COMPOSITIO MATHEMATICA

WINFRIED KOHNEN

On Siegel modular forms

Compositio Mathematica, tome 103, nº 2 (1996), p. 219-226
<http://www.numdam.org/item?id=CM_1996__103_2_219_0>

Compositio Mathematica **103**: 219–226, 1996.

On Siegel modular forms

WINFRIED KOHNEN
Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany

Received 18 April 1994: accepted in final form 30 August 1994

1. Introduction and statement of result

Let F be a holomorphic cusp form of integral weight k on the Siegel modular group $\Gamma = \mathrm{Sp}_g(\mathbb{Z})$ of genus g and denote by $a(T)$ (T a positive definite symmetric even integral (g, g)-matrix) its Fourier coefficients.

If $g = 1$ and $k \geqslant 2$, then by Deligne's theorem (previously the Ramanujan-Petersson conjecture) one has

$$a(T) \ll_{\varepsilon,F} T^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0),$$

and since by [10]

$$\limsup_{T\to\infty} |a(T)|/T^{k/2-1/2} = \infty,$$

this bound is best possible.

For arbitrary $g \geqslant 2$ our knowledge of how to obtain good bounds for the coefficients $a(T)$ in terms of $\det(T)$ is still extremely limited. For $g \geqslant 2$ and $k > g + 1$ Böcherer and the author in [4] proved that

$$a(T) \ll_{\varepsilon,F} (\det(T))^{k/2-\delta_g+\varepsilon}, \quad (\varepsilon > 0), \tag{1}$$

where

$$\delta_g := \frac{1}{2g} + \left(1 - \frac{1}{g}\right) \frac{1}{4(g-1) + 4[(g-1)/2] + 2/(g+2)}.$$

The bound (1) for arbitrary g seems to be the best one known so far. Note, however, that for $g \to \infty$ it is still of the same order of magnitude as Hecke's bound $a(T) \ll_F (\det(T))^{k/2}$.

In the present paper we shall prove

THEOREM. *Suppose that* $4/g$. *Then there exists* $\kappa = \kappa(g) \in \mathbb{N}$ *with the following property: for each* $N \in \mathbb{N}$ *there is an integer* $k \in \{N, N+1, \ldots, N+\kappa-1\}$ *and a non-zero cusp form* F *of weight* k *on* Γ_g *whose Fourier coefficients* $a(T)$ *satisfy*

$$a(T) \ll_{\varepsilon,F} (\det(T))^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0). \tag{2}$$

The proof of the Theorem will be given in Section 2. The functions F will be constructed as theta series attached to a positive definite quadratic form of rank $2g$ with certain harmonic forms. For some general comments we refer the reader to Section 3.

NOTATION. If A and B are real resp. complex matrices of appropriate sizes we put $A[B] := B'AB$ resp. $A\{B\} := \bar{B}'AB$; here B' is the transpose of B.

If S is a real symmetric matrix we write $S \geqslant 0$ resp. $S > 0$ if S is positive semi-definite resp. positive definite. If S is real of size m and $S > 0$, we denote by $S^{1/2}$ the unique real symmetric positive definite matrix of size m satisfying $(S^{1/2})^2 = S$.

2. Proof of theorem

For $\nu, m \in \mathbb{N}$ denote by $H_\nu(m, g)$ the $\mathbb{C}$-linear space of harmonic forms $P\colon \mathbb{C}^{(m,g)} \to \mathbb{C}$ of degree ν, i.e. of polynomial functions $P(X)(X = (x_{ij})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant g}})$ which satisfy $P(XU) = (\det(U))^\nu P(X)$ for all $U \in \mathrm{Gl}_g(\mathbb{C})$ and which are annihilated by the Laplace operator $\Sigma_{i,j}(\partial^2/\partial x_{ij}^2)$ [6, 8]. For $m \geqslant 2g$ the space $H_\nu(m, g)$ is generated by the forms $(\det(L'X))^\nu$ where L is a complex (m, g)-matrix with $L'L = 0$ [8].

Let S be a fixed positive definite symmetric even integral matrix of size m with determinant 1 (such a matrix exists if and only if $8|m$). Then the generalized theta series

$$\vartheta_{S,P}(Z) = \sum_{G \in \mathbb{Z}^{(m,g)}} P(S^{1/2}G)\, \mathrm{e}^{\pi i \cdot \mathrm{tr}(S[G]Z)},$$

$(Z \in \mathbf{H}_g =$ Siegel upper half-space of degree $g)$

is a cusp form of weight $m/2 + \nu$ on Γ_g [5, Kap. II, Sect. 3].

We now specialize to the case $m = 2g$ (supposing $4|g$). Take $L = \binom{E}{iE}$ where E is the unit matrix of size g and define

$$P_\nu(X) := (\det(L'X))^\nu, \quad (\nu \in \mathbb{N}).$$

We write

$$a_\nu(T) = \sum_{\substack{G \in \mathbb{Z}^{(2g,g)} \\ S[G]=T}} (\det((E\ iE)S^{1/2}G))^\nu, \quad (T > 0) \tag{3}$$

for the Fourier coefficients of ϑ_{S,P_ν}.

Put

$$H := T + iR$$

where

$$R := J[S^{1/2}G], \quad J := \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

and observe that R is skew-symmetric and H is hermitian.

From

$$\begin{pmatrix} E \\ iE \end{pmatrix} (E \; -iE) = E + iJ, \quad T = S[G]$$

we see that

$$H = G'S^{1/2} \begin{pmatrix} E \\ iE \end{pmatrix} \cdot \left(G'S^{1/2} \begin{pmatrix} E \\ iE \end{pmatrix} \right)';$$

in particular

$$\det(H) = |\det((E \; iE)S^{1/2}G)|^2. \tag{4}$$

Choose a unitary matrix $U \in \mathrm{GL}_g(\mathbb{C})$ such that $iR\{T^{-1/2}U\} = D$ is a real diagonal matrix with diagonal entries the eigenvalues of the hermitian matrix $iR[T^{-1/2}]$. As $R[T^{-1/2}]$ is real, these eigenvalues occur in pairs $\pm\alpha_\nu (\nu = 1, \ldots, g/2)$.

From

$$H\{T^{-1/2}U\} = E + D$$

we find that

$$\det(H\{T^{-1/2}U\}) = \prod_{\nu=1}^{g/2} (1 - \alpha_\nu^2) \leqslant 1,$$

hence by (4) we obtain

$$\begin{aligned} |\det((E \; iE)S^{1/2}G)|^2 &= \det(H\{T^{-1/2}U\}\{U^{-1}T^{1/2}\}) \\ &\leqslant \det(E\{U^{-1}T^{1/2}\}) \\ &= \det(T). \end{aligned}$$

From (3) we now infer that

$$|a_\nu(T)| \leqslant (\det(T))^{\nu/2} \cdot r_S(T), \tag{5}$$

where

$$r_S(T) := \#\{G \in \mathbb{Z}^{(2g,g)} | S[G] = T\}$$

is the number of representations of T by S.

Denote by

$$r_S^*(T) := \#\{G \in \mathbb{Z}^{(2g,g)} \mid G \text{ primitive}, \ S[G] = T\},$$

the number of primitive representations of T by S. By elementary divisor theory we have

$$r_S(T) = \sum_{\substack{D \in \mathrm{GL}_g(\mathbb{Z}) \backslash \mathbb{Z}_*^{(g,g)} \\ T[D^{-1}]>0 \text{ even integral}}} r_S^*(T[D^{-1}])$$

where $\mathbb{Z}_*^{(g,g)}$ denotes the set of integral (g,g)-matrices of rank g.

Let $S_1, \dots, S_h (h = h(2g))$ be a set of representatives of classes of (the genus of) unimodular positive definite symmetric even integral $(2g, 2g)$-matrices and $\varepsilon(S_\mu)$ $(\mu = 1, \dots, h)$ be the number of units of S_μ. Then according to the 'primitive' version of Siegel's main theorem on quadratic forms [11] one has

$$\left(\sum_{\mu=1}^{h} \frac{1}{\varepsilon(S_\mu)}\right)^{-1} \sum_{\mu=1}^{h} \frac{1}{\varepsilon(S_\mu)} r_{S_\mu}^*(T)$$
$$= c_g \cdot (\det(T))^{g-(g+1)/2} \cdot \prod_p \alpha_{p,T}^*,$$

where c_g is a constant depending only on g, and where the $\alpha_{p,T}^*$ are certain local densities which satisfy

$$\prod_{p \nmid \det(T)} \alpha_{p,T}^* \ll 1$$

and

$$\alpha_{p,T}^* \leqslant 2, \quad (p \mid \det(T))$$

(cf. [1; Sect. 2, esp. (2.6), (2.7b), (2.7d); Sect. 3] in combination with [2; 1.1ff.], and [7; Sect. 6.8, Thm. 6.8.1, iii)]; note that in the product in formulas (2.7b) and (2.7d) in [1] the index j should start with 0 (rather than 1) and that the algebraic calculations given in [1; Sect. 3] remain valid also without the assumption on the weight imposed there).

We therefore conclude that

$$r_S^*(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon}, \quad (\varepsilon > 0),$$

hence that

$$r_S(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon} \sum_{\substack{D\in \mathrm{GL}_g(\mathbb{Z})\backslash\mathbb{Z}_*^{(g,g)} \\ T[D^{-1}]>0 \text{ even integral}}} \frac{1}{|\det(D)|^{g-1+2\varepsilon}}, \quad (\varepsilon > 0). \tag{6}$$

The condition $T[D^{-1}]$ integral implies that $(\det(D))^2 \mid \det(T)$. Hence the sum on the right of (6) is majorized by

$$\sum_{d^2|\det(T)} \alpha_g(d)/d^{g-1+2\varepsilon},$$

where for any $n \in \mathbb{N}$ we have put

$$\alpha_n(d) := \#\{D \in \mathrm{GL}_n(\mathbb{Z})\backslash\mathbb{Z}_*^{(n,n)} \mid |\det(D)| = d\}, \quad (d \in \mathbb{N}).$$

As is well-known and easy to see one has

$$\sum_{d\geqslant 1} \alpha_n(d)/d^s = \zeta(s)\zeta(s-1)\dots\zeta(s-n+1), \quad (\mathrm{Re}(s) > n).$$

From the latter equality one easily checks by induction on n that

$$\alpha_n(d) \ll_\varepsilon d^{n-1+\varepsilon}.$$

Thus

$$\sum_{d^2|\det(T)} \alpha_n(d)/d^{g-1+2\varepsilon} \ll_\varepsilon (\det(T))^\varepsilon,$$

and by (6) it follows that

$$r_S(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon}$$

for any $\varepsilon > 0$.

Together with (5) this implies

$$a_\nu(T) \ll_\varepsilon (\det(T))^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0),$$

where $k = g + \nu$ is the weight of ϑ_{S,P_ν}.

To complete the proof we proceed as follows. Suppose that ϑ_{S,P_ν} is identically zero for all $\nu \geqslant 1$, so

$$\sum_{\substack{G\in\mathbb{Z}^{(2g,g)} \\ S[G]=T}} (\det((E\ iE)S^{1/2}G)^\nu = 0 \tag{7}$$

for all $T \geqslant 0$ and all $\nu \geqslant 1$. Identity (7) implies that

$$\det((E\, iE)S^{1/2}G) = 0 \tag{8}$$

for all $G \in \mathbb{Z}^{(2g,g)}$; in fact, this follows from the well-known more general result that if $\Sigma_{n=1}^{\infty} c_n$ is an absolutely convergent series of complex numbers such that $\Sigma_{n=1}^{\infty} c_n^{\nu} = 0$ for all $\nu \in \mathbb{N}$, then $c_n = 0$ for all n.

By (4) and the definition of H, (8) is equivalent to

$$\det((S + iJ[S^{1/2}])[G]) = 0. \tag{9}$$

for all $G \in \mathbb{Z}^{(2g,g)}$. Since the left-hand side of (9) is a polynomial in the components of G, equality (9) must hold for all $G \in \mathbb{R}^{(2g,g)}$. Replacing G by $S^{-1/2}G$ we find

$$\det((E + iJ)[G]) = 0$$

for all G, a contradiction (take e.g. $G = \binom{G_1}{0}$ with G_1 invertible). Therefore there exists $\nu \in \mathbb{N}$ with $\vartheta_{S,P_\nu} \neq 0$ (of course, we could have also used the slightly different reasoning suggested by Maass, cf. [9, p. 154f.]).

Repeating the above argument with ν replaced by $N\nu$ where N is an arbitrary positive integer, we deduce inductively that there are infinitely many ν with $\vartheta_{S,P_\nu} \neq 0$.

To obtain the slightly stronger assertion of the Theorem, we follow Maass [9, loc. cit.]. Assume that $a_{\nu_0}(T_0) \neq 0$, say and denote by $b_1, \dots, b_\kappa$ the distinct non-zero numbers of the form $\det((E\, iE)S^{1/2}G)$ as G runs over all $G \in \mathbb{Z}^{(2g,g)}$ with $S[G] = T_0$. Then there exist $n_1, \dots, n_\kappa \in \mathbb{N}$ such that

$$a_\nu(T_0) = \sum_{j=1}^{\kappa} n_j b_j^{\nu}$$

for all $\nu \geqslant 1$. Supposing that

$$a_N(T_0) = a_{N+1}(T_0) = \cdots = a_{N+\kappa-1}(T_0) = 0,$$

we obtain $n_1 = n_2 = \cdots n_\kappa = 0$ (Vandermonde determinant), a contradiction.

3. Comments

We conclude the paper with a few general comments.

(i) Certainly the estimate (2) can be proved for the Fourier coefficients of theta series with more general harmonics than the special forms P_ν considered in Section 2, and eventually it would be true for all $P \in H_\nu(2g, g)$. However, we have not checked this, mainly for the following reason: for $\nu \to \infty$ the dimension of $H_\nu(2g, g)$ grows like $\nu^{g(g+1)/2-g}$ ([6], cf. also [3, formula XI.1]), while the dimension of the space of cusp forms of weight $g+\nu$ on Γ_g grows like $(g+\nu)^{g(g+1)/2}$; hence there is no hope to eventually proving (2) for all cusp forms on Γ_g of weight $k \gg g$ by the method of this paper.

(ii) If in (2) one drops the condition that S is unimodular (and hence also the condition that $4|g$), one obtains cusp forms on subgroups of Γ_g of finite index with a multiplier system. The same method as before can be applied to estimate their Fourier coefficients. For example, take the simplest case $g=1$ and let $S=\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$. Then

$$\vartheta_{S,P_\nu}(z) = \sum_{x_1,x_2\in\mathbb{Z}} (x_1+ix_2)^\nu\, \mathrm{e}^{2\pi i(x_1^2+x_2^2)z}, \quad (z\in\mathbf{H}:=\mathbf{H}_1;\nu\in\mathbb{N})$$

is a cusp form of weight $1+\nu$ on $\Gamma_0(4)=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_1 \mid 4/c\}$ with character $(\frac{-4}{\cdot})$ (Legendre symbol). If $4|\nu$ it is not identically zero (the coefficient of $\mathrm{e}^{2\pi iz}$ is equal to 4). Since (as is of course well-known) $r_S(T)\ll T^\varepsilon(\varepsilon>0)$, we obtain Deligne's bound

$$a(T)\ll_\varepsilon T^{\nu/2+\varepsilon}, \quad (\varepsilon>0)$$

for the Fourier coefficients $a(T)$ of ϑ_{S,P_ν}.

(iii) One should observe that in general (i.e. for $m\neq 2g$) the coefficients of theta functions with harmonic forms in $H_\nu(m,g)$ cannot be estimated directly in a good way. In fact, for $m<2g$ one has

$$H_\nu(m,g)=\{0\} \quad \text{if} \quad m<g, \quad \text{all} \quad \nu\geqslant 1,$$
$$H_\nu(m,g)=\{0\} \quad \text{if} \quad g\leqslant m<2g \quad \text{and} \quad \nu\neq 1$$

(cf. [3, p.13]). On the other hand, for $m>2g$ an estimate with the same method as in Section 2 leads to a bound which is even worse than the usual Hecke bound.

Acknowledgements

The author would like to thank S. Böcherer for a useful conversation.

References

1. Böcherer, S.: Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen, *Manuscripta Math.* 45 (1984) 273–288.
2. Böcherer, S. and Raghavan, S.: On Fourier coefficients of Siegel modular forms, *J. Reine Angew. Math.* 384 (1988) 80–101.
3. Böcherer, S.: Siegel modular forms and theta series, *Proc. Sympos. Pure Maths. AMS*, vol. 49, part 2 (1989) 3–17.
4. Böcherer, S. and Kohnen, W.: Estimates for Fourier coefficients of Siegel cusp forms, *Math. Ann.* 297 (1993) 499–517.
5. Freitag, E.: Siegelsche Modulformen, *Grundl. Math. Wiss.* vol. 254, Springer, Berlin-Heidelberg-New York, 1983.

6. Kashiwara, M. and Vergne, M.: On the Segal-Shale-Weil representations and harmonic polynomials, *Invent. Math.* 84 (1978) 1–47.
7. Kitaoka, Y.: Arithmetic of quadratic forms. Cambridge Texts in Maths, no. 106, Cambridge University Press, 1993.
8. Maass, H.: Harmonische Formen in einer Matrixvariablen, *Math. Ann.* 252 (1980) 133–140.
9. Raghavan, S.: Cusp forms of degree 2 and weight 3, *Math. Ann.* 224 (1976) 149–156.
10. Rankin, R.-A.: An Ω-result for the Fourier coefficients of cusp forms, *Math. Ann.* 203 (1973) 239–250.
11. Siegel, C. L.: Über die analytische Theorie der quadratischen Formen. In: *Collected Works I* (eds.: K. Chandrasekharan and H. Maass), pp. 326–405. Springer, Berlin-Heidelberg-New York, 1966.