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A note on Zariski pairs

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Abstract. A pair of complex projective plane curves of a same degree is said to make a Zariski pair if the singularities of them have a same topological type, while their embeddings in the projective plane are topologically different. Such pairs are interesting because they show that an equisingular family of plane curves may fail to be connected. A method for constructing examples of Zariski pairs is presented. In our examples, the topological types of the embeddings are distinguished by means of fundamental groups of complements to the curves. Two infinite series of examples with explicit defining equations of the curves are given.

Key words: Projective plane curve, Zariski pair, fundamental group of complement to plane curve, plane curve singularities

1. Introduction

In [1], Artal Bartolo defined the notion of *Zariski pairs* as follows:

DEFINITION. A couple of complex reduced projective plane curves C_1 and C_2 of a same degree is said to make a Zariski pair, if there exist tubular neighborhoods $T(C_i) \subset \mathbf{P}^2$ of C_i for $i = 1, 2$ such that $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are diffeomorphic, while the pairs (\mathbf{P}^2, C_1) and (\mathbf{P}^2, C_2) are not homeomorphic; that is, the singularities of C_1 and C_2 are topologically equivalent, but the embeddings of C_1 and C_2 into \mathbf{P}^2 are not topologically equivalent.

The first example of Zariski pair was discovered and studied by Zariski in [12] and [14]. He showed that there exist projective plane curves C_1 and C_2 of degree 6 with 6 cusps and no other singularities such that $\pi_1(\mathbf{P}^2 \setminus C_1)$ and $\pi_1(\mathbf{P}^2 \setminus C_2)$ are not isomorphic. Indeed, the placement of the 6 cusps on the sextic curve has a crucial effect on the fundamental group of the complement. Let C_1 be a sextic curve defined by an equation $f^2 + g^3 = 0$, where f and g are general homogeneous polynomials of degree 3 and 2, respectively. Then C_1 has 6 cusps lying on a conic defined by $g = 0$. In [12], it was shown that $\pi_1(\mathbf{P}^2 \setminus C_1)$ is isomorphic to the free product $\mathbf{Z}/(2) * \mathbf{Z}/(3)$ of cyclic groups of order 2 and 3. On the other hand, in [14], it was proved that there exists a sextic curve C_2 with 6 cusps which are not lying on any conic, and that the fundamental group $\pi_1(\mathbf{P}^2 \setminus C_2)$ is cyclic of order 6. In [6], Oka gave an explicit defining equation of C_2 . In [1], Artal Bartolo presented a simple way to construct (C_1, C_2) from a cubic curve C by means of a Kummer

covering of \mathbf{P}^2 of exponent 2 branched along three lines tangent to C at its points of inflection.

Except for this example, very few Zariski pairs are known ([1], [10]). In [7], and independently in [9], infinite series of Zariski pairs have been constructed from the above example of Zariski by means of covering tricks of the plane.

In this paper, we present a method to construct Zariski pairs, which yields two infinite series of new examples of Zariski pairs as special cases.

A germ of curve singularity is called of type (p, q) if it is locally defined by $x^p + y^q = 0$.

Series I. This series consists of pairs $(C_1(q), C_2(q))$ of curves of degree $3q$, where q runs through the set of integers ≥ 2 prime to 3. Each of $C_1(q)$ and $C_2(q)$ has $3q$ singular points of type $(3, q)$ and no other singularities. The fundamental group $\pi_1(\mathbf{P}^2 \setminus C_1(q))$ is non-Abelian, while $\pi_1(\mathbf{P}^2 \setminus C_2(q))$ is Abelian. When $q = 2$, this example is nothing but the classical one of the sextic curves due to Zariski.

Series II. This series consists of pairs $(D_1(q), D_2(q))$ of curves of degree $4q$, where q runs through the set of odd integers > 2 . Each of $D_1(q)$ and $D_2(q)$ has $8q$ singular points of type $(2, q)$ – that is, rational double points of type A_{q-1} – and no other singularities. The fundamental group $\pi_1(\mathbf{P}^2 \setminus D_1(q))$ is non-Abelian, while $\pi_1(\mathbf{P}^2 \setminus D_2(q))$ is Abelian.

Our method is a generalization of Artal Bartolo's method for re-constructing the classical example of Zariski to higher dimensions and arbitrary exponents of the Kummer covering. Indeed, when $q = 2$ in Series I, our construction coincides with his.

Instead of the computation of the first Betti number of the cyclic branched covering of \mathbf{P}^2 , which was employed in [1], we use the fundamental groups of the complements in order to distinguish two embeddings of curves in \mathbf{P}^2 . For the calculation of the fundamental groups, we use Theorem 1 of [8] and a result of [4] and [9].

For the Zariski pair $(C_1(q), C_2(q))$ in Series I, the non-isomorphism of $\pi_1(\mathbf{P}^2 \setminus C_1(q))$ and $\pi_1(\mathbf{P}^2 \setminus C_2(q))$ can be seen also from the theory of Alexander modules by Libgober [2]. This argument is sketched in Section 3.

2. A Method of Constructing Zariski Pairs

2.1. NON-ABELIAN MEMBERS

Let p and q be integers ≥ 2 prime to each other. We choose homogeneous polynomials $f \in H^0(\mathbf{P}^2, \mathcal{O}(pk))$ and $g \in H^0(\mathbf{P}^2, \mathcal{O}(qk))$, where k is an integer ≥ 1 . Suppose that f and g are generally chosen. Consider the projective plane curve

$$C_{p,q,k}: f^q + g^p = 0$$

of degree pqk (cf. [3]). It is easy to see that the singular locus of this curve consists of pqk^2 points of type (p, q) . In [9; Introduction], the following is shown.

PROPOSITION 1. *The fundamental group $\pi_1(\mathbf{P}^2 \setminus C_{p,q,k})$ is isomorphic to the group $\langle a, b, c \mid a^p = b^q = c, c^k = 1 \rangle$. In particular, it is non-Abelian.*

See also [4], in which the fundamental groups of the complements of curves of this type are calculated. There the groups are presented in a different way.

The calculation of $\pi_1(\mathbf{P}^2 \setminus C_{p,q,k})$ has a long history from Zariski's work on $C_{2,3,1}$ in [12]. This group had been calculated for $C_{2,3,k}$ by Turpin in [11], and for $C_{p,q,1}$ by Oka in [5].

This curve $C_{p,q,k}$ will be a member C_1 of a Zariski pair.

2.2. ABELIAN PARTNERS

We shall construct the other member C_2 of the Zariski pair such that $\pi_1(\mathbf{P}^2 \setminus C_2)$ is Abelian.

Let p, q and k be integers as above. We put $n = pk$. Interchanging p and q if necessary, we may assume that $n \geq 3$. Let $S_0 \subset \mathbf{P}^{n-1}$ be a hypersurface of degree n defined by $F_0(X_1, \dots, X_n) = 0$. We consider a linear pencil of hypersurfaces

$$S_t : F_0(X_1, \dots, X_n) + t \cdot X_1 \cdots X_n = 0$$

which is spanned by S_0 and $S_\infty := \{X_1 \cdots X_n = 0\}$. We put $H_i = \{X_i = 0\}$ ($i = 1, \dots, n$). We consider the morphism $\phi_q: \mathbf{P}^{n-1} \rightarrow \mathbf{P}^{n-1}$ given by

$$(Y_1 : \dots : Y_n) \longmapsto (X_1 : \dots : X_n) = (Y_1^q : \dots : Y_n^q)$$

which is a covering of degree q^{n-1} branched along S_∞ .

PROPOSITION 2. *Suppose that (1) every member S_t is reduced, and that (2) S_0 contains none of the hyperplanes H_i . Then $\pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(S_t))$ is Abelian for a general member S_t .*

Proof. Let \mathbf{P}^1 be the t -line, and we put $\mathbf{A}^1 := \mathbf{P}^1 \setminus \{\infty\}$. Let $\mathcal{W} \subset \mathbf{P}^{n-1} \times \mathbf{A}^1$ be the divisor defined by

$$X_1 \cdots X_n \cdot (F_0(X_1, \dots, X_n) + t \cdot X_1 \cdots X_n) = 0$$

which is the union of $S_\infty \times \mathbf{A}^1$ and the universal family of the affine part $\{S_t; t \in \mathbf{A}^1\}$ of the pencil. For $t \in \mathbf{A}^1$, we denote by $W_t \subset \mathbf{P}^{n-1}$ the divisor obtained from the scheme theoretic intersection $(\{t\} \times \mathbf{P}^{n-1}) \cap \mathcal{W}$, which is equal with the divisor $S_t + S_\infty$.

First, we shall show that $\pi_1(\mathbf{P}^{n-1} \setminus W_t)$ is Abelian for a general t . Remark that the assumption (2) implies that S_t contains none of H_i unless $t = \infty$. Combining this with the assumption (1), we see that W_t is reduced for all $t \in \mathbf{A}^1$. Hence,

by Theorem 1 of [8], the inclusion $\mathbf{P}^{n-1} \setminus W_t \hookrightarrow (\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W}$ induces an isomorphism on the fundamental groups for a general t . Therefore, it is enough to show that $\pi_1((\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W})$ is Abelian. In order to prove this, we consider the first projection

$$p: (\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W} \rightarrow \mathbf{P}^{n-1} \setminus S_\infty.$$

Since $\{S_t; t \in \mathbf{P}^1\}$ is a pencil whose base locus is contained in S_∞ , there is a unique member $S_{t(P)}$ ($t(P) \neq \infty$) containing P for each point $P \in \mathbf{P}^{n-1} \setminus S_\infty$. Therefore $p^{-1}(P)$ is a punctured affine line $\mathbf{A}^1 \setminus \{t(P)\}$ for every $P \in \mathbf{P}^{n-1} \setminus S_\infty$. Consequently, p is a locally trivial fiber space. Moreover, p has a section

$$s: \mathbf{P}^{n-1} \setminus S_\infty \rightarrow (\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W}$$

which is given by, for example, $s(P) = (P, t(P) + 1)$. Hence the homotopy exact sequence of p splits. Combining this with the fact that the image of the injection $\pi_1(\mathbf{A}^1 \setminus \{t(P)\}) \rightarrow \pi_1((\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W})$ is contained in the center, we see that

$$\pi_1((\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W}) \cong \pi_1(\mathbf{P}^{n-1} \setminus S_\infty) \times \pi_1(\mathbf{A}^1 \setminus \{\text{a point}\}).$$

This shows that $\pi_1((\mathbf{P}^{n-1} \times \mathbf{A}^1) \setminus \mathcal{W})$ is Abelian.

Note that $\phi_q: \mathbf{P}^{n-1} \rightarrow \mathbf{P}^{n-1}$ is étale over $\mathbf{P}^{n-1} \setminus W_t$ for every t . Hence the natural homomorphism

$$\phi_{q*}: \pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \rightarrow \pi_1(\mathbf{P}^{n-1} \setminus W_t)$$

is injective. This implies that $\pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(W_t))$ is Abelian for a general t . On the other hand, since $\mathbf{P}^{n-1} \setminus \phi_q^{-1}(W_t)$ is a Zariski open dense subset of $\mathbf{P}^{n-1} \setminus \phi_q^{-1}(S_t)$, the inclusion induces a surjective homomorphism

$$\pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \twoheadrightarrow \pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(S_t)).$$

Thus $\pi_1(\mathbf{P}^{n-1} \setminus \phi_q^{-1}(S_t))$ is also Abelian for a general t . □

PROPOSITION 3. *Suppose the following; (3) $S_0 \cap H_i$ is a non-reduced divisor pD_i of H_i of multiplicity p , where D_i is a reduced divisor of H_i , none of whose irreducible components is contained in $H_i \cap (\cup_{j \neq i} H_j)$, and (4) the singular locus of S_t is of codimension ≥ 2 in S_t for a general t . Then the general plane section $\mathbf{P}^2 \cap \phi_q^{-1}(S_t)$ of $\phi_q^{-1}(S_t)$ is a curve of degree pqk , and its singular locus consists of pqk^2 points of type (p, q) for a general t .*

Proof. Note that the assumption (3) implies that $S_t \cap H_i$ is also equal with pD_i for $t \neq \infty$. Let P be a general point of any irreducible component of D_i , and let Q be a point such that $\phi_q(Q) = P$, which lies on the hyperplane defined by $Y_i = 0$. By the assumption (3), Q is not contained in any of the other hyperplanes

defined by $Y_j = 0$ ($j \neq i$). Hence there exist analytic local coordinate systems (w_1, \dots, w_{n-1}) and (z_1, \dots, z_{n-1}) of \mathbf{P}^{n-1} with the origins P and Q , respectively, such that H_i is given by $w_1 = 0$, $\phi_q^{-1}(H_i)$ is given by $z_1 = 0$, and ϕ_q is given by

$$(z_1, \dots, z_{n-1}) \longmapsto (w_1, \dots, w_{n-1}) = (z_1^q, z_2, \dots, z_{n-1}).$$

Let $t \in \mathbf{A}^1$ be general. By the assumption (3), the defining equation of S_t at P is of the form

$$u(w) \cdot w_1 + v(w_2, \dots, w_{n-1})^p = 0.$$

By the assumption (4), S_t is non-singular at P , because P is a general point of an irreducible component of D_i . This implies that $u(P) \neq 0$. On the other hand, the divisor D_i , which is defined by $v(w_2, \dots, w_{n-1}) = 0$ on the hyperplane $H_i = \{w_1 = 0\}$, is non-singular at P , because D_i is reduced by the assumption (3) and P is general. Hence we have

$$\frac{\partial v}{\partial w_j}(P) \neq 0 \quad \text{at least for one } j \geq 2.$$

The defining equation of $\phi_q^{-1}(S_t)$ is then of the form

$$\tilde{u}(z) \cdot z_1^q + v(z_2, \dots, z_{n-1})^p = 0, \quad \text{where } \tilde{u}(Q) \neq 0.$$

Then, it is easy to see that, in terms of suitable analytic coordinates $(\tilde{z}_1, \dots, \tilde{z}_{n-1})$ with the origin Q , this equation can be written as follows:

$$\tilde{z}_1^q + \tilde{z}_2^p = 0.$$

Thus, when we cut $\phi_q^{-1}(S_t)$ by a general 2-dimensional plane passing through Q , a germ of curve singularity of type (p, q) appears at Q .

Since the degree of D_i is $k = n/p$, the inverse image $\phi_q^{-1}(D_i)$ is a reduced hypersurface of degree qk in the hyperplane defined by $Y_i = 0$. Moreover $\phi_q^{-1}(D_i)$ and $\phi_q^{-1}(D_j)$ have no common irreducible components when $i \neq j$ because of the assumption (3). Hence the intersection points of $\phi_q^{-1}(\sum_{i=1}^n D_i)$ with a general plane $\mathbf{P}^2 \subset \mathbf{P}^n$ is pqk^2 in number. Moreover, $\mathbf{P}^2 \cap \phi_q^{-1}(S_t)$ is non-singular outside of these intersection points, because of the assumption (4). □

2.3. SUMMARY

Suppose that we have constructed a hypersurface $S_0 \subset \mathbf{P}^{n-1}$ of degree $n \geq 3$ which satisfies the assumptions (1)-(4) in Propositions 2 and 3. Let C_2 be a general plane section of $\phi_q^{-1}(S_t)$, where t is general. Because of Zariski's hyperplane section

theorem [13] and Propositions 1, 2 and 3, we see that the curve C_2 has the same type of singularities as that of $C_{p,q,k}$, but the fundamental group $\pi_1(\mathbf{P}^2 \setminus C_2)$ is Abelian. Hence (C_1, C_2) is a Zariski pair, with $C_1 = C_{p,q,k}$.

3. Construction of Series I

We carry out the construction of the previous section with $p = 3$, $k = 1$, $n = 3$ and q an arbitrary integer ≥ 2 prime to 3.

We fix a homogeneous coordinate system $(X : Y : Z)$ of \mathbf{P}^2 , and put

$$L_1 = \{X = 0\}, \quad L_2 = \{Y = 0\}, \quad L_3 = \{Z = 0\},$$

and

$$R_1 = (0 : 1 : -1) \in L_1, \quad R_2 = (1 : 0 : -1) \in L_2.$$

Let $\mathbf{P}_*(\Gamma(\mathbf{P}^2, \mathcal{O}(3)))$ be the space of all cubic curves on \mathbf{P}^2 , which is isomorphic to the projective space of dimension 9, and let $\mathcal{F} \subset \mathbf{P}_*(\Gamma(\mathbf{P}^2, \mathcal{O}(3)))$ be the family of cubic curves C which satisfy the following conditions:

- (a) C intersects L_1 at R_1 with multiplicity ≥ 3 ,
- (b) C intersects L_2 at R_2 with multiplicity ≥ 3 , and
- (c) C intersects L_3 at a point with multiplicity ≥ 3 .

(We consider that C intersects a line L_i with multiplicity ∞ , if L_i is contained in C .)

PROPOSITION 4. *The family \mathcal{F} consists of 3 projective lines. They meet at one point corresponding to $C_\infty := \{XYZ = 0\}$.*

Proof. Let $F(X, Y, Z) = 0$ be the defining equation of a member C of this family \mathcal{F} . By the condition (a), F is of the form

$$F(X, Y, Z) = A(Y + Z)^3 + X \cdot G(X, Y, Z),$$

where A is a constant, and $G(X, Y, Z)$ is a homogeneous polynomial of degree 2. By the condition (b), we have $F(X, 0, Z) = A(Z + X)^3$, and hence we get

$$G(X, Y, Z) = A(3Z^2 + 3ZX + X^2) + Y \cdot H(X, Y, Z),$$

where $H(X, Y, Z)$ is a homogeneous polynomial of degree 1. By the condition (c), we have $F(X, Y, 0) = A(Y + \alpha X)^3$ for some α . Then α must be a cubic root of unity, and we get

$$H(X, Y, Z) = 3A\alpha^2 X + 3A\alpha Y + BZ,$$

where B is a constant. Combining all of these, we get

$$\begin{aligned} F(X, Y, Z) &= A(X^3 + Y^3 + Z^3) \\ &\quad + 3A(\alpha^2 X^2 Y + \alpha X Y^2 + Y^2 Z + Y Z^2 + Z^2 X + Z X^2) + B X Y Z \\ &= A(X + Y + Z)^3 + 3A(\alpha^2 - 1)X^2 Y \\ &\quad + 3A(\alpha - 1)X Y^2 + (B - 6A)X Y Z. \end{aligned}$$

This curve $C = \{F = 0\}$ intersects L_3 at

$$R_3 = R_3(\alpha) := (1 : -\alpha : 0) \in L_3$$

with multiplicity ≥ 3 . This means that the family \mathcal{F} consists of three lines $\mathcal{L}(1)$, $\mathcal{L}(\omega)$ and $\mathcal{L}(\omega^2)$ in the projective space $\mathbf{P}_*(\Gamma(\mathbf{P}^2, \mathcal{O}(3)))$, where $\omega = \exp(2\pi i/3)$, such that a general cubic C in $\mathcal{L}(\alpha)$ intersects L_3 at $R_3(\alpha)$ with multiplicity 3. The ratio of the coefficients $t := B/A$ gives an affine coordinate on each line $\mathcal{L}(\alpha)$. The three lines $\mathcal{L}(1)$, $\mathcal{L}(\omega)$, $\mathcal{L}(\omega^2)$ intersect at one point $t = \infty$ corresponding to the cubic $C_\infty = L_1 + L_2 + L_3$. \square

Hence we get three pencils of cubic curves $\{C(1)_t; t \in \mathcal{L}(1)\}$, $\{C(\omega)_t; t \in \mathcal{L}(\omega)\}$, and $\{C(\omega^2)_t; t \in \mathcal{L}(\omega^2)\}$. It is easy to check that these pencils satisfy the assumptions (2), (3) and (4) in the previous section. Note that the pencil $\mathcal{L}(1)$ does not satisfy the assumption (1) because $C(1)_6$ is a triple line. However, the other two satisfy (1). Indeed, if a cubic curve C in the family \mathcal{F} is non-reduced, then the conditions (a)–(c) imply that it must be a triple line. Therefore the three points R_1 , R_2 and $R_3(\alpha)$ are co-linear, which is equivalent to $\alpha = 1$. Consequently, C must be a member of $\mathcal{L}(1)$.

Now, by using the pencil $\mathcal{L}(\omega)$ or $\mathcal{L}(\omega^2)$, we complete the construction of Series I.

Note that, if $C(1)_a$ is a non-singular member of $\mathcal{L}(1)$, then $\pi_1(\mathbf{P}^2 \setminus \phi_q^{-1}(C(1)_a))$ is isomorphic to the free product $\mathbf{Z}/(3) * \mathbf{Z}/(q)$. Indeed, since $C(1)_a$ is defined by

$$(X + Y + Z)^3 + (a - 6)XYZ = 0,$$

the pull-back $\phi_q^{-1}(C(1)_a)$ is defined by

$$(U^q + V^q + W^q)^3 + (a - 6)(UVW)^q = 0$$

which is of the form $\tilde{f}^3 + \tilde{g}^q = 0$. The polynomials \tilde{f} and \tilde{g} are not general by any means. However, since the type of singularities of $\phi_q^{-1}(C(1)_a)$ is the same as that of $C_{3,q,1}$, we have an isomorphism $\pi_1(\mathbf{P}^2 \setminus \phi_q^{-1}(C(1)_a)) \cong \pi_1(\mathbf{P}^2 \setminus C_{3,q,1})$.

We can show that $\pi_1(\mathbf{P}^2 \setminus C_1(q))$ and $\pi_1(\mathbf{P}^2 \setminus C_2(q))$ are not isomorphic by showing that the real Alexander modules $\pi_1/\pi_1'' \otimes \mathbf{R}$, which are the modules over the group ring $\mathbf{R}[\pi_1/\pi_1'] \cong \mathbf{R}[t, t^{-1}]$, are different for $C_1(q)$ and $C_2(q)$. (See [2] for the precise definition.) By Theorem 5.1 of [2], these modules decompose into direct sums of

$$R(\kappa) := \mathbf{R}[t, t^{-1}] / ((t - \exp(2\pi i \kappa))(t - \exp(-2\pi i \kappa))),$$

where κ runs through the constants of quasi-adjunction of the singular point $x^3 + y^q = 0$, and the multiplicity of $R(\kappa)$ in $\pi_1/\pi_1'' \otimes \mathbf{R}$ can be calculated by the superabundance of a certain linear system on \mathbf{P}^2 . In the case of the largest constant $\kappa_{\max} := 1 - 1/3 - 1/q$, the multiplicity is given by the superabundance of the linear system of curves of degree q passing through every singular point of $C_i(q)$. Since we have coordinates of the singular points quite explicitly, we can readily calculate the superabundance. It turns out to be 1 for $C_1(q)$, while it is 0 for $C_2(q)$.

4. Construction of Series II

It is enough to show the following:

PROPOSITION 5. *The quartic surface*

$$S_0: F_0(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2)^2 + 2x_3x_4(x_1^2 - x_2^2) + x_3^2x_4^2 = 0$$

in \mathbf{P}^3 satisfies the assumptions (1)–(4) with $p = 2$ and $k = 2$.

Proof. The assumptions (2) and (3) can be trivially checked. To check the assumptions (1) and (4), we put

$$F_t := F_0 + t \cdot x_1x_2x_3x_4,$$

and calculate the partial derivatives $\partial F_t / \partial x_i$ for $i = 1, \dots, 4$. Let $Q_t \subset \mathbf{P}^3$ be the quadric surface defined by

$$2x_1^2 - 2x_2^2 + 2x_3x_4 + tx_1x_2 = 0.$$

It is easy to see that Q_t is irreducible for all $t \neq \infty$. It is also easy to see that Q_t is the unique common irreducible component of the two cubic surfaces

$$\frac{\partial F_t}{\partial x_3} = 0, \quad \text{and} \quad \frac{\partial F_t}{\partial x_4} = 0.$$

Suppose that a surface $S_a = \{F_a = 0\}$ in this pencil contains a non-reduced irreducible component mT ($m \geq 2$). Then, both of $\partial F_a / \partial x_3$ and $\partial F_a / \partial x_4$ must vanish on T . Hence T must coincide with Q_a , and we get $S_a = 2Q_a$. Comparing

the defining equations of S_a and $2Q_a$, we see that there are no such a . Thus the assumption (1) is satisfied. To check the assumption (4), we remark that the condition $\dim \text{Sing } S_t \leq 0$ is an open condition for t . Hence it is enough to prove, for example, $\dim \text{Sing } S_2 = 0$. It is easy to show that $\text{Sing } S_2$ consists of four points $(1 : \pm\sqrt{-1} : 0 : 0)$, $(0 : 0 : 0 : 1)$ and $(0 : 0 : 1 : 0)$. \square

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