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APPROXIMATION OF CONTROL PROBLEMS INVOLVING ORDINARY AND IMPULSIVE CONTROLS

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Abstract. In this paper we study an approximation scheme for a class of control problems involving an ordinary control v, an impulsive control u and its derivative $\dot{u}$. Adopting a space-time reparametrization of the problem which adds one variable to the state space we overcome some difficulties connected to the presence of $\dot{u}$. We construct an approximation scheme for that augmented system, prove that it converges to the value function of the augmented problem and establish an error estimates in L^∞ for this approximation. Moreover, a characterization of the limit of the discrete optimal controls is given showing that it converges (in a suitable sense) to an optimal control for the continuous problem.

Résumé. Dans ce papier nous étudions un schéma d'approximation pour une classe de problèmes de contrôle où la dynamique contient un contrôle mesurable v, un contrôle impulsionnel u et sa dérivée $\dot{u}$. En utilisant une reparamétrisation espace-temps, du problème qui introduit une nouvelle variable d'état, nous arrivons à résoudre les difficultés liées à la présence de $\dot{u}$. Nous proposons un schéma d'approximation pour le système augmenté, prouvons qu'il converge vers la fonction valeur du système augmenté et nous donnons une estimation *a priori* dans L^∞ pour cette approximation. Nous donnons aussi une caractérisation de la limite des contrôles discrets et nous montrons qu'elle converge (dans un sens à définir) vers un contrôle optimal du problème continu.

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1. Introduction

We deal with a controlled system governed by

$$\begin{cases} \dot{x}(t) = g_0(x(t), u(t), t, v(t)) + \sum_{j=1}^{d} g_j(x(t), u(t), t, v(t))\ \dot{u}_j(t)\ , \quad t \in (\bar{t}, T] \\ x(\bar{t}) = \bar{x}, \qquad u(\bar{t}) = \bar{u}. \end{cases} \tag{E}$$

The state x of the system belongs to $\mathbb{R}^D$ and the control laws u and v go from $[\bar{t}, T]$ into a closed, arcwise connected set $U \subset \mathbb{R}^d$ and respectively into a compact set $V \subset \mathbb{R}^q$. The control u is subject to two additional constraints. A directional constraint $\dot{u} \in C$, where C is a closed cone in $\mathbb{R}^d$, and a constraint on the total

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variation of u over the interval $[\bar{t}, T]$, namely $V_{\bar{t}}^{T}(u) \leq K - \bar{k}$, where K is a fixed positive constant and $\bar{k} \in [0, K]$. Note that in (E) the initial value of the impulsive control is also fixed.

We want to minimize

$$\mathcal{L}(\bar{x}, \bar{u}, \bar{t}, \bar{k}) = \inf\{\Phi(x(T), u(T)) : (u, v) \text{ admissible control laws}\}$$

for $(\bar{x}, \bar{u}, \bar{t}, \bar{k}) \in \mathbb{R}^D \times U \times [0, T) \times [0, K]$, where for an admissible control law we intend a pair $(v(\cdot), u(\cdot))$ such that v is Borel measurable, u is absolutely continuous and the previous constraints are satisfied. Note that, for a given admissible control law, the notion of solution to (E) is given by the Caratheodory theorem. The function Φ is a continuous function from $\mathbb{R}^D \times U$ in $\mathbb{R}$ and $(x(T), u(T))$ is the final state of the system (E).

Since the dynamics depends linearly on the derivative of u, the state of the system can jump in correspondence of the discontinuities of the map $\dot{u}$. Therefore the problem has an impulsive character and the control u is called impulsive control. Control problems with impulsive controls arise in many applications, for example in mechanics [7, 24], space navigation [20], economics [13] and advertising models [14, 15]. Let us mention that those impulse control problems are not covered by classical impulse control theory, as formulated for example in [3, 6].

For the presence of the variables x, u, v in the argument of the vector fields g_j, we cannot consider equation (E) in the sense of measures, since this definition doesn't preserves the continuity of the input-output map. However this difficulty has been recently solved in [8, 21–23] embedding the original problem into a space-time system (the so-called augmented system) where time plays the role of a new control. The control problem for the augmented system is then considered. That extension of the original control problem is proper, *i.e.* the infimum of the original problem turns out to coincide with the infimum of the space-time problem. Moreover, the value function of the extended problem satisfies in the viscosity sense an Hamilton-Jacobi-Bellman equation with appropriate boundary conditions (as it has been proved in [5, 22]).

In this paper, we consider a numerical scheme for this class of problems. We extend here the discretization techniques adopted for the approximation of standard control problems (see *e.g.* Capuzzo Dolcetta-Falcone [10], Bardi-Falcone [2], Camilli-Falcone [9], Falcone [16]), constructing a direct approximation of the augmented problem. We introduce a family of control problems which are discrete in the time and in the variation variables and satisfy, as in the continuous case, a bound on the total variation of the discrete impulsive control.

Convergence to the value function and an error estimate of order $h^{1/2}$ are proved modifying standard technique in the theory of viscosity solution. Moreover, we prove that the sequence of the continuous controls obtained interpolating on the the discrete optimal controls converges in a suitable sense (see Ths. 4.1 and 4.2 for precise results) to an optimal control for the continuous problem.

In the last section we present the basic features of the space discretization (with respect to the variables x and u) and the convergence of a fully discrete approximation scheme.

We will not recall definitions and properties of viscosity solutions referring to [1] for a detailed introduction to this theory in the case of first order PDEs.

2. Space-time control problem and viscosity solutions

Let us briefly describe the extended problem introduced in [8] and studied in the context of dynamic programming by Motta-Rampazzo in [23]. Their technique requires to embed the original system (E) with impulsive controls into a larger autonomous system with bounded controls. For this system the state is $y = (x, u, t)$ and the dynamics is given by

$$\begin{cases} y'(s) = \hat{g}_0(y(s), v(s))\, t'(s) + \sum_{j=1}^{d} \hat{g}_j(y(s), v(s))\, u_j'(s), & s \in [0, 1] \\ y(0) = (\bar{x}, \bar{u}, \bar{t}) \end{cases} \tag{2.1}$$

where the vector fields $\hat{g}_i$, for $i = 0, .., d$, are the columns of the matrix

$$\hat{G}(y,v) = \begin{pmatrix} g_{01} & \dots & \dots & g_{d1} \\ \dots & \dots & \dots & \dots \\ g_{0D} & \dots & \dots & g_{dD} \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

In this new setting, the variable t plays the role of a control variable. Admissible controls for the system (2.1) are the triple (t, u, v): $[0, 1] \longrightarrow [\,\overline{t}, T] \times U \times V$ which satisfy

(i) $(t, u)(0) = (\overline{t}, \overline{u})$;
(ii) (t, u) is Lipschitz continuous and $u'(s) \in C$ for a.e. $s \in [0, 1]$;
(iii) $t : [0, 1] \longrightarrow [\,\overline{t}, T]$ is nondecreasing and surjective;
(iv) $v : [0, 1] \longrightarrow V$ is Borel-measurable.

The class of the admissible space-time controls will be denoted by $\Gamma(\overline{t}, \overline{u})$. Using a reparametrization of s it is always possible to assume that the control (t, u, v) is in *canonical form*, which means that $|(t', u')(s)| = const$ a.e. in $[0, 1]$ and the constant can be assumed to be equal to $V_0^1(u, t)$.

Let us observe that, in this new formulation, jumps of the original control u correspond to interval of instantaneous evolution of the first component of the control (u, t), *i.e.* intervals of the parameter s where $t'(s) = 0$ while the function u describes an arc on the plane $t = const$ connecting the initial and the final point of the jump.

The set of bounded–variation controls for the system (2.1) related to the original problem is given by

$$\Gamma_{K-\overline{k}}(\overline{u}, \overline{t}) = \Big\{(u, t, v) \in \Gamma(\overline{u}, \overline{t}) : V_0^1(u) \leq K - \overline{k}\Big\}.$$

The minimum problem for the space-time system is given by

$$\mathcal{V}(\overline{x}, \overline{u}, \overline{t}, \overline{k}) = \inf_{\Gamma_{K-\overline{k}}(\overline{u}, \overline{t})} \Phi(y(1))$$

for all $(\overline{x}, \overline{u}, \overline{t}, \overline{k}) \in \overline{\Omega} = \mathbb{R}^D \times U \times \mathbb{R}^+ \times [0, K]$ ($\Phi(y)$ is written in place of $\Phi(x, u)$ to remind that this functional is referred to the space-time control problem).

In the following theorems we summarize the main results proved by Motta–Rampazzo [22] for the space-time control problem. From now on we will assume that the following conditions are always satisfied:

- there exists two positive constants M and L such that

$$|g_i(x_1, u_1, t_1, v)| \leq M, \quad |g_i(x_1, u_1, t_1, v) - g_i(x_2, u_2, t_2, v)| \leq L|(x_1, u_1, t_1) - (x_2, u_2, t_2)| \tag{H1}$$

for $i = 0, .., d$, (x_1, u_1, t_1), $(x_2, u_2, t_2) \in \mathbb{R}^D \times U \times \mathbb{R}^+$, $v \in V$.

- there exist two positive constants M_Φ, L_ϕ such that

$$|\Phi(x_1, u_1)| \leq M_\Phi, \quad |\Phi(x_1, u_1) - \Phi(x_2, u_2)| \leq L_\Phi|(x_1, u_1) - (x_2, u_2)| \tag{H2}$$

for (x_1, u_1), $(x_2, u_2) \in \mathbb{R}^D \times U$.

- we have either

$$U = \mathbb{R}^d \tag*{$(H3)_C$}$$

or

$$\begin{cases} C=\mathbb{R}^d \text{ and for all } \epsilon>0,\ u_1\in U, \text{ there exists } \delta>0 \text{ such that} \\ \text{for all } u_2\in U\cap B(u_1,\delta) \text{ there exists an absolutely continuous} \\ \text{function } \gamma_{12}:[0,1]\to U, \text{ such that } \gamma_{12}(0)=u_1, \gamma_{12}(1)=u_2 \\ \text{and } \int_0^1|\gamma_{12}'(s)|ds\le\epsilon. \end{cases} \tag{$(H3)_U$}$$

Theorem 2.1. *The infimum for the original control problem and for the extended control problem coincide, i.e. for any initial condition* $(x,u,t,k)\in\mathbb{R}^D\times U\times[0,T)\times[0,K]$

$$\mathcal{V}(x,u,t,k)=\mathcal{L}(x,u,t,k). \tag{2.2}$$

Let us define the Hamilton-Jacobi-Bellman equation associated to the extended control problem

$$-H(x,u,t,\nabla\mathcal{V})=0 \qquad (x,u,t,k)\in\Omega \tag{DPE}$$

where $\nabla\mathcal{V}=(\nabla_x\mathcal{V},\nabla_u\mathcal{V},\mathcal{V}_t,\mathcal{V}_k)$, the Hamiltonian H is given by

$$H(x,u,t,\nabla\phi)=\min_{\substack{v\in V\\(w_0,w)\in S^d_+}}\left\{\left(\frac{\partial\phi}{\partial t}+\nabla_x\phi\cdot g_0(x,u,t,v)\right)w_0+\sum_{j=1}^d\left(\nabla_x\phi\cdot g_j(x,u,t,v)+\frac{\partial\phi}{\partial u_j}\right)w_j+\frac{\partial\phi}{\partial k}|w|\right\}$$

and the set S^d_+ is the intersection between $[0,+\infty)\times C$ and the unit sphere $S^d=\{(w_0,w)\in\mathbb{R}^{1+d}:|(w_0,w)|=1\}$.

Since equation (DPE) does not admit, in general, classical solution, it is usually studied in the framework of viscosity solution theory.

Theorem 2.2. *i)* $\mathcal{V}$ *is Lipschitz continuous in the variables* x,t,k *and uniformly continuous in the variable* u *in* $\overline{\Omega}$. *Moreover,* $\mathcal{V}$ *is a viscosity solution of the equation* (DPE) *in* Ω *and satisfies the following boundary conditions*

$$\mathcal{V}(x,u,t,k)\le\Phi(x,u) \quad (x,u,t,k)\in\partial_T\Omega=\mathbb{R}^D\times U\times\{T\}\times[0,K] \tag{$(BC)_1$}$$

$$\begin{aligned}&\mathcal{V} \textit{ is a supersolution on } \partial'\Omega=\partial\Omega\setminus\partial_T\Omega \textit{ and for all}\\ &(x,u,t,k)\in\partial_T\Omega \textit{ such that } \mathcal{V}(x,u,t,k)<\Phi(x,u).\end{aligned} \tag{$(BC)_2$}$$

ii) Let u *be a viscosity subsolution of* (DPE) *in* Ω *satisfying* $(BC)_1$, v *be a viscosity supersolution of* (DPE) *in* Ω *satisfying* $(BC)_2$. *Then*

$$u\le v \qquad \textit{on } \overline{\Omega}.$$

An immediate consequence of the previous theorem is that $\mathcal{V}$ is the unique viscosity solution of (DPE) satisfying the boundary conditions $(BC)_1$ and $(BC)_2$. Moreover, if $(H3)_C$ holds true, it is possible to prove that $\mathcal{V}$ is Lipschitz continuous also in the variable u.

Remark 2.1. For simplicity we develop the theory for a problem without running costs. However, the results can be extended easily to problems with a running cost $f(x,u,t,v,\dot{u})$ provided the function f is linear in the impulsive control, *i.e.*

$$f(x,u,t,v,\dot{u})=f_1(x,u,t,v)+f_2(x,u,t,v)\dot{u}$$

and the functions f_i, $i=1,2$, satisfy $(H1)$.

3. Discretization in time and variation variables

We will start discretizing simultaneously the variables t and k with the same discretization step h. That discretization can also be interpreted as the result of a discretization of the pseudo-time s appearing in the space-time system (2.1). However, for technical and numerical reasons, we prefer to work directly with the dependent variables t and k and leave s in the background. Note that the variable k plays, in some sense, the role of a second time variable. Moreover, t and k are strongly connected by the bound on the total variation of the control u, *i.e.* $V_{\overline{t}}^{T}(u) \leq K - \overline{k}$ for the original system and the corresponding constraint for extended system.

We assume for simplicity that there exist two integers M and N such that $Nh = T$ and $Mh = K$. This assumption is not restrictive. In fact, the case when $T - Nh$ and $K - Mh$ tend to 0 can also be treated extending the proof of Theorem 4.1 in [18], Chapter IX to the case when two variables are discretized with the same discretization step. We consider the following Euler scheme approximation of (2.1)

$$\begin{cases} x_{n+1} = x_n + hg_0(x_n, u_n, t_n, v_n)w_{0n} + \sum_{j=1}^{d} hg_j(x_n, u_n, t_n, v_n)w_{jn} \\ u_{n+1} = u_n + hw_n, \\ t_{n+1} = t_n + hw_{0n}, \\ (x_0, u_0, t_0) = (\overline{x}, \overline{u}, \overline{n}h) \end{cases} \tag{3.1}$$

where $n = 0, .., P$. The class of admissible controls for the discrete problem is given by

$$\Gamma^h_{(M-\overline{m})h}(u, \overline{n}h) = \left\{ \{(w_{0n}, w_n)\}_{n=0}^{P} : (w_{0n}, w_n) \in \{0; 1\} \times B_d(0, 1) \quad \text{and} \quad \text{satisfies } (3.2) - (3.4) \right\}$$

where

$$\sum_{n=0}^{P} hw_{0n} = (N - \overline{n})h, \quad \sum_{n=0}^{P} h|w_n| \leq (M - \overline{m})h, \tag{3.2}$$

$$|(w_{0n}, w_n)| = 1 \text{ and } w_n \in C \text{ for any } n = 0, .., P \tag{3.3}$$

$$\overline{u} + h \sum_{i=0}^{n} w_i \in U \text{ for any } n = 0, .., P. \tag{3.4}$$

Note that the integer P is not fixed a priori, since it depends on the particular control. In fact, P must be such that the condition (3.2) is satisfied so that the global bound $P \leq M + N$ holds true.

The minimum problem for the discrete system is given by

$$\mathcal{V}_h(\overline{x}, \overline{u}, \overline{n}h, \overline{m}h) = \inf_{\Gamma^h_{(M-\overline{m})h}(\overline{u}, \overline{n}h)} \{\Phi(x_{P+1}, u_{P+1})\}. \tag{3.5}$$

Remark 3.1. Let us explain the previous definitions. The control variables for the discrete problem are an approximation of the derivative of u and t, since $w_{0n} = (t_{n+1} - t_n)/h$ and $w_n = (u_{n+1} - u_n)/h$. The discrete dynamics implies that the total variation k_n is incremented by $|w_n|$ at the $n-th$ step. This corresponds implicitly to the discretization of the differential equation $k'(s) = |u'(s)|$. Moreover, the normalization condition (3.3) means essentially that we are approximating canonical control laws with a total variation on the single step equal to h.

The first condition in (3.2) is imposed to guarantee that the final time is equal to T, while the second condition is a bound on the total variation of the discrete control. Condition (3.4) is a state constraint condition on the variable u.

Note that for $n = N$, we get a standard optimal stopping problem (where the variation plays the role of the time), while for $m = M$ we obtain a pure evolutive control problem. These two conditions correspond exactly to the boundary conditions added in [5] to the continuous equation.

Let us define

$$A^h(n,m,u) = \{(w_0,w)\in\{0,1\}\times B_d(0,1) : |(w_0,w)|=1, \\ nh+hw_0\le T, mh+h|w|\le K, u+hw\in U, w\in C\}. \tag{3.6}$$

By applying a standard argument based on the dynamic programming principle for the discrete problem, we find the following set of equations depending on the values of n and m.

For $n<N$ and $m\le M$,

$$\mathcal{V}_h(x,u,nh,mh) = \min_{\substack{v\in V\\(w_0,w)\in A^h(n,m,u)}} \Big\{ \mathcal{V}_h(x+hg_0(x,u,nh,v)w_0 + \\ +\sum_{j=1}^{d} hg_j(x,u,nh,v)w_j, u+hw, nh+hw_0, mh+h|w|) \Big\}. \tag{DDP_1}$$

For $n=N$ and $m<M$,

$$\mathcal{V}_h(x,u,Nh,mh) = \min_{\substack{v\in V\\(w_0,w)\in A^h(N,m,u)}} \Big\{ \mathcal{V}_h(x+\sum_{j=1}^{d} hg_j(x,u,Nh,v)w_j,\ u+hw, Nh, mh+h) \Big\} \wedge \Phi(x,u). \tag{DDP_2}$$

For $n=N$ and $m=M$,

$$\mathcal{V}_h(x,u,Nh,Mh) = \Phi(x,u)\,. \tag{DDP_3}$$

We will refer in the sequel to the set of equations (DDP_1)–(DDP_3) as the system (DDP). Note that the scheme corresponding to (DDP) is explicit, *i.e.* the value on the right hand side depends only on values which have been previously computed. A standard discrete dynamic programming principle (see [1] p. 388) implies that the solution of (DDP) coincides with the value function of the discrete control problem.

Let us give a regularity result for the discrete value function. The result below is the discrete analogue of Theorem 2.2.

Proposition 3.1. *For any x_1, $x_2\in\mathbb{R}^D$, $u\in U$, i, $j\le N$ and m,$n\le M$ we have*

$$|\mathcal{V}_h(x_1,u,ih,mh)-\mathcal{V}_h(x_2,u,jh,nh)|\le C(|x_1-x_2|+|i-j|h+|m-n|h). \tag{3.7}$$

Moreover, if $(H3)_C$ holds, then $\mathcal{V}_h$ is Lipschitz continuous in u, uniformly in the other variables.

If $(H3)_U$ holds and the boundary of U is sufficiently smooth, then $\mathcal{V}_h$ is uniformly continuous in u.

Proof. The proof of (3.7) and of the regularity with respect to u under hypothesis $(H3)_C$ is obtained iterating in (DDP). As an example let us sketch the proof with respect to the first variable (the argument is very similar for the other variables).

For $i=j=N$ and $m=n=M$ we have

$$|\mathcal{V}_h(x_1,u,Nh,Mh)-\mathcal{V}_h(x_2,u,Nh,Mh)| = |\Phi(x_1,u)-\Phi(x_2,u)|\le L_\Phi|x_1-x_2|.$$

For $i = j = N$ and $m = n = M - 1$ we have

$$|\mathcal{V}_h(x_1, u, Nh, (M-1)h) - \mathcal{V}_h(x_2, u, Nh, (M-1)h)| \leq \left| \Phi \left(x_1 + \sum_{j=1}^{d} hg_j(x_1, u, Nh, \overline{v})\overline{w}_j, u + h\overline{w} \right) - \right.$$
$$\left. + \Phi \left(x_2 + \sum_{j=1}^{d} hg_j(x_2, u, Nh, \overline{v})\overline{w}_j, u + h\overline{w} \right) \right|$$

where the couple $(\overline{v}, \overline{w})$ is optimal for $\mathcal{V}_h(x_2, u, Nh, (M-1)h)$. By the Lipschitz continuity of the data we get

$$|\mathcal{V}_h(x_1, u, Nh, (M-1)h) - \mathcal{V}_h(x_2, u, Nh, (M-1)h)| \leq L_\Phi \left(|x_1 - x_2| + h \sum_{j=1}^{d} L|x_1 - x_2| \right).$$

For $i = j = N$ and $m = n = M - 2$ we have

$$\begin{aligned} &|\mathcal{V}_h(x_1, u, Nh, (M-2)h) - \mathcal{V}_h(x_2, u, Nh, (M-2)h)| \\ &\qquad \leq \left| \mathcal{V}_h \left(x_1 + \sum_{j=1}^{d} hg_j(x_1, u, Nh, \overline{v})\overline{w}_j, u + h\overline{w}, Nh, (M-1)h \right) + \right. \\ &\qquad \left. -\mathcal{V}_h \left(x_2 + \sum_{j=1}^{d} hg_j(x_2, u, Nh, \overline{v})\overline{w}_j, u + h\overline{w}, Nh, (M-1)h \right) \right| \\ &\qquad \leq L_\Phi(1 + hdL) \left| x_1 + \sum_{j=1}^{d} hg_j(x_1, u, Nh, \overline{v})\overline{w}_j - x_2 - \sum_{j=1}^{d} hg_j(x_2, u, Nh, \overline{v})\overline{w}_j \right| \\ &\qquad \leq L_\Phi(1 + hdL)^2 \, |x_1 - x_2|. \end{aligned}$$

The result is finally obtained iterating backward.

Concerning the proof under hypothesis $(H3)_U$, observe that, if $u \in \partial U$ and ν_u is the exterior normal to U at u, we can find a control $(w_0, w) \in S^d_+$ such that $w \cdot \nu_u < 0$. This means that the domain U satisfies a viability condition for the dynamics (2.1), relatively to the components $n+1, .., n+d$. Therefore we can apply the result in [9], where the uniform continuity of the state constraint discrete value function was proved, to get the same result in the present situation. □

We conclude this section proving the convergence of the scheme to the continuous value function. The result depends on a general stability property of viscosity solutions (see [1,4]).

Rewrite system (DDP) in compact form as

$$\mathcal{R}_h(X, \mathcal{V}_h(X), \mathcal{V}_h) = 0 \tag{3.8}$$

where $X = (x, u, nh, mh)$, $\mathcal{R}_h = -\left(\frac{\mathcal{S}_h}{h} \wedge \mathcal{T}_h\right)$, and the operators $\mathcal{S}_h$ and $\mathcal{T}_h$ are defined for $X \in \overline{\Omega}$, $r \in \mathbb{R}$, $\phi : \overline{\Omega} \to \mathbb{R}$ in the following way

$$\mathcal{S}_h(X, r, \phi) = \min_{\substack{v \in V \\ A^h(n,m,u)}} \left\{ \phi(x + hg_0(x, u, nh, v)w_0 + \sum_{j=1}^{d} hg_j(x, u, nh, v)w_j, \right.$$
$$\left. u + hw, nh + hw_0, mh + h|w|) - r \right\}$$

and

$$\mathcal{T}_h(X,r) = \begin{cases} \Phi(x,u) - r & \text{if } n = N \\ 0 & \text{otherwise} \end{cases}$$

($\mathcal{T}_h$ is the part of the operator $\mathcal{R}_h$ which takes into account the boundary condition.) It is easy to check that the scheme satisfies the following properties:

(i) *Monotonicity*
If ϕ, ψ are bounded measurable functions such that $\phi \geq \psi$ in $\overline{\Omega}$, then

$$\mathcal{R}_h(X,r,\phi) \leq \mathcal{R}_h(X,r,\psi).$$

(ii) *Stability*
There exists a constant M, indipendent of h, such that

$$|\mathcal{V}_h(x,u,nh,mh)| \leq M.$$

(iii) *Consistency*
For all $\phi \in C^\infty(\overline{\Omega})$,

$$\liminf_{\substack{h\to 0^+\\ Y\to X}} \mathcal{R}_h(Y,\phi(Y),\phi) \geq -H(X,\nabla\phi(X)), \qquad \text{for } X \in \Omega$$

and

$$\limsup_{\substack{h\to 0^+\\ Y\to X}} \mathcal{R}_h(Y,\phi(Y),\phi) \leq -H(X,\nabla\phi(X)), \quad X \in \Omega \cup \partial'\Omega$$

$$\limsup_{\substack{h\to 0^+\\ Y\to X}} \mathcal{R}_h(Y,\phi(Y),\phi) \leq -H(X,\nabla\phi(X)) \vee (\phi(X) - \Phi(x,u)), \quad X \in \partial_T\Omega.$$

The previous properties have been introduced in [4] as general conditions guaranteeing the convergence of a sequence of solutions of approximating problems to the viscosity solution of the limit equation. Here we split the discrete operator $\mathcal{R}_h$ into two parts. The first part takes into account the approximation of the equation, while the second part refers to the approximation of the boundary conditions. Note that, for the consistency condition, only the first part has to be divided by h before passing to the limit.

The proof of the convergence theorem, given the previous properties, is essentially the same of the convergence theorem in [4] and it consists in verifying that the upper and lower weak limits of the sequence of the discrete value functions are respectively sub and supersolution of equation (DPE) with boundary conditions $(BC)_1$ and $(BC)_2$. The comparison Theorem 2.2 implies that these limits coincide and therefore the sequence of the approximating solutions converges to the unique viscosity solution of (DPE). In conclusion, the following result holds true.

Theorem 3.2. *For $h \to 0^+$, the sequence of the approximating value functions $\mathcal{V}_h$ converges to the value function of the extended control problem $\mathcal{V}$, locally uniformly in $\overline{\Omega}$.*

4. Error estimate

This section is devoted to establish an estimate of the approximation error for the scheme described in Section 3. This estimate is derived comparing the continuous and the discrete dynamic programming equations and it is obtained adapting the techniques in [11, 26].

Theorem 4.1. *Let us assume that $\mathcal{V}$ and $\mathcal{V}_h$ are Lipschitz continuous respect to all the variables and let us denote by $L_{\mathcal{V}}$ and $L_{\mathcal{V}_h}$ their Lipschitz constants. Then, there exists a positive constant C such that*

$$\sup_{\substack{n\le N, m\le M\\ (x,u)\in\mathbb{R}^D\times U}} |\mathcal{V}(x,u,nh,mh)-\mathcal{V}_h(x,u,nh,mh)| \le C\sqrt{h}. \tag{4.1}$$

Proof. We will prove only the estimate $\mathcal{V}(x,u,nh,mh)-\mathcal{V}_h(x,u,nh,mh) \le C\sqrt{h}$, since the reverse inequality can be obtained similarly.

Let $\overline{\Omega}_h = \mathbb{R}^D \times U \times \{ih : i = 0,1,..,N\} \times \{jh : j = 0,1,..,M\}$ and

$$\sigma = \sup_{\substack{(x,u)\in\mathbb{R}^D\times U\\ 0\le n\le N\\ 0\le m\le M}} \{\mathcal{V}(x,u,nh,mh)-\mathcal{V}_h(x,u,nh,mh)\}\cdot$$

We may assume without loss of generality that $\sigma > 0$. Define the function $\psi : \overline{\Omega}_h \times \overline{\Omega}_h \to \mathbb{R}$ by

$$\begin{aligned}\psi(x,u,t,k,y,z,s,l) &= \mathcal{V}(x,u,t,k)-\mathcal{V}_h(y,z,s,l)\\ &\quad + 4M\beta_\varepsilon(x-y,u-z,t-s,k-l) + \frac{\sigma}{4(T+K)}(k+l+s+t)\end{aligned}$$

where

$$\beta_\varepsilon(x,u,t,k) = \beta\left(\frac{x}{\varepsilon},\frac{u}{\varepsilon},\frac{t}{\varepsilon},\frac{k}{\varepsilon}\right)$$

and β is a smooth function such that $0 \le \beta \le 1$ and

$$\beta(x,u,t,k) = \begin{cases} 1-\frac{1}{2}(|x|^2+|u|^2+t^2+k^2), & \text{if } |x|^2+|u|^2+t^2+k^2 < 1\\ \le \frac{1}{2} & \text{elsewhere.}\end{cases}$$

Let $M = M_\Phi$, therefore $\|\mathcal{V}\|_{L^\infty}, \|\mathcal{V}_h\|_{L^\infty} \le M$. Let us assume that $(X_0,Y_0) = ((x_0,u_0,t_0,k_0),(y_0,z_0,s_0,l_0))$ is a maximum point of $\psi(X,Y)$ in $\overline{\Omega}_h \times \overline{\Omega}_h$. We will make use of the following Lemma, which will be proved at the end of this section.

Lemma 4.2. *The following inequalities hold true*

$$\psi(X_0,Y_0) \ge \sigma + 4M \tag{4.2}$$

$$|X_0-Y_0|^2 \le \varepsilon^2 \tag{4.3}$$

$$\max\{|x_0-y_0|,|u_0-z_0|,|t_0-s_0|,|k_0-l_0|\} \le \left(\frac{L_{\mathcal{V}}}{2M}+\frac{M}{T+K}\right)\varepsilon^2 \tag{4.4}$$

$$-4M\nabla_x\beta_\varepsilon(X_0-Y_0) \text{ and } -4M\nabla_u\beta_\varepsilon(X_0-Y_0) \text{are bounded by } L_{\mathcal{V}}+L_{\mathcal{V}_h} \tag{4.5}$$

$$\nabla_x\beta_\varepsilon(X_0-Y_0) = -\frac{(X_0-Y_0)}{\varepsilon^2}\cdot \tag{4.6}$$

We divide the proof of Theorem 4.1 into several cases.

Case 1. $t_0, s_0 < T$, $l_0, k_0 < K$.

Since X_0 is a maximum point for $\mathcal{V}(X) - \mathcal{V}_h(Y_0) + 4M\beta_\varepsilon(X - Y_0) + \sigma(k + t + l_0 + s_0)/(4(T + K))$, by definition of viscosity subsolution we have

$$-H(X_0, \nabla\psi(X_0, Y_0)) = -\min_{\substack{v\in V\\(w_0,w)\in S_+^d}} \left\{ \left(-4M\frac{\partial\beta_\varepsilon}{\partial t}(X_0 - Y_0) - 4M\nabla_x\beta_\varepsilon(X_0 - Y_0) \cdot \right.\right.$$
$$\left. g_0(x_0, u_0, t_0, v) \right) w_0 - 4M\frac{\partial\beta_\varepsilon}{\partial k}(X_0 - Y_0)|w| - 4M\sum_{j=1}^{d}\left(g_j(x_0, u_0, t_0, v)\right.$$
$$\left.\left. \nabla_x\beta_\varepsilon(X_0 - Y_0) + \frac{\partial\beta_\varepsilon}{\partial u_j}(X_0 - Y_0) \right) w_j \right\} + \frac{\sigma}{4(T + K)} \le 0. \tag{4.7}$$

Let us assume that, for example, the minimum in (DDP) is obtained for $(w_0, w) = (1, 0)$ (if the minimum is obtained for $(w_0, w) = (0, 1)$ the proof is very similar). Therefore, we have

$$\mathcal{V}_h(Y_0) = \min_{v\in V} \{\mathcal{V}_h(y_0 + hg(y_0, z_0, s_0, v),\ z_0,\ s_0 + h,\ l_0)\} \cdot$$

Set $s_0 = n_0 h$, $l_0 = m_0 h$. Since $\psi(X_0, Y_0) \ge \psi(X_0, Y)$, we get

$$\mathcal{V}_h(Y) \ge \mathcal{V}_h(Y_0) - 4M\beta_\varepsilon(X_0 - Y_0) + 4M\beta_\varepsilon(X_0 - Y) + -\frac{\sigma}{4(T + K)}\left[(n_0 - n)h + (m_0 - m)h\right].$$

Take $y = y_0 + hg_0(y_0, z_0, n_0 h, v)$, $z = z_0$, $n = n_0 + 1$ and $m = m_0$ in the previous inequality to get

$$\min_{v\in V}\left\{ \mathcal{V}_h(y_0 + hg_0(y_0, z_0, n_0 h, v), z_0, (n_0 + 1)h,\ m_0 h) \right\} \ge \mathcal{V}_h(y_0, z_0, n_0 h, m_0 h) +$$
$$- 4M\beta_\varepsilon(X_0 - Y_0) + \frac{\sigma}{4(T + K)}h + \min_{v\in V}\left\{ 4M\beta_\varepsilon(x_0 - y - hg_0(y, z, n_0 h, v),\ u_0 - z, t_0 - s_0 - h,\ k_0 - l_0) \right\}$$

and therefore

$$\frac{\sigma}{4(T + K)}h \le 4M\left[\beta_\varepsilon(X_0 - Y_0) - \beta_\varepsilon(x_0 - y_0,\ u_0 - z_0, t_0 - s_0 - h,\ k_0 - l_0)\right] +$$
$$- \min_{v\in V}\left\{ 4M\left[\beta_\varepsilon(x_0 - y_0 - hg_0(y_0, z_0, n_0 h, v),\ u_0 - z_0,\ t_0 - s_0 - h,\ k_0 - l_0) + \right.\right.$$
$$\left.\left. - \beta_\varepsilon(x_0 - y_0,\ u_0 - z_0,\ t_0 - s_0 - h,\ k_0 - l_0) \right] \right\} \cdot$$

By (4.2)–(4.5) of Lemma 4.2, it follows that

$$-\frac{1}{h}\left[\beta_\varepsilon(x_0 - y_0 - hg_0(y_0, z_0, s_0, v),\ u_0 - z_0,\ t_0 - s_0 - h,\ k_0 - l_0) + \right.$$
$$\left. - \beta_\varepsilon(x_0 - y_0,\ u_0 - z_0,\ t_0 - s_0 - h, k_0 - l_0) \right] \le g_0(y_0, z_0, s_0, v) \cdot \nabla_x\beta_\varepsilon(X_0 - Y_0) + C\frac{h}{\varepsilon^2}$$

for some positive constant C. Choosing $\varepsilon = h^{1/4}$, the two previous inequalities give

$$\frac{\sigma}{4(T+K)} \le 4M\frac{\partial \beta_\varepsilon}{\partial t}(X_0 - Y_0) - \min_{v\in V}\{-4M\nabla_x\beta_\varepsilon(X_0-Y_0)\cdot g_0(y_0,z_0,s_0,v)\} + Ch^{\frac{1}{2}}. \tag{4.8}$$

Let $\overline{v} \in V$ be a control which attains the maximum in (4.8). Taking the control $(w_0, w) = (1, 0)$ in equation (4.7) and subtracting (4.8), we finally get

$$\frac{\sigma}{2(T+K)} \le 2L(g_0(x_0,u_0,t_0,\overline{v}) - g_0(y_0,z_0,s_0,\overline{v})) + C\varepsilon^2 \le C(|x_0-y_0|+|u_0-z_0|+|t_0-s_0|) + C\varepsilon^2 \le Ch^{1/2}.$$

Case 2. We have the following possibilities:

(A) $s_0 = T$, $t_0 \le T$, and either $k_0 \le K, l_0 = K$ or $k_0 = K,\ l_0 < K$ or $k_0 < K, l_0 < K$.

(B) $s_0 < T$, $t_0 \le T$, and either $k_0 \le K, l_0 = K$ or $k_0 = K,\ l_0 < K$ or $k_0 < K, l_0 < K$.

Let us analyze the first case, the other cases will follow in a similar way. Assume that $s_0 = T, t_0 \le T$, $k_0 \le K$, $l_0 = K$. We have

$$\begin{aligned}\sigma + 4M &\le \psi(X_0,(y_0,z_0,T,K)) = \mathcal{V}(X_0) - \mathcal{V}_h(Y_0)\\ &+ \frac{\sigma}{4(T+K)}(T+K+t_0+k_0) + 4M\beta_\varepsilon(x_0-y_0,\ u_0-z_0,\ t_0-T,\ k_0-K).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\sigma}{2} &\le \mathcal{V}(x_0,u_0,t_0,k_0) - \mathcal{V}_h(y_0,z_0,T,K)\\ &\le |\mathcal{V}(x_0,u_0,t_0,k_0) - \mathcal{V}(y_0,z_0,t_0,k_0)| + |\mathcal{V}(y_0,z_0,t_0,k_0) - \Phi(y_0,z_0)|\\ &\le L_{\mathcal{V}}(|x_0-y_0|+|u_0-z_0|) + L_{\mathcal{V}_h}(|T-t_0|+|K-k_0|).\end{aligned} \tag{4.9}$$

We need to give an estimate for $|T - t_0|$ (the estimate for $|K - k_0|$ will be obtained in the same way). Since

$$\psi(X_0,(y_0,z_0,T,K)) \ge \psi((x_0,u_0,T,k_0),(y_0,z_0,T,K)),$$

we get

$$\begin{aligned}\mathcal{V}(x_0) &+ \frac{\sigma}{4(T+K)}(t_0+k_0+T+K) + 4M\beta_\varepsilon(x_0-y_0,u_0-z_0,t_0-T,k_0-K)\\ &\ge \mathcal{V}(x_0,u_0,T,k_0 + \frac{\sigma}{4(T+K)}(k_0+2T+K) + 4M\beta_\varepsilon(x_0-y_0,\ u_0-z_0,\ 0,\ k_0-K)\end{aligned}$$

and

$$\begin{aligned}4M[\beta_\varepsilon(x_0-y_0,\ u_0-z_0,\ 0,\ k_0-K) &- \beta_\varepsilon(x_0-y_0,\ u_0-z_0,\ t_0-T,\ k_0-K)]\\ &\le \mathcal{V}(x_0,u_0,t_0,k_0) - \mathcal{V}(x_0,u_0,T,k_0)\end{aligned} \tag{4.10}$$

which implies that $|t_0 - T| \le \frac{2L}{M}\varepsilon^2$.

Substituting the above estimate and those for $|x_0 - y_0|, |u_0 - z_0|$ given by (4.4) in inequality (4.9), we get the estimate on σ and then (4.1) for $\varepsilon = h^{\frac{1}{4}}$.

Now assume that $s_0 = T$, $t_0 \leq T \quad k_0 = K$, $l_0 < K$. We have

$$\sigma + 4M \leq \psi(x_0, u_0, t_0, K, y_0, z_0, T, l_0) = \mathcal{V}(x_0, u_0, t_0, K) - \mathcal{V}_h(y_0, z_0, T, l_0)$$
$$+ \frac{\sigma}{4(T+K)}\,[t_0 + K + T + l_0] + 4M\beta_\varepsilon(X_0 - Y_0).$$

Then

$$\begin{aligned}\frac{\sigma}{2} &\leq \mathcal{V}(x_0, u_0, t_0, K) - \mathcal{V}_h(y_0, z_0, T, l_0)\\ &\leq |\mathcal{V}(x_0, u_0, t_0, K) - \mathcal{V}(x_0, u_0, T, K)| + |\Phi(x_0, u_0) - \mathcal{V}_h(y_0, z_0, T, l_0)|\\ &\leq L_{\mathcal{V}}|t_0 - T| + L_{\mathcal{V}_h}(|x_0 - y_0| + |u_0 - z_0| + |K - l_0|).\end{aligned} \tag{4.11}$$

To get an estimate for $|K - l_0|$, consider that

$$\psi(x_0, u_0, t_0, K, y_0, z_0, T, l_0) \geq \psi(x_0, u_0, t_0, K, y_0, z_0, T, l_0 + h)\ .$$

Hence

$$\begin{aligned}\mathcal{V}(X_0) - \mathcal{V}_h(Y_0) &+ \frac{\sigma}{4(T+K)}\,[t_0 + K + T + l_0] + 4M\beta_\varepsilon(X_0 - Y_0)\\ &\geq \mathcal{V}(X_0) - \mathcal{V}_h(y_0, z_0, T, l_0 + h) + \frac{\sigma}{4(T+K)}\,[t_0 + K + T + l_0 + h]\\ &+ 4M\beta_\varepsilon(x_0 - y_0, u_0 - z_0, t_0 - T, K - l_0 - h).\end{aligned}$$

That implies

$$\begin{aligned}4M[\beta_\varepsilon(x_0 - y_0, u_0 - z_0, t_0 - T, K - l_0 - h) - \beta_\varepsilon(x_0 - y_0, u_0 - z_0, t_0 - T, K - l_0)]&\\ \leq \mathcal{V}_h(y_0, z_0, T, l_0 + h) - \mathcal{V}_h(y_0, z_0, T, l_0) - \frac{\sigma}{4(T+K)}h&\end{aligned}$$

and therefore

$$\frac{4M}{\varepsilon^2}\,(K - l_0)h \leq \frac{4M}{\varepsilon^2}[|K - l_0|^2 - |K - l_0 - h|^2] \leq Lh,$$

that is

$$|K - l_0| \leq \frac{L}{4M}\,\varepsilon^2\ .$$

The estimate for $|T - t_0|$ can be obtained similarly. Estimate for $|u_0 - z_0|$, $|x_0 - y_0|$ follows from Lemma 4.2. Finally, substituting in (4.11) we get the result.

The last case $s_0 = T$, $t_0 \leq T$, $l_0 < K$ and $k_0 < K$ can be treated in similar way. □

Remark 4.1. The Lipschitz continuity of the function $\mathcal{V}$ with respect to the variable u holds under the hypothesis $(H3)_C$. However, when $\mathcal{V}$ is only continuous (f.e. under the hypothesis $(H3)_U$), the previous theorem gives an estimate of order $C\sqrt{\omega(h)}$ where ω is the modulus of continuity of the function $\mathcal{V}$ respect to the variable u.

Proof of Lemma 4.2 Let us start from (4.2). By $\psi(X_0, Y_0) \geq \psi(X, X)$, it follows that

$$\psi(X_0, Y_0) \geq \mathcal{V}(x) - \mathcal{V}_h(x) + 4M + \frac{\sigma}{4(T+K)}\,[t + k + s + l]$$

which gives (4.2).

To prove (4.3) observe that, if $|X_0 - Y_0|^2 > \varepsilon^2$, then

$$\beta_\varepsilon(X_0 - Y_0) \leq \frac{1}{2}$$

and $\psi(X_0, Y_0) \leq 2M + 2M + \frac{\sigma}{2} \leq 4M + \sigma$. This gives a contradiction and therefore (4.3) is true.

Let us prove (4.4). We first estimate $|x_0 - y_0|$. Since

$$\psi(X_0, Y_0) \geq \psi(y_0, u_0, t_0, k_0, y_0, z_0, s_0, l_0),$$

then

$$\mathcal{V}(X_0) + 4M\beta_\varepsilon(X_0 - Y_0) \geq \mathcal{V}(y_0, u_0, t_0, k_0) + 4M\beta_\varepsilon(0, u_0 - z_0, t_0 - s_0, k_0 - l_0),$$

and therefore

$$4M[\beta_\varepsilon(0, u_0 - z_0, t_0 - s_0, k_0 - l_0) - \beta_\varepsilon(X_0 - Y_0)] \leq L|x_0 - y_0|.$$

The last inequality gives $4M|x_0 - y_0|^2/2\varepsilon^2 \leq L|x_0 - y_0|$. The estimate for $|u_0 - z_0|$ can be proved similarly.

To estimate $|t_0 - s_0|$, observe that $\psi(X_0, Y_0) \geq \psi(x_0, u_0, s_0, k_0, y_0, v_0, s_0, l_0)$ implies

$$\mathcal{V}(x_0) + 4M\beta_\varepsilon(X_0 - Y_0) + \frac{\sigma}{2(T+K)}(t_0 + s_0) \geq \mathcal{V}(x_0, u_0, s_0, k_0)$$
$$+ 4M\beta_\varepsilon(x_0 - y_0, u_0 - v_0, 0, k_0 - l_0) + \frac{\sigma}{2(T+K)}(s_0 + s_0).$$

Hence

$$4M[\beta_\varepsilon(x_0 - y_0, u_0 - v_0, 0, k_0 - l_0) - \beta_\varepsilon(X_0 - Y_0)] \leq L|t_0 - s_0| + \frac{\sigma}{2(T+K)}|t_0 - s_0|$$

which yelds

$$4\frac{M}{2\varepsilon^2}|t_0 - s_0|^2 \leq (L + \frac{M}{(T+K)})|t_0 - s_0|.$$

The proof of (4.5) and (4.6) are immediate. □

5. Convergence of optimal controls

The main goal in the approximation of optimal control problems is to obtain informations valid for the optimal control of the continuous problem. How the discrete optimal control w_h^* is related to the optimal control w^* for the continuous problem? Does it converge to w^* for h going to 0? Usually the answer to those questions is rather difficult and one can only prove convergence in the sense of weak convergence of probability measures (see [19]). Nonetheless, we are able to prove a stronger convergence result for the problem under study taking advantage of its particular structure.

Fix $(x, u, t, k) \in \Omega$ and let $(x, u, n_h h, m_h h)$ be a sequence such that $n_h h \to t$ and $m_h h \to k$. By solving iteratively the minimum problem in the approximating equation, we can find a control $\{(w_{0n}^*, w_n^*), v_n^*\}_{n=1}^{P_h^*}$, depending on h, which is optimal for the discrete control problem, *i.e.*

$$\mathcal{V}_h(x, u, nh, mh) = \Phi(y_{P_h^*+1}(x, u, nh;\, w_{0n}^*, w_n^*, v_n^*)) \;. \tag{5.1}$$

Corresponding to any admissible discrete control $\{(w_{0n}, w_n), v_n\}_{n=1}^P$, we can define a continuous control $(u_h(s), t_h(s), v_h(s))$ in the following way

$$\begin{cases} t_h(s) = t_n + h(P+1)(s - \frac{n}{P+1})w_{0n} \\ u_h(s) = u_n + h(P+1)(s - \frac{n}{P+1})w_n \\ v_h(s) = v_n \end{cases} \tag{5.2}$$

for $s \in [\frac{n}{P+1}, \frac{n+1}{P+1})$ and $n = 0, .., P$, where (u_n, t_n) is the discrete trajectory corresponding to the discrete control. The previous definition and the admissibility of the discrete control imply that

$$t^h(0) = nh\ ,\ t_h(1) = T\ ,\quad t_h \text{ is non decreasing,}$$
$$u^h(0) = u\ ,\quad V_0^1(u_h) \le (M-m)h,$$
$$|(u_h'(s), t_h'(s))| = h(P+1) \qquad \text{for any } s \in [0,1].$$

Therefore $(u_h, t_h, v_h) \in \Gamma_{(M-m)h}(u, nh)$ and it is in canonical form.

5.1. Pure impulsive control problem

Let us assume that the control problem does not depend on the ordinary control v (this is the case in the Vidale-Wolfe type of models, see [14]). We have the following result

Theorem 5.1. *Let $\{(w_{0n}, w_n)\}$ be an optimal control for the discrete problem and (t_h, u_h) be the continuous control defined in* (5.2)*. Then, there exist a subsequence $h_p \to 0^+$ and an admissible control (u^*, t^*) such that $(u_{h_p}, t_{h_p}) \to (u^*, t^*)$ locally uniformly in $U \times [0, T]$ and (u^*, t^*) is an optimal control for the extended control problem, i.e.*

$$\mathcal{V}(x, u, t, k) = \Phi(y[x, u, t; u^*, t^*](1))\ . \tag{5.3}$$

Proof. Since the controls (u_h, t_h) are in canonical form, the sequence $\{(u_h, t_h)\}$ is equicontinuous and equibounded. Then, for a subsequence, there exists (u^*, t^*) such that (u_h, t_h) converges locally uniformly to (u^*, t^*) in $U \times [0, T]$. It is easy to see that $(u^*, t^*) \in \Gamma_{K-k}(u, t)$.

Let $y_h(s) = y[x, nh, u;\ u_h, t_h](s)$. Since we have

$$\lim_{h\to 0} \mathcal{V}_h(x, u, nh, mh) = \lim_{h\to 0} \Phi(y_{P_h^*+1}[x, u, nh;\ w_{0n}^*, w_n^*]) = \mathcal{V}(x, u, t, k)$$

it can be proved that

$$\lim_{h\to 0} \Phi(y[x, u, nh;\ u_h, t_h](1)) = \mathcal{V}(x, u, t, k)\ . \tag{5.4}$$

By the hypotheses $(H1)$, $(H2)$ and the Gronwall's inequality, we obtain

$$|y_h^i(s)| \le M \qquad \text{for any } s \in [0,1],\ i = 1, .., n$$

where M is independent of h. Since $|(u_h, t_h)| \le T + K + |u|$, the sequence $\{y_h(s)\}$ is also equibounded. Then, there exists a function y^* such that $y_h(s)$ converges uniformly to $y^*(s)$. It is obvious that $y_{n+j}^*(s) = u_j^*(s)$, $j = 1, \ldots, d$ and $y_{n+d+1}^*(s) = t^*(s)$.

The uniform convergence of (u_h, t_h) to (u^*, t^*) in $[0, 1]$ implies that $(u_h'(s), t_h'(s))$ converges to $(u'(s), t'(s))$ in the weak-star topology of $L^\infty([0, 1], \mathbb{R}^{1+d})$. Passing to the limit in the equation satisfied by $y_h(s)$, we get that

$$y^*(s) = y[x, u, t;\ u^*, t^*](s), \qquad \text{for } s \in [0,1]\ .$$

Finally, since

$$\lim_{h\to 0} \Phi(y[x,u,nh;\, u_h,t_h](1)) = \Phi(y[x,u,t;\, u^*,t^*](1))$$

the above equality with (5.4) gives (5.3). If the optimal control for the continuous problem is unique, then the entire sequence of discrete optimal controls converges to it.

5.2. **The general case**

It is well known that a sequence of ordinary controls may have no limit, due to the possible highly oscillating behavior. Therefore we need to consider relaxed controls in place of ordinary controls (see [11] for a similar approach in a problem without impulsive controls). The class of the relaxed control laws is given by $V_r = \{\mu : [0,1] \to \mathcal{T}(V)\}$ where $\mathcal{T}(V)$ is the class of Radon measure on V. The dynamics of the extended process under the action of relaxed controls does not change for the components y_{n+j}, $j = 1,\dots,d$, and y_{n+d+1}, while is given by

$$y_i(s) = x + \int_0^t \int_V g_{i0}(y(s),v(s))t'(s)d\mu_s(v)ds + \int_0^t \int_V \sum_{j=1}^m g_{ij}(y(s),v(s))u_j'(s)d\mu_s(v)ds$$

for $i = 1,..,n$. Let us denote by $\mathcal{V}^r$ the value function corresponding to the infimum problem with relaxed control laws.

An ordinary control v can be identified with a relaxed control defined by $\mu^v : s \to \delta_{v(s)}$. We have the following result, which can be proved combining the technique of Theorem 5.1 and the convergence result in [11].

Theorem 5.2. *Let $\{(w_{0n}, w_n, v_n)\}$ be an optimal control for the discrete problem and (t_h, u_h, v_h) the continuous control defined in (5.2). Then there exists a subsequence $h_p \to 0^+$ and an admissible relaxed control (u^*, t^*, μ^*) such that*

i) *$\mu^{v_h} \to \mu^*$ in the weak-star topology of $L^\infty([0,1], \mathcal{T}(V))$;*
ii) *$(u_h, t_h) \to (u^*, t^*)$ locally uniformly in $U \times [0,T]$;*
iii) *$\mathcal{V}^r(x,u,t,k) = \Phi(y[x,u,t;\, u^*,t^*,\mu^*](1))$, i.e. the control (u^*,t^*,μ^*) is optimal for the extended control problem.*

6. Fully discrete scheme: discretization in x and u

In this section, we describe the fully discrete scheme which we get by a further discretization of the problem with respect to the variables x and u. The technique used here is based on a piecewise linear local reconstruction of the value function (like in P^1 finite element schemes) applied to the (DDP) system of equation. That technique has been successfully applied to other optimal control problems, we refer the interested reader to the recent survey [17].

Let us observe first that up to now we have considered a problem in $\mathbb{R}^D$. To solve it numerically we need to restrict x to some bounded set $\mathcal{D}$. In some cases (as for example in the Vidale-Wolfe advertising problem [14,15]) the problem itself is settled in a bounded set so that some natural boundary conditions are given. However, if the problem is on the whole $\mathbb{R}^D$, we must introduce an arbitrary domain $\mathcal{D}$ adding some artificial boundary condition, as for example an homogeneous Neumann boundary condition or an absorbing boundary condition on the boundary of $\mathcal{D}$. We also remark that modifying in a suitable way the scheme described in the previous section, it is possible to approximate problems with boundary conditions (see, for example [2,9,25] respectively for Dirichlet, state constraints and Neumann boundary conditions). Concerning the set of controls U, when it is unbounded, we may solve the problem in $U^R = U \cap B_d(0,R)$ and then send R to $+\infty$. Under reasonable assumptions, one can prove that the value function $\mathcal{V}^R$ corresponding to the set U^R converges locally uniformly to $\mathcal{V}$.

Since we just want to sketch here the main points of the algorithm, let us consider for simplicity the case where $\mathcal{D}$ and U are bounded set and the dynamics is invariant with respect to $\mathcal{D}$. Fix two regular triangulation

$\mathcal{T}_\mathcal{D}$ and $\mathcal{T}_\mathcal{U}$ of these sets. Let $\{x_p : p = 1,..,P\}$ and $\{u_q : q = 1,..,Q\}$ be the corresponding vertices of the triangulation. We define

$$\delta = |\mathcal{T}_\mathcal{D}| \qquad \eta = |\mathcal{T}_\mathcal{U}|$$

where $|\cdot|$ denotes here the maximum diameter of the simplices of the triangulation. For all $x \in \mathcal{D}$ and $u \in U$, we have

$$\begin{cases} x = \sum_{p=1}^{P} \lambda_p(x) x_p, & 0 \le \lambda_p \le 1, \quad \sum_{p=1}^{P} \lambda_p(x) = 1 \\ u = \sum_{q=1}^{Q} \mu_q(x) u_q, & 0 \le \mu_q \le 1, \quad \sum_{q=1}^{Q} \mu_q(x) = 1. \end{cases} \tag{6.1}$$

The approximating fully discrete problem consists in the computation of a function $\mathcal{W}(\S, \sqcap, \backslash\langle, \Updownarrow\langle)$ (dependent on h, η and δ), such that $\mathcal{W}$ is continuous in (x,u) for $(x,u) \in \mathcal{D} \times \mathcal{U}$, piecewise linear on the simplices of the two triangulations and satisfies the equation (DDP) at the grid points (x_p, u_q, nh, mh), $p = 1,..,P$, $q = 1,..,Q$, $n = 1,..,N$, $m = 1,..,M$.

Using (6.1) and the linearity of $\mathcal{W}$ in the variables x and u, it possible to prove (see [9, 17] for details) that the previous approximating problem is equivalent to find a vector $W \in \mathbb{R}^{P+Q+N+M}$ (which actually represents the value of the function $\mathcal{W}$ at the point (x_p, u_q, nh, mh)) given by the following recursive scheme

$$\begin{cases} W_{p,q,n,m} = \min\limits_{\substack{v \in V \\ (w_0,w) \in A^h(n,m,u_q)}} \left\{ \sum\limits_{\substack{r=1,..,P \\ s=1,..,Q}} \Delta^n_{pqrs}(w_0, w, v) W_{r,s,n+w_0,m+|w|} \right\}, & n < N, \\ & m \le M, \\ W_{p,q,N,m} = \min\limits_{\substack{v \in V \\ (w_0,w) \in A^h(N,m,u_q)}} \left\{ \sum\limits_{\substack{r=1,..,P \\ s=1,..,Q}} \Delta^N_{pqrs}(w_0, w, v) W_{r,s,N,m+1} \right\}, & m < M \\ W_{p,q,N,M} = \Phi(x_p, u_q) \end{cases}$$

where $A^h(n, m, u_q)$ in defined as in (3.6) (note that $|w|$ can only take values in $\{0, 1\}$).

For any $n = 1,..,N$, the matrix Δ^n is defined by the following relations

$$\Delta^n_{pqrs}(w_0, w, v) = B^n_{pqr}(w_0, w, v) C_{qs}(w),$$

$$x_p + hg_0(x_p, u_q, v, nh) w_0 + \sum_{j=1}^{d} hg_j(x_p, u_q, nh, v) w_j = \sum_{r=1}^{P} B^n_{pqr} x_r,$$

$$u_q + hw = \sum_{r=1}^{Q} C_{qr}(w) u_r.$$

The scheme is explicit, since the components of the vector W which appear on the right end side of the equation have the last two indices equal either to $(n+1, m)$ or to $(n, m+1)$ and therefore they are always available when we make one iteration.

Note that the matrix C does not depend on n, x_i and v so that we just need to compute it once, at the first iteration. On the contrary, the matrix B has to be up-dated at each iteration.

The following theorem gives an estimate of the distance between the solution of the approximating equation (DDP) and the solution of the fully discrete equation (the proof can be obtained as in [16]).

Theorem 6.1. *If $\mathcal{V}_h$ is the solution of the equation $(DPE)_h$, we have*

$$|\mathcal{V}_h(x, u, nh, mh) - \mathcal{W}(\S, \sqcap, \backslash\langle, \Updownarrow\langle)| \le \mathcal{C} \left(\frac{\omega_\langle(\eta + \delta)}{\langle} + \omega_\langle(\eta + \delta) \right) \tag{6.2}$$

where C is constant depending only on the coefficients of the problem and ω_h is the continuity modulus of the function $\mathcal{V}_h$.

The above theorem implies that the sequence of the solutions to the fully discrete approximating equations converges to the value function of the extended control problem provided h, η, δ go to 0 with $\omega_h(\eta + \delta)/h$.

We conclude discussing some implementation issues. Two additional discretizations are necessary to implement the scheme: the discretization for the space V of ordinary controls and of the set $B_d(0,1)$ where the control w takes values. Those discretizations allow to compute the minimum in the discrete operator just comparing a finite number of values. Other strategies are possible (f.e. descent methods) but they can result in a very expensive algorithm since the minimization procedure takes place for every iteration at each node of the grid.

When the set of ordinary controls V is not finite, it is usually replaced with a finite set $\widehat{V}$ (f.e. using another triangulation). It can be proved that the discretization of V produces an error of order $\mathcal{H}(\mathcal{V}, \widehat{\mathcal{V}})$ to be added to the error estimate (here $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff distance between two closed sets).

Finally, the discretization of $\partial B_d(0,1)$ for the impulsive control is obtained in the simplest way. For example when $d = 2$, we take an integer L and fix L directions on the unit ball given by the angles $\frac{2\pi}{L}i$, $i = 0, .., L-1$. The error given by this discretization is of order $\frac{1}{L}$. When $d = 1$, there are only two possible directions and we do not need any discretization.

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