

WELL POSEDNESS AND CONTROL OF SEMILINEAR WAVE EQUATIONS WITH ITERATED LOGARITHMS*

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Abstract. Motivated by a classical work of Erdős we give rather precise necessary and sufficient growth conditions on the nonlinearity in a semilinear wave equation in order to have global existence for all initial data. Then we improve some former exact controllability theorems of Imanuvilov and Zuazua.

Résumé. Motivé par un travail classique d'Erdős on donne des conditions nécessaires et suffisantes de croissance de la non linéarité dans une équation des ondes semilinéaire pour l'existence des solutions globales pour toutes les données initiales. Ensuite on améliore certains théorèmes antérieurs de contrôlabilité exacte de Imanuvilov et de Zuazua.

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1. INTRODUCTION AND FORMULATION OF THE MAIN RESULTS

Consider the problem

$$\begin{cases} u'' - \Delta u - f(u) = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma \times (0, T), \\ u(0) = u_0 \text{ and } u'(0) = u_1 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where

- Ω is a nonempty bounded open domain in \mathbb{R}^N having a boundary Γ of class C^2 ;

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- $f: \mathbb{R} \rightarrow \mathbb{R}$ is a given function of class C^1 ;
- T is a given positive number.

For the existence of a unique maximal solution

$$u \in C([0, T']; H_0^1(\Omega)) \cap C^1([0, T']; L^2(\Omega)), \quad 0 < T' \leq T,$$

in case $N \geq 2$ we need the additional property

$$\begin{cases} |f'(s)| \leq c(1 + |s|^{2/(N-2)}) & \text{if } N \geq 3, \\ |f'(s)| \leq c(1 + |s|^\alpha) & \text{for some finite } \alpha \text{ if } N = 2 \end{cases} \quad (1.2)$$

for all $s \in \mathbb{R}$. No condition of this type is needed if $N = 1$. We refer, *e.g.*, to [2], Chapter 6, for proof.

First we study the existence of a (unique) *global* solution

$$u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \quad (1.3)$$

for every given

$$u_0 \in H_0^1(\Omega) \quad \text{and} \quad u_1 \in L^2(\Omega), \quad (1.4)$$

without any sign condition on f . We need another growth assumption on f . For this we introduce the primitive F of f defined by

$$F(s) = \int_0^s f(z) dz, \quad s \in \mathbb{R}.$$

Furthermore, motivated by [3], let us introduce the *iterated logarithm* functions \log_j defined by the formulae

$$\log_0 s := s \quad \text{and} \quad \log_j s := \log(\log_{j-1} s), \quad j = 1, 2, \dots$$

Define e_j by the equation $\log_j e_j = 2$.

Theorem 1.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^1 . Assume (1.2) if $N \geq 2$. Furthermore, assume that there exist a positive integer k and a number $\beta > 0$ such that*

$$|F(s)| \leq \beta \prod_{j=0}^k \log_j^2 |s| \quad \text{for all } |s| > e_k. \quad (1.5)$$

Then for every u_0, u_1 satisfying (1.4) the problem (1.1) has a unique global solution satisfying (1.3).

Our proof will show that the quantity

$$\int_{\Omega} (u'(t))^2 + |\nabla u(t)|^2 - 2F(u(t)) dx$$

does not depend on $t \in [0, T]$.

We will also show that the condition (1.5) is essentially optimal.

Theorem 1.2. *Assume that there exist a positive integer k , a real number $p > 2$ and two positive numbers α_0 and c such that*

$$F(s) \geq \alpha_0 \left(\prod_{j=0}^{k-1} \log_j^2 s \right) \log_k^p s \quad \text{for all } s > c. \quad (1.6)$$

Then there exist initial data

$$u_0 \in H_0^1(\Omega) \quad \text{and} \quad u_1 \in L^2(\Omega)$$

such that (1.1) has no global solution satisfying (1.3).

Remarks.

- Theorem 1.1 was first proved by Cazenave and Haraux [1] in the case where f is a constant multiple of the function $s \log s$. They considered the case $\Omega = \mathbb{R}^N$, but their method equally works for bounded domains.
- Subsequently Zuazua [10] noticed that their method also applies under the weaker assumption (1.5) for $k = 1$, and he also proved Theorem 1.2 for $k = 1$.
- One can readily verify that if

$$f(s) = O\left(|s| \prod_{j=1}^k \log_j^2 |s|\right) \quad \text{as } |s| \rightarrow \infty,$$

then condition (1.5) is satisfied.

- Zuazua asked whether the condition (1.5) can be weakened by replacing the product by an infinite linear combination of the products indexed by k , with a suitable sequence of coefficients converging sufficiently quickly to zero. In order to not make this paper too long, we shall study this question in a later work.
- If there exist a positive integer k , a real number $p > 2$ and two positive numbers α_1 and c_1 such that

$$f(s) \geq \alpha_1 s \left(\prod_{j=1}^{k-1} \log_j^2 s \right) \log_k^p s \quad \text{for all } s > c_1,$$

then condition (1.6) is satisfied. Indeed, we have

$$F(s) = \int_0^s f(t) dt \geq \int_{s/2}^s f(t) dt \geq \alpha_1 \frac{s}{2} \frac{s}{2} \left(\prod_{j=1}^{k-1} \log_j^2 \frac{s}{2} \right) \log_k^p \frac{s}{2}.$$

If s is sufficiently large, say $s > c$, then

$$\log_j \frac{s}{2} > \frac{\log_j s}{2} \quad \text{for } j = 1, \dots, k$$

and therefore

$$F(s) \geq \alpha_1 2^{-2-2(k-1)-p} \left(\prod_{j=0}^{k-1} \log_j^2 s \right) \log_k^p s,$$

i.e. (1.6) is satisfied with

$$\alpha_0 = \alpha_1 2^{-2-2(k-1)-p}.$$

Next we study the boundary controllability of the system

$$\begin{cases} u'' - \Delta u - f(u) = 0 & \text{in } \Omega \times (0, T), \\ u = h & \text{on } \Gamma \times (0, T), \\ u(0) = u_0 \text{ and } u'(0) = u_1 & \text{in } \Omega. \end{cases}$$

We only consider the one-dimensional case

$$\Omega = (a, b),$$

so we rewrite it in the following form:

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } (a, b) \times (0, T), \\ u(t, a) = h_a(t) \text{ and } u(t, b) = h_b(t) & \text{for } t \in (0, T), \\ u(0, x) = u_0(x) \text{ and } u_t(0, x) = u_1(x) & \text{for } x \in (a, b). \end{cases} \quad (1.7)$$

Definition. The problem (1.7) is *exactly controllable at time T* if for every

$$(u_0, u_1), (v_0, v_1) \in H^1(\Omega) \times L^2(\Omega)$$

there exist *control functions*

$$h_a, h_b \in H^1(0, T)$$

such that (1.7) has a global solution satisfying the *final conditions*

$$u(T) = v_0 \text{ and } u'(T) = v_1 \text{ in } \Omega.$$

Theorem 1.3. *Assume that f satisfies the growth condition (1.5) for some positive integer k . If*

$$T > b - a, \quad (1.8)$$

then the problem (1.7) is exactly controllable at time T .

Remarks.

- This theorem answers in particular a question of Zuazua [10]. He proved the exact controllability of (1.1) by assuming instead of (1.5) that

$$|F(s)| < \beta_0 s^2 \log^2 |s| \text{ for all } |s| > c$$

for a sufficiently small β_0 and for a sufficiently large $c > 0$, and he asked whether the same conclusion holds for large β_0 . Our theorem provides, in particular, an affirmative answer to such a question. Note, however, that Zuazua also obtained analogous results by controlling at only one endpoint of the interval (a, b) , and for a related *internal* controllability problem.

- The optimality of condition (1.8) follows from the finite propagation speed for the wave equation (at least if $f \equiv 0$).

To prove Theorem 1.1 we shall apply the approach of Cazenave and Haraux [1]. In particular, we shall need a general nonlinear version of Gronwall's inequality (see Th. 2.1 below), which may be interesting itself.

The proof of Theorem 1.2 will use the ideas of Zuazua [10] by showing that the solution blows up before T .

We were unable to adapt Zuazua's fixed point method for the proof of Theorem 1.3. Instead, we apply a former method of Imanuvilov [4], based on characteristics.

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2. GRONWALL TYPE INEQUALITIES AND ITERATED LOGARITHMS

Theorem 2.1. *Let $g: [0, \infty) \rightarrow \mathbb{R}$ be a continuous and non-decreasing function. Assume that there exists a positive number c such that*

$$g > 0 \text{ in } (c, \infty) \quad \text{and} \quad \int_c^\infty \frac{1}{g(s)} ds = \infty. \quad (2.1)$$

Let $\varphi: [0, T] \rightarrow \mathbb{R}$ be a continuous, nonnegative function, satisfying for some numbers $A, B > 0$ the inequalities

$$\varphi(t) \leq A + B \int_0^t g(\varphi(s)) ds \quad \text{for all } t \in [0, T]. \quad (2.2)$$

Then φ is bounded on $[0, T]$ by a constant depending only on g , A , B and on T .

Remarks.

1) More precisely, putting

$$G(t) := \int_c^t \frac{1}{g(s)} ds, \quad t \in (0, \infty),$$

we have

$$\varphi(t) \leq G^{-1}(G(A) + Bt), \quad \text{for all } t \in [0, T].$$

2) The original Gronwall lemma corresponds to the choice $g(s) = s$.

3) Cazenave and Haraux [1] considered the case $g(s) = s \log(1 + s)$.

4) We shall use later the more general case where

$$g(s) = (e_k + s) \log(e_k + s) \dots \log_k(e_k + s) = \prod_{j=0}^k \log_j(e_k + s)$$

for some positive integer k and with e_k as defined above by $\log_k e_k = 2$.

Proof of Theorem 2.1. It follows from our assumptions that G is of class C^1 , strictly increasing, and it maps the interval $(0, \infty)$ onto $(G(0+), \infty)$. We have

$$\frac{d}{dt} \left[G \left(A + B \int_0^t g(\varphi(s)) ds \right) \right] = G' \left(A + B \int_0^t g(\varphi(s)) ds \right) B g(\varphi(t)) = B \frac{g(\varphi(t))}{g \left(A + B \int_0^t g(\varphi(s)) ds \right)} \leq B.$$

Therefore

$$G(\varphi(t)) \leq G \left(A + B \int_0^t g(\varphi(s)) ds \right) \leq G(A) + Bt$$

and hence

$$\varphi(t) \leq G^{-1}(G(A) + Bt), \quad t \in [0, T]$$

as claimed. □

Now we generalize a crucial technical lemma in [1] to iterated logarithms. For this we introduce the iterated exponential functions $\exp_j: \mathbb{R} \rightarrow \mathbb{R}$ by the formulas

$$\exp_0 x = x \quad \text{and} \quad \exp_j x = \exp(\exp_{j-1} x), \quad j = 1, 2, \dots$$

Given an integer $k \geq 0$, set $e_k = \exp_k 2$ and

$$L_k(x) = \prod_{j=0}^k \log_j(e_j + |x|), \quad x \in \mathbb{R}.$$

This function is well defined, *even*, and *increasing* for $x \geq 0$. Furthermore, we deduce from the inequality

$$L_k(x) \geq (2 + |x|) \prod_{j=1}^k \log_j e_j = 2^k (2 + |x|) \geq 2^{k+1} > 0$$

that

$$L_k(x) \geq |x| \quad \text{and} \quad L_k(x) \geq 2 > 0$$

for all x and k . We shall use these properties several times.

In order to simplify the notations henceforth we denote by $\|\cdot\|_p$ the usual norm of $L^p(\Omega)$ and we write simply $\|\cdot\|$ instead of $\|\cdot\|_2$.

We need a generalization of a lemma in [1].

Lemma 2.2. *Given $k \geq 0$ and $\varepsilon > 0$ arbitrarily, there is a constant $c(\varepsilon)$ such that*

$$\|L_k(u)\| \leq \varepsilon \|\nabla u\| + c(\varepsilon) L_k(\|u\|) \tag{2.3}$$

for all $u \in H_0^1(\Omega)$, where $\|\cdot\|$ denotes the usual norm in $L^2(\Omega)$.

Proof. Assume for simplicity that $N \geq 3$: the cases $N = 1, 2$ are analogous and simpler. We recall that by the Sobolev imbedding theorem there exists a constant S such that

$$\|u\|_{2N/(N-2)}^2 \leq S \|\nabla u\|^2$$

for all $u \in H_0^1(\Omega)$.

Given $\delta > 0$ arbitrarily, there exists a constant $c(\delta) > 0$ such that

$$L_k(x)^2 \leq \delta |x|^{(2N+4)/N} + c(\delta)$$

for all real x . Since

$$\frac{N}{2N+4} = \alpha \frac{1}{2} + (1-\alpha) \frac{N-2}{2N}$$

if $\alpha = 2/(N+2)$, applying the interpolational inequality we have (denoting by $|\Omega|$ the volume of Ω)

$$\begin{aligned} \|L_k(u)\|^2 &\leq \delta \|u\|_{(2N+4)/N}^{(2N+4)/N} + c(\delta) |\Omega| \leq \delta (\|u\|^{2/(N+2)} \|u\|_{2N/(N-2)}^{N/(N+2)})^{(2N+4)/N} + c(\delta) |\Omega| \\ &= \delta \|u\|^{4/N} \|u\|_{2N/(N-2)}^2 + c(\delta) |\Omega| \leq \delta S \|u\|^{4/N} \|\nabla u\|^2 + c(\delta) |\Omega|. \end{aligned}$$

Since $L_k > 1$ everywhere, in case $\|u\| \leq 1$ hence we deduce the estimate

$$\|L_k(u)\|^2 \leq \delta S \|\nabla u\|^2 + c(\delta) |\Omega| L_k(\|u\|)^2, \quad (2.4)$$

and (2.3) follows by choosing $\delta = S^{-1}\varepsilon^2$.

Henceforth assume that $\|u\| > 1$. Let us note that

$$\log(c + ab) \leq \log(c + a) + \log(c + b)$$

for all $a, b \geq 0$ and $c \geq 1$, and that

$$\log(a + b) \leq \log a + \log b$$

for all $a, b \geq 2$. Hence for every $a, b \geq 0$ we have the inequalities

$$\begin{aligned} \log_j(e_j + ab) &\leq \log_{j-1}[\log(e_j + a) + \log(e_j + b)] \leq \log_{j-2}[\log_2(e_j + a) + \log_2(e_j + b)] \\ &\leq \dots \leq 2 \max\{\log_j(e_j + a), \log_j(e_j + b)\} \end{aligned}$$

for $j = 1, 2, \dots, k$. Distinguishing the cases $a \geq b$ and $a < b$ hence we deduce the inequality

$$\begin{aligned} \left(\prod_{j=1}^k \log_j(e_j + ab) \right)^2 &\leq 4^k \max \left\{ \left(\prod_{j=1}^k \log_j(e_j + a) \right)^2, \left(\prod_{j=1}^k \log_j(e_j + b) \right)^2 \right\} \\ &\leq 4^k \left(\prod_{j=1}^k \log_j(e_j + a) \right)^2 + 4^k \left(\prod_{j=1}^k \log_j(e_j + b) \right)^2. \end{aligned} \quad (2.5)$$

Now given $u \in H_0^1(\Omega)$ such that $\|u\| > 1$, setting $v := u/\|u\|$ and applying (2.5) we have

$$\begin{aligned} \int_{\Omega} L_k(u)^2 dx &= \int_{|u| \leq \|u\|} L_k(u)^2 dx + \int_{|u| > \|u\|} L_k(u)^2 dx \leq |\Omega| L_k(\|u\|)^2 \\ &+ \int_{|u| > \|u\|} (2 + |u|)^2 \left(\prod_{j=1}^k \log_j(e_j + |u|) \right)^2 dx \leq |\Omega| L_k(\|u\|)^2 \\ &+ 4^k \int_{|u| > \|u\|} (2 + |u|)^2 \left(\prod_{j=1}^k \log_j(e_j + |v|) \right)^2 dx \\ &+ 4^k \int_{|u| > \|u\|} (2 + |u|)^2 \left(\prod_{j=1}^k \log_j(e_j + \|u\|) \right)^2 dx =: |\Omega| L_k(\|u\|)^2 + I_1 + I_2. \end{aligned}$$

Since $\|u\| > 1$ implies

$$2 + |u| \leq \|u\|(2 + |v|),$$

we have

$$I_1 \leq 4^k \int_{|u| > \|u\|} \|u\|^2 L_k(v)^2 dx \leq 4^k \|u\|^2 \|L_k(v)\|^2.$$

Furthermore, since $|u| > \|u\| > 1$ implies

$$2 + |u| \leq 3|u|,$$

we have

$$I_2 \leq 9 \cdot 4^k \int_{|u| > \|u\|} |u|^2 dx \left(\prod_{j=1}^k \log_j(e_j + \|u\|) \right)^2 \leq 9 \cdot 4^k L_k(\|u\|)^2.$$

Substituting this into the above inequality we find that

$$\|L_k(u)\|^2 \leq |\Omega| L_k(\|u\|)^2 + 4^k \|u\|^2 \|L_k(v)\|^2 + 9 \cdot 4^k L_k(\|u\|)^2.$$

Applying (2.4) for v and using the inequality $L_k(x) \geq |x|$ hence we obtain that

$$\begin{aligned} \|L_k(u)\|^2 &\leq \{|\Omega| + 9 \cdot 4^k\} L_k(\|u\|)^2 + 4^k \|u\|^2 \delta S \|\nabla v\|^2 + 4^k \|u\|^2 c(\delta) |\Omega| L_k(1)^2 \\ &\leq 4^k \delta S \|\nabla u\|^2 + \{|\Omega| + 4^k c(\delta) |\Omega| L_k(1)^2 + 9 \cdot 4^k\} L_k(\|u\|)^2. \end{aligned}$$

Choosing $\delta = 4^{-k} S^{-1} \varepsilon^2$ the lemma follows. \square

3. PROOF OF THEOREM 1.1

According to the general Theorem 6.1.4 in [8], it suffices to show that the maximal solution cannot blow up before $t = T$. Thus Theorem 1.1 will follow from the

Lemma 3.1. *Consider the unique maximal solution of (1.1), defined on some interval $[0, T')$ with $0 < T' \leq T$. Then*

$$u \in L^\infty(0, T'; H_0^1(\Omega)), \quad u' \in L^\infty(0, T'; L^2(\Omega)), \quad F(u) \in L^\infty(0, T'; L^1(\Omega)).$$

Proof. Multiplying the equation in (3.1) by u' and integrating by parts we obtain that the quantity

$$\int_{\Omega} (u'(t))^2 + |\nabla u(t)|^2 - 2F(u(t)) dx \tag{3.1}$$

is conserved. Applying Lemma 2.2 we obtain that

$$\|u'\|^2 + \|\nabla u\|^2 \leq c + \|L_k(u)\|^2 \leq c + \varepsilon \|\nabla u\|^2 + c(\varepsilon) L_k(\|u\|)^2. \tag{3.2}$$

Choosing $\varepsilon = 1$ and using the estimate

$$\|u(t)\| \leq \|u_0\| + \int_0^t \|u'(s)\| ds =: \|u_0\| + \varphi(t) \tag{3.3}$$

it follows that

$$\varphi'(t) = \|u'(t)\| \leq c + c(1) L_k(\|u_0\| + \varphi(t))$$

and therefore

$$\varphi(t) \leq cT + c \int_0^t L_k(\|u_0\| + \varphi(s)) ds, \quad 0 \leq t < T'.$$

Now choose $k' > k$ such that

$$L_k(\|u_0\| + x) \leq L_{k'}(x)$$

for all $x \geq 0$, and apply Theorem 2.1 with

$$g(s) = L_k(\|u_0\| + s).$$

We obtain that

$$\varphi \in L^\infty(0, T').$$

Using (3.3) it follows that

$$u \in L^\infty(0, T'; L^2(\Omega))$$

and then, applying (3.2) again, this time with $\varepsilon < 1$, we conclude that

$$u \in L^\infty(0, T'; H_0^1(\Omega))$$

and

$$u' \in L^\infty(0, T'; L^2(\Omega)).$$

Finally, by the conservation of the quantity (3.1) we have

$$F(u) \in L^\infty(0, T'; L^1(\Omega)),$$

as stated. □

4. PROOF OF THEOREM 1.2

It is sufficient to prove Theorem 1.2 for small T 's. Set

$$x_0 = \frac{a+b}{2}$$

and fix

$$0 < T < R < \frac{b-a}{2}$$

arbitrarily. Thanks to (1.5) we have

$$\int_c^\infty \frac{ds}{\sqrt{F(s)}} \leq c_1 \int_c^\infty \frac{ds}{s \log s \dots \log_{k-1} s (\log_k s)^{p/2}} = c_1 \left(1 - \frac{p}{2}\right)^{-1} [(\log_k s)^{1-\frac{p}{2}}]_c^\infty = \frac{2c_1}{p-2} (\log_k c)^{1-\frac{p}{2}} < \infty.$$

Therefore we can choose a positive constant $\alpha > c$ such that

$$\int_\alpha^\infty \frac{ds}{\sqrt{F(s)}} < \sqrt{2}T,$$

and choose $u_0 \in H_0^1(\Omega)$ such that

$$u_0 = \alpha \text{ in } (x_0 - R, x_0 + R).$$

Furthermore, choose another positive constant β , and choose $u_1 \in L^2(\Omega)$ such that

$$u_1 = \beta \text{ in } (x_0 - R, x_0 + R).$$

We claim that the corresponding solution of (1.1) blows up before T .

Assume on the contrary that u is defined in $(a, b) \times (0, T)$. It follows from the finite propagation property that for any fixed $x \in (x_0 - R + T, x_0 + R - T)$, the function $v(t) := u(x, t)$ satisfies the ordinary differential equation

$$v'' - f(v) = 0 \quad \text{in } (0, T)$$

with the initial conditions

$$v(0) = \alpha \quad \text{and} \quad v'(0) = \beta.$$

(In particular, it does not depend on the particular choice of x .)

It follows from the differential equation that the quantity

$$E(t) := |v'(t)|^2 - 2F(v(t))$$

is in fact independent of $t \in (0, T)$. Hence

$$v'(t) = \sqrt{E(0) + 2F(v(t))}$$

and therefore

$$T = \int_{\alpha}^{v(T)} \frac{dv}{\sqrt{E(0) + 2F(v)}} \leq \int_{\alpha}^{\infty} \frac{dv}{\sqrt{2F(v)}}.$$

But this is impossible because the last integral is less than T by the choice of α .

5. PROOF OF THEOREM 1.3

We apply a method of Imanuvilov [4]. First we deduce from Theorem 1.1 an auxiliary result on the Cauchy problem in a triangle. Given a bounded interval $(-d, d)$, set

$$Q = \{(x, t) \in \mathbb{R}^2 : -d < x < d, 0 < t < d - |x|\},$$

and

$$S = \{(x, t) \in \mathbb{R}^2 : -d < x < d, t = d - |x|\}.$$

Then Q is a triangle with basis $(-d, d)$ and altitude d , and S is the union of the other two sides; see Figure 1.

Consider the problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } Q, \\ u(0, x) = u_0(x) \text{ and } u_t(0, x) = u_1(x) & \text{for } x \in (-d, d). \end{cases} \quad (5.1)$$

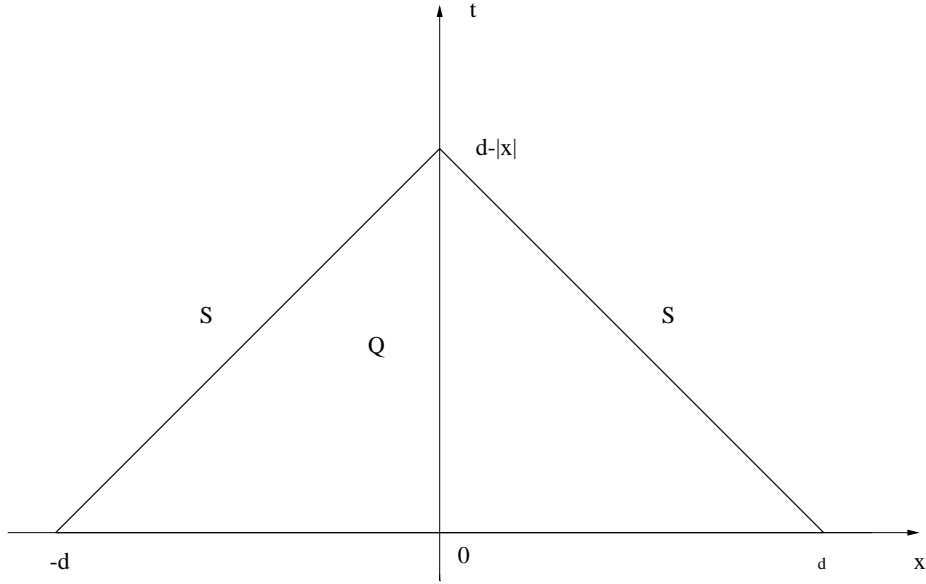


FIGURE 1

Proposition 5.1. *Assume that f satisfies for some positive integer k the condition (1.5), and let*

$$u_0 \in H^1(-d, d), \quad u_1 \in L^2(-d, d).$$

Then the problem (5.1) has a unique solution $u \in H^1(Q)$ for which the traces $u_t(\cdot, t)$, $u_x(\cdot, t)$ are well defined in $L^2(t-d, d-t)$ for every $0 \leq t \leq d$, and the function

$$t \mapsto \int_{t-d}^{d-t} u_t^2 + u_x^2 dx$$

is continuous (hence bounded) on $[0, d]$.

Furthermore, the trace of the solution on S satisfies

$$u|_S \in H^1(S). \tag{5.2}$$

Remark.

Note that the above properties imply that solutions are bounded: $u \in L^\infty(Q)$.

Proof. Let us choose a bounded interval $\Omega = (a, b)$ containing $[-d, d]$ in its interior, and extend u_0 and u_1 to Ω such that

$$u_0 \in H_0^1(a, b), \quad u_1 \in L^2(a, b).$$

Then by Theorem 1.1 the problem (1.1) has a (unique) global solution

$$\bar{u} \in C([0, d]; H^1(a, b)) \cap C^1([0, d]; L^2(a, b)).$$

Its restriction u to Q has the desired properties.

Let us note that if f is *globally* Lipschitz continuous and if the initial data are more regular:

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \quad \text{and} \quad u_1 \in H_0^1(\Omega),$$

then by standard regularity results we have

$$\bar{u} \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega)).$$

Its restriction u to Q will be called a *strong* solution.

For the proof of the uniqueness, let u^1, u^2 be two solutions and set $w = u^1 - u^2$, so that

$$\begin{cases} w_{tt} - w_{xx} = \varphi w & \text{in } Q, \\ w(0, x) = w_t(0, x) = 0 & \text{for } x \in (-d, d) \end{cases}$$

where

$$\varphi = \int_0^1 f'(\lambda u^1 + (1 - \lambda)u^2) d\lambda \in L^\infty(Q).$$

Therefore

$$w(t, x) = \frac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \varphi w d\sigma d\tau.$$

Define

$$W(t) = \sup_{|x| < d-t} |w(t, x)|, \quad 0 \leq t \leq d,$$

then

$$W(t) \leq \|\varphi\|_\infty \int_0^t W(\tau) d\tau, \quad 0 \leq t \leq d,$$

and so $W \equiv 0$ by the usual Gronwall lemma.

For the proof of (5.2), set

$$E(t) = \int_{t-d}^{d-t} u_t^2 + u_x^2 dx, \quad 0 \leq t \leq d.$$

Since the solution is bounded, modifying f outside a sufficiently large interval we may assume that f is globally Lipschitz continuous. Then for every strong solution (see the definition above) we have

$$\begin{aligned} E'(t) &= -(u_t^2 + u_x^2)(t, d-t) - (u_t^2 + u_x^2)(t, t-d) + \int_{t-d}^{d-t} 2u_t u_{tt} + 2u_x u_{xt} dx \\ &= -(u_t - u_x)^2(t, d-t) - (u_t + u_x)^2(t, t-d) + \int_{t-d}^{d-t} 2u_t(u_{tt} - u_{xx}) dx \\ &= -2u_t^2(t, d-t) - 2u_x^2(t, t-d) + \int_{t-d}^{d-t} 2f(u)u_t dx, \end{aligned}$$

where u_τ denotes the tangential derivative along S . Since $E(d) = 0$, integrating we obtain the identity

$$\sqrt{2} \int_S u_\tau^2 dS = E(0) + \int_Q F(u) dx. \quad (5.3)$$

This identity remains valid for all solutions by an approximation argument.

Since the right-hand side of the identity (5.3) is finite, the property (5.2) follows. \square

Remark. Let us note for further reference that the proof of identity (5.3) remains valid for every function $u \in C^2(Q) \cap C^2(\bar{Q})$ satisfying

$$u_{tt} - u_{xx} - f(u) = 0 \quad \text{in } Q.$$

Now, using the same notation as in Proposition 5.1, consider the Goursat problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } Q, \\ u = \psi & \text{on } S. \end{cases} \quad (5.4)$$

Proposition 5.2. *Assume that f satisfies condition (1.5) for some positive integer k , and let $\psi \in H^1(S)$. Then the problem (5.4) has a unique solution $u \in H^1(Q)$ such that the traces $u_t(\cdot, t)$, $u_x(\cdot, t)$ are well defined in $L^2(t-d, d-t)$ for every $0 \leq t \leq d$, and the function*

$$t \mapsto \int_{t-d}^{d-t} u_t^2 + u_x^2 dx$$

is continuous (hence bounded) on $[0, d]$.

Proof. Uniqueness. We note that, as in the preceding proposition, solutions belong to $L^\infty(Q)$. If u^1, u^2 solve (5.4), then $w := u^1 - u^2$ satisfies

$$\begin{cases} w_{tt} - w_{xx} = \varphi w & \text{in } Q, \\ w = 0 & \text{on } S \end{cases}$$

where

$$\varphi = \int_0^1 f'(\lambda u^1 + (1-\lambda)u^2) d\lambda \in L^\infty(Q).$$

We have to show that w vanishes identically. More generally, we will show that if a function w , having the same regularity as the solutions in the formulation of the proposition, satisfies

$$\begin{cases} w_{tt} - w_{xx} = \varphi w & \text{in } Q, \\ w = \psi & \text{on } S \end{cases}$$

for some $\varphi \in L^\infty(Q)$ and $\psi \in H^1(S)$, then

$$\|w\|_{H^1(Q)} + \|w_t\|_{L^2(0,d;L^2(t-d,d-t))} + \|w_x\|_{L^2(0,d;L^2(t-d,d-t))} \leq c' \|\psi\|_{H^1(S)} \quad (5.5)$$

with a constant c' depending only on R if $\|\varphi\|_{L^\infty(Q)} \leq R$.

First, from the inequalities

$$|w(x, t)| \leq |w(x, t) - w(t-d, t)| + |\psi(t-d, t)|$$

and

$$|w(x, t) - w(t - d, t)| \leq \|w_x\|_{L^1(t-d, d-t)} \leq c \|w_x\|_{L^2(t-d, d-t)}$$

we deduce the estimate

$$\|w(\cdot, t)\|_{L^2(t-d, d-t)} \leq c \|w_x\|_{L^2(t-d, d-t)} + c |\psi(t - d, t)|. \quad (5.6)$$

Next, setting

$$E(t) = \int_{t-d}^{d-t} w_t^2 + w_x^2 \, dx$$

and repeating the computation of the preceding proof, we obtain

$$E'(t) = -2\psi_\tau^2(t, d-t) - 2\psi_\tau^2(t, t-d) + \int_{t-d}^{d-t} 2\varphi w w_t \, dx.$$

Since $E(d) = 0$, it follows that

$$\begin{aligned} E(t) &= 2 \int_t^d \psi_\tau^2(s, d-s) + \psi_\tau^2(s, t-s) \, ds - \int_t^d \int_{s-d}^{d-s} 2(\varphi w w_t)(x, s) \, dx \, ds \\ &\leq \sqrt{2} \|\psi_\tau\|_{L^2(S)}^2 + \int_t^d \int_{s-d}^{d-s} w^2(x, s) + \|\varphi\|_{L^\infty(Q)}^2 w_t^2(x, s) \, dx \, ds. \end{aligned}$$

Hence, using (5.6), we deduce that

$$\begin{aligned} E(t) &\leq \sqrt{2} \|\psi_\tau\|_{L^2(S)}^2 + c \int_t^d \psi^2(s-d, s) \, ds + \int_t^d \int_{s-d}^{d-s} c w_x^2(x, s) + \|\varphi\|_{L^\infty(Q)}^2 w_t^2(x, s) \, dx \, ds \\ &\leq c \|\psi\|_{H^1(S)}^2 + c(1 + \|\varphi\|_{L^\infty(Q)}^2) \int_t^d E(s) \, ds. \end{aligned}$$

Applying the usual Gronwall lemma, the above inequality yields

$$\sup_{0 \leq t \leq d} E(t) \leq c' \|\psi\|_{H^1(S)}^2 \quad (5.7)$$

with a constant c' depending only on R if $\|\varphi\|_{L^\infty(Q)} \leq R$. Since (5.6) implies that

$$\|w\|_{L^2(Q)}^2 \leq c' (\|\psi\|_{H^1(S)}^2 + \sup_{0 \leq t \leq d} E(t)), \quad (5.8)$$

the desired inequality (5.5) follows from (5.7) and (5.8).

Existence. Assume first that f is globally Lipschitz continuous. Then for every $\psi \in C^1(S)$ the problem (5.4) has a solution $u \in C^1(\bar{Q})$, defined in a suitable weak sense. Indeed, the proof given in [9] (lecture 5) for the linear case remains valid for the globally Lipschitz case.

Now, given $\psi \in H^1(S)$ arbitrarily, choose a sequence (ψ^n) in $H^1(S)$ such that the corresponding solutions u^n of (5.4) belong to $C^1(\bar{Q}) \cap C^2(Q)$ and $\psi^n \rightarrow \psi$ in $H^1(S)$. Then the estimate (5.5) holds for all differences $w := u^n - u^m$ with a constant $c' = c'(R)$ (where R is the Lipschitz constant of f), which is easily seen to be independent of n . Therefore, (u_n) converges to a solution of (5.2), having the required regularity properties.

Let us note that we also have the estimate

$$\|u\|_{L^\infty(Q)} \leq c'' \quad (5.9)$$

with a constant c'' depending only on $\|\psi\|_{H^1(S)}$ and on the constant β in the condition (1.5), but not on the Lipschitz constant of f . Indeed, applying the identity (5.3) for each n (see the remark following the proof of the preceding proposition), we have

$$\|u_n(0)\|_{L^\infty(-d,d)}^2 \leq cE_n(0) \leq c+c \int_Q |F(u_n)| dx dt \leq c+c \int_Q L(u_n)^2 dx dt \leq c+c \int_0^d L(\|u_n(t)\|_{L^\infty(t-d,d-t)}^2) dt.$$

Since the time 0 plays no special role here, we have more generally

$$\|u_n(s)\|_{L^\infty(s-d,d-s)}^2 \leq c+c \int_s^d L(\|u_n(t)\|_{L^\infty(t-d,d-t)}^2) dt$$

for all $0 \leq s \leq t$, with the same constant c . Applying Theorem 2.1 we conclude that $\|u_n\|_{L^\infty(Q)}$ is bounded by some constant depending only on β and on $\|\psi_n\|_{H^1(S)}$. Letting $n \rightarrow \infty$ we obtain the desired estimate (5.9).

Now let us turn to the case where f is *not* globally Lipschitz continuous. Let us denote by v^n the solution of (5.1) where f is replaced by the globally Lipschitz continuous function f_n defined by

$$f_n(s) := \begin{cases} n & \text{if } f(s) > n, \\ -n & \text{if } f(s) < -n, \\ f(s) & \text{otherwise.} \end{cases}$$

By the preceding estimate (5.9) the sequence (v^n) is bounded in $L^\infty(Q)$, because all functions f_n satisfy the same growth conditions as f . It follows that if n is larger than this common bound, then v_n also solves (5.1) with the original f . \square

Turning to the proof of Theorem 1.3, let us set

$$\begin{aligned} A &= (a, 0), \\ B &= (a, T), \\ C &= (b, T), \\ D &= (b, 0), \\ E &= \left(\frac{a+b}{2}, \frac{b-a}{2} \right), \\ F &= \left(\frac{a+b}{2}, T - \frac{b-a}{2} \right). \end{aligned}$$

Furthermore, let us denote by K_1 the triangle ADE , by K_2 the triangle BFC , by K_3 the trapezoid $AEFB$ and by K_4 the trapezoid $CFED$; see Figure 2.

By Proposition 5.1 the Cauchy problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_1, \\ u(0, x) = u_0(x) \quad \text{and} \quad u_t(0, x) = u_1(x) & \text{for } x \in (a, b) \end{cases}$$

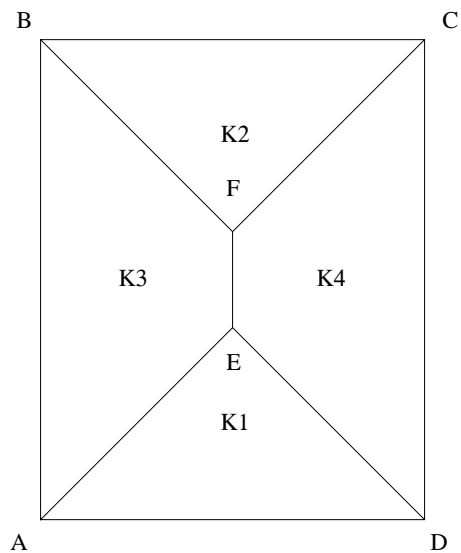


FIGURE 2

has a unique solution u^1 , and its trace on the line segments AE , DE belongs to $H^1(AE)$ and $H^1(DE)$, respectively.

Similarly, the Cauchy problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_2, \\ u(0, x) = v_0(x) \text{ and } u_t(0, x) = v_1(x) & \text{for } x \in (a, b) \end{cases}$$

has a unique solution u^2 , and its trace on the line segments BF , CF belongs to $H^1(BF)$ and $H^1(CF)$, respectively.

Now choose an arbitrary function $z_0 \in H^1(EF)$ which has the same values at the endpoints E and F of the segment EF as u^1 and u^2 . Furthermore, fix a function $z_1 \in L^2(EF)$ arbitrarily.

We are going to prove that the Goursat problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_3, \\ u = u^1 & \text{on } AE, \\ u = z_0 & \text{on } EF, \\ u = u^2 & \text{on } BF, \\ \partial u / \partial x = z_1 & \text{on } EF \end{cases} \quad (5.10)$$

has a unique solution u^3 , and similarly, that the Goursat problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_4, \\ u = u^1 & \text{on } DE, \\ u = z_0 & \text{on } EF, \\ u = u^2 & \text{on } CF, \\ \partial u / \partial x = z_1 & \text{on } EF \end{cases}$$

has a unique solution u^4 . By symmetry, we only consider the first one in the trapezoid K_3 .

Let us denote by G the intersection of the lines AE and BF , and let us denote by K_5 the triangle EGF . Exchanging the role of the variables t and x and applying Proposition 5.1 again, we obtain that the problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_5, \\ u = z_0 & \text{on } EF, \\ \partial u / \partial x = z_1 & \text{on } EF, \end{cases}$$

has a unique solution whose traces on the line segments FG and EG belong to $H^1(FG)$ and $H^1(EG)$, respectively.

Now consider the usual Goursat problem

$$\begin{cases} u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_6, \\ u = \varphi & \text{on } S, \end{cases} \quad (5.11)$$

where K_6 denotes the triangle AGB , S denotes the union of the four line segments BF , FG , GE and EA , and φ is given by the restrictions to these four line segments of the solutions of the above three Cauchy problems in the triangles K_1 , K_2 and K_5 . By Proposition 5.1 we have $\varphi \in H^1(S)$. Applying Proposition 5.2 (we exchange again the sense of the space and time variables), the problem (5.11) has a unique solution, and its trace on the side AB belongs to $H^1(AB)$.

In order to prove uniqueness, let u_1 and u_2 be two solutions of (5.10). Then the above method allows us to extend them to two solutions \tilde{u}_1 and \tilde{u}_2 of (5.11) with the same boundary data φ . We conclude by using the uniqueness part of Proposition 5.2 that $\tilde{u}_1 \equiv \tilde{u}_2$ and thus $u_1 \equiv u_2$.

Finally, we claim that the formula

$$u := u^i \quad \text{in } K_i \quad (i = 1, \dots, 4)$$

defines a solution of (1.7) with $h_a, h_b \in H^1(0, T)$ given by the traces of u , and satisfying the desired final conditions by construction. The only property to verify is that the differential equation in (1.7) is satisfied in the whole rectangle $R := (a, b) \times (0, T)$ and not only in the subdomains K_1 , K_2 , K_3 and K_4 . More precisely, we have to prove the

Lemma 5.3. *The equation*

$$\int_R u(\partial_t^2 - \partial_x^2)\varphi \, dx \, dt = \int_R f(u)\varphi \, dx \, dt \quad (5.12)$$

is satisfied for all $\varphi \in C_0^\infty(R)$.

The proof of this lemma involves nothing else than the theory of linear hyperbolic systems. For the reader's convenience we give a proof in an Appendix.

6. APPENDIX. PROOF OF LEMMA 5.3

We proceed in two steps.

Lemma 6.1. *The equation (5.12) is satisfied for all $\varphi \in C_0^\infty(R')$ where $R' = R - \{E, F\}$.*

Proof. Set

$$S_{ij} = \overline{K_i} \cap \overline{K_j}, \quad 1 \leq i \neq j \leq 4$$

and

$$S = \cup_{i \neq j} S_{ij}.$$

Let us fix $\varphi \in C_0^\infty(R')$. We can cover $\text{supp } \varphi$ with a finite collection B_1, \dots, B_N of open balls contained in R such that

$$E, F \notin \cup_{k=1}^N \overline{B_k},$$

and that each $\overline{B_k}$ intersects at most one of the sets S_{ij} .

Let $1 = \sum_{k=1}^N \psi_k$ be a partition of unity related to such a collection. Then

$$\varphi = \sum_{k=1}^N \varphi_k, \quad \varphi_k = \varphi \psi_k, \quad \text{supp } \varphi_k \subset \overline{B_k}.$$

It suffices to show that

$$\int_R u(\partial_t^2 - \partial_x^2) \varphi_k \, dx \, dt = \int_R f(u) \varphi_k \, dx \, dt, \quad k = 1, \dots, N. \quad (6.1)$$

If $\overline{B_k} \cap S = \emptyset$, this equality follows directly from the assumptions. Let us assume that $\overline{B_k} \cap S \neq \emptyset$, then $\overline{B_k} \cap S = \overline{B_k} \cap S_{ij}$ for some i, j .

If $S_{ij} = S_{34}$, then the equality follows because u_3 and u_4 agree on S_{34} together with their normal derivatives.

The remaining four cases are similar: suppose, for instance, that $S_{ij} = S_{13}$. Then

$$\int_R u(\partial_t^2 - \partial_x^2) \varphi_k \, dx \, dt = \left(\int_{B_k \cap K_1} + \int_{B_k \cap K_3} \right) u(\partial_t^2 - \partial_x^2) \varphi_k \, dx \, dt.$$

Moreover, using the normal vector $\nu = (1, -1)$ on S_{13} ,

$$\begin{aligned} \int_{B_k \cap K_1} u(\partial_t^2 - \partial_x^2) \varphi_k \, dx \, dt &= \int_{B_k \cap K_1} (\partial_t - \partial_x)[u(\partial_t + \partial_x) \varphi_k] - (\partial_t - \partial_x)u(\partial_t + \partial_x) \varphi_k \, dx \, dt \\ &= \int_{B_k \cap S_{13}} u(\partial_t + \partial_x) \varphi_k (\nu_t - \nu_x) \, dS - \int_{B_k \cap K_1} (\partial_t + \partial_x) \varphi_k (\partial_t - \partial_x) u \, dx \, dt \\ &= 2 \int_{B_k \cap S_{13}} u(\partial_t + \partial_x) \varphi_k \, dS - \int_{B_k \cap S_{13}} \varphi_k (\partial_t - \partial_x) u (\nu_t + \nu_x) \, dS \\ &\quad + \int_{B_k \cap K_1} \varphi_k f(u) \, dx \, dt. \end{aligned}$$

Since $\nu_t + \nu_x = 0$ on S_{13} , we conclude that

$$\int_{B_k \cap K_1} u(\partial_t^2 - \partial_x^2) \varphi_k \, dx \, dt = 2 \int_{B_k \cap S_{13}} u(\partial_t + \partial_x) \varphi_k \, dS + \int_{B_k \cap K_1} \varphi_k f(u) \, dx \, dt. \quad (6.2)$$

Similarly,

$$\begin{aligned} \int_{B_k \cap K_3} u(\partial_t^2 - \partial_x^2)\varphi_k \, dx \, dt &= \int_{B_k \cap S_{13}} u(\partial_t + \partial_x)\varphi_k(\nu_x - \nu_t) \, dS - \int_{B_k \cap S_{13}} \varphi_k(\partial_t - \partial_x)u(-\nu_t - \nu_x) \, dS \\ &\quad + \int_{B_k \cap K_3} \varphi_k f(u) \, dx \, dt; \end{aligned}$$

since $\nu_x - \nu_t = -2$ and $-\nu_t - \nu_x = 0$ on S_{13} , we conclude that

$$\int_{B_k \cap K_3} u(\partial_t^2 - \partial_x^2)\varphi_k \, dx \, dt = -2 \int_{B_k \cap S_{13}} u(\partial_t + \partial_x)\varphi_k \, dS + \int_{B_k \cap K_3} \varphi_k f(u) \, dx \, dt. \quad (6.3)$$

Adding together the equalities (6.2) and (6.3), the boundary terms cancel each other and we obtain (6.1).

The same computation works if $S_{ij} = S_{24}$. If $S_{ij} = S_{14}$ or $S_{ij} = S_{23}$, then we can similarly argue, exchanging the roles of the operators $\partial_t - \partial_x$ and $\partial_t + \partial_x$. \square

The twofold application of following lemma will complete the proof of Lemma 5.3. First we apply it with $g = f(u)$, $\Omega = R - \{E\}$ and $P = F$, and then with $g = f(u)$, $\Omega = R$ and $P = E$.

Lemma 6.2. *Let Ω be an open set in \mathbb{R}^2 and let $P \in \Omega$. If for some $u \in H^1(\Omega)$ and $g \in L^2(\Omega)$ the equality*

$$\int_{\Omega} u(\partial_t^2 - \partial_x^2)\varphi \, dx \, dt = \int_{\Omega} g\varphi \, dx \, dt$$

is satisfied for all $\varphi \in C_0^\infty(\Omega - \{P\})$, then it is also satisfied for all $\varphi \in C_0^\infty(\Omega)$.

Proof. Since the points have zero capacity in \mathbb{R}^2 , there exists for every $\varepsilon > 0$ a function $\varphi_\varepsilon \in C_0^\infty(B_\varepsilon(P))$ and a number $0 < \rho_\varepsilon < \varepsilon$ such that

$$0 \leq \varphi_\varepsilon(x, t) \leq 1, \quad \varphi_\varepsilon(x, t) = 1 \quad \text{if } |(x, t) - P| \leq \rho_\varepsilon, \quad \int_{B_\varepsilon(P)} |\nabla \varphi_\varepsilon|^2 \, dx \, dt \leq \varepsilon.$$

(See, e.g., Maz'ja [7] for proof.)

Let $\varphi \in C_0^\infty(\Omega)$, then

$$\varphi = \varphi\varphi_\varepsilon + \varphi(1 - \varphi_\varepsilon) =: \varphi_\varepsilon^1 + \varphi_\varepsilon^2$$

with

$$\varphi_\varepsilon^2 \in C_0^\infty(\Omega - \{P\}).$$

Therefore

$$\begin{aligned} \int_{\Omega} u(\partial_t^2 - \partial_x^2)\varphi \, dx \, dt &= \int_{\Omega} u(\partial_t^2 - \partial_x^2)\varphi_\varepsilon^1 \, dx \, dt + \int_{\Omega} u(\partial_t^2 - \partial_x^2)\varphi_\varepsilon^2 \, dx \, dt \\ &= - \int_{\Omega} (\partial_t u \partial_t \varphi_\varepsilon^1 - \partial_x u \partial_x \varphi_\varepsilon^1) \, dx \, dt + \int_{\Omega} g\varphi_\varepsilon^2 \, dx \, dt. \end{aligned}$$

Now, notice that

$$\|\varphi - \varphi_\varepsilon^2\|_{L^2(\Omega)} = \|\varphi\varphi_\varepsilon\|_{L^2(B_\varepsilon(P))} \rightarrow 0$$

as $\varepsilon \rightarrow 0$, and that

$$\left| \int_{\Omega} (\partial_t u \partial_t \varphi_{\varepsilon}^1 - \partial_x u \partial_x \varphi_{\varepsilon}^1) dx dt \right| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi_{\varepsilon}^1\|_{L^2(\Omega)}.$$

Using the choice of φ_{ε} we have

$$\|\nabla \varphi_{\varepsilon}^1\|_{L^2(\Omega)} \leq \|\nabla \varphi\|_{L^2(\Omega)} \|\varphi_{\varepsilon}\|_{L^2(B_{\varepsilon}(P))} + \|\varphi\|_{L^2(\Omega)} \|\nabla \varphi_{\varepsilon}\|_{L^2(B_{\varepsilon}(P))} \rightarrow 0$$

as $\varepsilon \rightarrow 0$, and so

$$\left| \int_{\Omega} u(\partial_t^2 - \partial_x^2)\varphi dx dt - \int_{\Omega} g\varphi dx dt \right| \leq \|g\|_{L^2(\Omega)} \|\varphi - \varphi_{\varepsilon}^2\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi_{\varepsilon}^1\|_{L^2(\Omega)} \rightarrow 0.$$

Hence the lemma follows. □

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