* $\mu_\Lambda(b)^\natural$ *of* $\pi_1(G)_\Gamma$ *are identical.*
(ii) *We have the following relation between elements of* $\mathcal{N}(G)$,

$$\bar{\nu}(\bar{b}) \prec \bar{\mu}_\Lambda(b).$$

Before giving the proof we give an example.

EXAMPLE 4.3 Let $G = GL(V)$. The elements of $\mathcal{V}$ are just O_L-lattices in $V \otimes L$. Let Λ and Λ' be two lattices. Then there are uniquely defined integers

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_j < 0 \leqslant a_{j+1} \leqslant \cdots \leqslant a_h, \quad h = \dim V,$$

such that

$$\Lambda/\Lambda \cap \Lambda' \simeq O_L/(\pi^{-a_1}) \oplus \cdots \oplus O_L/(\pi^{-a_j}),$$
$$\Lambda'/\Lambda \cap \Lambda' \simeq O_L/(\pi^{a_{j+1}}) \oplus \cdots \oplus O_L/(\pi^{a_h}).$$

Here π denotes a uniformizing element. Then

$$\mathrm{inv}(\Lambda, \Lambda') = (a_1, \ldots, a_h) \in X_*(T)/\Omega = \mathbf{Z}^n/S_n.$$

In case $\Lambda' = b\sigma(\Lambda)$, the element $\mathrm{inv}(\Lambda, \Lambda')$ is called the *Hodge polygon* of Λ and $b \in G(L)$. The case when $L = K_0$ and when $\Lambda' \subset \Lambda$ is considered in [Ka]. In this case it is proved in [Ka], 1.4.1 (Mazur's Theorem) that the Hodge polygon of Λ and b lies below the Newton polygon of b and that both polygons have the same end points. The proof of *loc. cit.* immediately generalizes to the slightly more general case considered here.

Proof of Theorem (4.2): (i) We first consider the case when the derived group of G is simply connected. In this case the assertion is equivalent to the statement that $\bar{b}$ and $\mu_\Lambda(b)$ have identical images in $B(G_{ab}) = X_*(G_{ab})_\Gamma$. The torus G_{ab} is unramified, let $G_{ab}(O_L)$ be the maximal compact subgroup of $G_{ab}(L)$. Then Λ defines a well-defined coset

$$\bar{t} \in G_{ab}(L)/G_{ab}(O_L).$$

The image of $\mu_\Lambda(b)$ in $B(G_{ab})$ is given by the σ-conjugacy class of

$$b_{ab} \cdot \sigma(t)t^{-1},$$

where t is a representative of $\bar{t}$ and where b_{ab} denotes the image of b in $G_{ab}(L)$. Since G_{ab} is unramified, the σ-conjugacy class of this element is well-determined (cf. (1.15), (ii)) and the assertion follows in this case. The general case is immediately reduced to the previous case by making use of an *unramified z-extension* $\tilde{G} \to G$, comp. [K3], Section 3.

(ii) Let $\varrho\colon G \to G' = GL(V)$ be a representation. By choosing an O_L-lattice in $V \otimes L$, rational over F and fixed under the stabilizer of Λ_0 in $G(L)$, we obtain a $G(L) \rtimes \langle\sigma\rangle$-equivariant map

$$\varrho\colon \mathcal{V} \to \mathcal{V}',$$

where $\mathcal{V}'$ denotes the set of O_L-lattices in $V \otimes L$. Let T' be a maximal torus of G' containing $\varrho(T)$ and let Ω' be its Weyl group. Then

$$\varrho(\mathrm{inv}(\Lambda, \Lambda')) = \mathrm{inv}(\varrho(\Lambda), \varrho(\Lambda')).$$

In particular by example (4.3) we have

$$\bar{\nu}_{G'}(\varrho(\bar{b})) = \varrho(\bar{\nu}_G(\bar{b})) \prec \varrho(\mu_\Lambda(b)) = \mu_{\varrho(\Lambda)}(\varrho(b)).$$

The last element on the right is of course σ-invariant. Therefore applying the previous result to all elements of the orbit of the representative of $\mu_\Lambda(b)$ under σ and adding up the results we obtain that

$$\varrho(\bar{\nu}(\bar{b})) \prec \varrho(\bar{\mu}_\Lambda(b)).$$

This being true for any representation ϱ, we thus conclude by appealing to (2.2), (iv).   □

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