

LINKS BETWEEN YOUNG MEASURES ASSOCIATED TO CONSTRAINED SEQUENCES

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Abstract. We give necessary and sufficient conditions which characterize the Young measures associated to two oscillating sequences of functions, u_n on $\omega_1 \times \omega_2$ and v_n on ω_2 satisfying the constraint $v_n(y) = \frac{1}{|\omega_1|} \int_{\omega_1} u_n(x, y) dx$. Our study is motivated by nonlinear effects induced by homogenization. Techniques based on equimeasurability and rearrangements are employed.

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1. INTRODUCTION

Given two sequences of functions, u_n on $\omega_1 \times \omega_2$ and v_n on ω_2 such that the constraint $v_n(y) = \frac{1}{|\omega_1|} \int_{\omega_1} u_n(x, y) dx$ holds, one wants to characterize the relationship between their associated Young measures.

Our main result (Th. 3.1) gives necessary and sufficient conditions for the above problem in terms of distribution measures and decreasing rearrangements. It can be interpreted as follows: Young measures capture something of the oscillating behaviour of the sequence (u_n) and the integration $\int_{\omega_1} u_n(x, y) dx$ destroys part of the oscillations of u_n . We use Young measure techniques introduced by Ball, Tartar, Balder, Valadier. Notions and results from Probability Theory are employed; particularly, the order relation \prec on the set of positive measures due to Choquet and Loomis, and related results (see Cartier *et al.* [4] and Meyer [6]). The corresponding preorder relation \prec on the set of nonnegative L^1 functions introduced by Hardy *et al.* is used, as well as properties of doubly stochastic operators proved by Ryff.

Our study was motivated by more complex questions in relation with nonlocal effects induced by homogenization.

Consider the degenerate elliptic equation studied by Amirat *et al.* in [1]:

$$\begin{cases} \frac{\partial}{\partial x} \left(a_n(x, y) \frac{\partial u_n}{\partial x}(x, y) \right) = f(x, y) \text{ in }]0, 1[\times]0, 1[, \\ u_n(0, y) = u_n(1, y) = 0 \text{ on }]0, 1[. \end{cases} \quad (1.1)$$

By homogenization a nonlocal effect appears expressed in terms of a kernel. We introduce a parameter γ (following an idea of Tartar [10]) by setting $a_n(x, y) := a_-(x, y)/(1 + \gamma b_n(x, y))$ where $1/a_n \rightharpoonup 1/a_-$ for the

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weak $*$ topology and $b_n := a_-/a_n - 1$. Then the above problem corresponds to the case $\gamma = 1$. The solution u_n writes in series of powers of γ like $u_{n,\gamma}(x, y) = v_n^0(x, y) + v_n^1(x, y)\gamma + \dots + v_n^k(x, y)\gamma^k + \dots$ and the analysis would be based on analyticity properties in the parameter γ . The kernel describing the nonlocal effect is the weak limit of

$$K_{n,\gamma}(x, \xi, y) = \frac{b_n(\xi, y)}{1 + \gamma \int_0^1 b_n(x, y) dx} \left(\int_0^1 b_n(x, y) dx - b_n(x, y) \right). \tag{1.2}$$

One would like to characterize the nature of kernels that may appear as weak limits of $K_{n,\gamma}$ in (1.2), in order to understand the constitutive laws of the homogenized materials. This is still an open problem.

We inscribe our contribution in the effort to characterize such kernels by solving the symplified problem presented in the beginning.

In Section 2 we make a brief recall on notions to be employed: distribution measures, equimeasurability, rearrangements, the relation \prec , Young measures.

Section 3 is dedicated to our main result: we state it and give its proof by making use of auxiliary results contained in Section 4.

2. PRELIMINARY NOTIONS

To describe precisely the type of questions we are considering here, we begin by recalling a few facts about distribution measures and equimeasurable functions. Let Ω be a bounded domain in \mathbb{R}^N , denote by \mathcal{L}_N the Lebesgue measure on \mathbb{R}^N and by $\mathcal{B}(\Omega)$ the Borel σ -field of Ω . Let $f : \Omega \rightarrow \mathbb{R}_+$ be a nonnegative, measurable function; we represent by μ_f its *distribution measure* defined by

$$\mu_f(B) := \mathcal{L}_N(f^{-1}(B)), \quad \forall B \in \mathcal{B}(\mathbb{R}_+).$$

We denote by f^* the *decreasing rearrangement* of f given by:

$$f^*(s) := \sup\{t > 0 : \mu_f(]t, \infty[) > s\}.$$

It is easily checked that f^* is the unique (up to modifications on Lebesgue zero-measure sets) nonincreasing function on $[0, \mathcal{L}_N(\Omega)[$ such that f^* and f have the same distribution measure μ_f . Finally, we will say that $f, g \geq 0$ are equimeasurable or, equivalently, that f is a rearrangement of g if they have the same distribution measure. We will denote this equivalence relation by $f \sim g$.

Hardy *et al.* introduced (see [5]) the following preorder relation on the set of nonnegative functions in $L^1(\Omega)$: for f and g in $L^1(\Omega)$

$$g \prec f \text{ iff } \int_{\Omega} F(g(t)) dt \leq \int_{\Omega} F(f(t)) dt,$$

for all convex, continuous functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}$. Observe that $f \prec g$ and $g \prec f$ is equivalent to $f \sim g$. We recall from [5] the following property of the preorder relation between functions in $L^\infty(\Omega)$, in terms of decreasing rearrangements

$$g \prec f \text{ iff } \begin{cases} \int_0^t g^*(s) ds \leq \int_0^t f^*(s) ds \quad \forall t \in [0, \mathcal{L}_N(\Omega)[, \\ \int_0^{\mathcal{L}_N(\Omega)} g^*(s) ds = \int_0^{\mathcal{L}_N(\Omega)} f^*(s) ds. \end{cases} \tag{2.1}$$

Actually we shall only use functions which take values in a compact interval.

On the set of positive finite measures, the corresponding order relation was introduced by Choquet and, in a different framework, by Loomis (see [4] and [6]).

Definition 2.1. Two positive finite measures on $([\alpha, \beta], \mathcal{B}([\alpha, \beta]))$, ν and μ , having the same total mass ($\mu([\alpha, \beta]) = \nu([\alpha, \beta])$) satisfy $\mu \prec \nu$ if and only if

$$\int_{\alpha}^{\beta} \phi(z) d\mu(z) \leq \int_{\alpha}^{\beta} \phi(z) d\nu(z),$$

for all continuous and convex functions $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$.

Remark 2.2. Given two positive functions such that $g \prec f$, their distribution measures μ_g and μ_f satisfy $\mu_g \prec \mu_f$. Conversely, given two measures that satisfy $\mu \prec \nu$, if g and f have the distribution measures μ and ν respectively, then $g \prec f$.

The “old” tool of Young measures has proven crucial for applications in asymptotic analysis. Young measures techniques were developed by Tartar [9], Ball [3], Balder [2], Valadier [11], etc.

In the sequel we make a brief recall on Young measures following the notations and framework from [11].

We call *Young measure* any positive measure μ on $\Omega \times S$ (S is a metrizable space) whose projection on Ω is \mathcal{L}_N . Let $\mathcal{Y}(\Omega \times S)$ be the set of all Young measures on $\Omega \times S$.

We will not distinguish μ from its disintegration $(\mu_x)_{x \in \Omega}$ which is a measurable family of probabilities on S such that for any $\psi : \Omega \times S \rightarrow \mathbb{R}$, μ -integrable,

$$\int_{\Omega \times S} \psi d\mu = \int_{\Omega} \int_S \psi(x, \xi) d\mu_x(\xi) d\mathcal{L}_N(x).$$

For each measurable function $a : \Omega \rightarrow S$ we associate a Young measure μ_a , with support in the graph of a , defined by

$$\langle \mu_a, \phi \rangle = \int_{\Omega \times S} \phi(x, \lambda) d\mu_a(x, \lambda) = \int_{\Omega} \phi(x, a(x)) dx,$$

for all positive Carathéodory integrands $\phi : \Omega \times S \rightarrow \mathbb{R}$.

On $\mathcal{Y}(\Omega \times S)$ we consider the *narrow* topology, *i.e.*, the weakest topology that makes continuous the maps

$$\mu \mapsto \int_{\Omega \times S} \phi(x, \lambda) d\mu(x, \lambda),$$

for all bounded Carathéodory integrands ϕ .

Remark 2.3. 1) The set of Young measures associated to a sequence of functions uniformly bounded in $L^1(\Omega; \mathbb{R}^d)$ is relatively compact in $\mathcal{Y}(\Omega \times \mathbb{R}^d)$.

2) If S is a metrizable compact space, then every set of Young measures associated to a sequence of measurable functions $a_n : \Omega \rightarrow S$ is relatively compact in $\mathcal{Y}(\Omega \times S)$.

Remark 2.4. 1) Given a uniformly bounded sequence (a_n) in $L^1(\Omega; \mathbb{R}^d)$, using Remark 2.3 1), we may assume that, up to a subsequence of (a_n) , the sequence of their associated Young measures, narrow converges to some $\mu = (\mu_x)_{x \in \Omega} \in \mathcal{Y}(\Omega \times \mathbb{R}^d)$.

2) For a sequence of functions $a_n : \Omega \rightarrow S$ we say that μ is the Young measure associated to the sequence (a_n) , or, that (a_n) gives rise to the Young measure μ , if the Young measures associated to (a_n) narrow converge to μ , *i.e.*, for all bounded Carathéodory integrand ϕ ,

$$\int_{\Omega} \phi(x, a_n(x)) dx \rightarrow \int_{\Omega \times S} \phi(x, \lambda) d\mu(x, \lambda).$$

3. MAIN RESULT

Theorem 3.1 below was conjectured by Tartar in some discussions we had together, during his visits to Lisbon, in December 1998.

Consider two domains $\omega_1 \subset \mathbb{R}^N$ and $\omega_2 \subset \mathbb{R}^M$ and a compact interval $[\alpha, \beta]$ in \mathbb{R} . We may assume without losing generality that $|\omega_1| = |\omega_2| = 1$.

Theorem 3.1. *Given are the sequences $u_n : \omega_1 \times \omega_2 \rightarrow [\alpha, \beta]$ and $v_n : \omega_2 \rightarrow [\alpha, \beta]$ such that $v_n(y) = \int_{\omega_1} u_n(x, y) dx$. Assume that the sequence (u_n) gives rise to the Young measure $\nu = (\nu_{x,y})_{(x,y) \in \omega_1 \times \omega_2}$ and that the sequence (v_n) gives rise to the Young measure $\mu = (\mu_y)_{y \in \omega_2}$. One defines a function $f : \omega_1 \times \omega_2 \times (0, 1) \rightarrow [\alpha, \beta]$ such that $f(x, y, \cdot)$ is nonincreasing and its distribution measure is $\nu_{x,y}$, and similarly, a function $g : \omega_2 \times (0, 1) \rightarrow [\alpha, \beta]$ such that $g(y, \cdot)$ is nonincreasing and its distribution measure is μ_y . Then one has*

$$\int_0^t g(y, s) ds \leq \int_{\omega_1} \int_0^t f(x, y, s) ds dx, \tag{3.1}$$

with equality for $t = 1$.

Conversely, if f and g satisfy the above inequality for all $t \in [0, 1)$ and the corresponding equality for $t = 1$, then, denoting by $\nu_{x,y}$ and μ_y the distribution measures of $f(x, y, \cdot)$ and $g(y, \cdot)$, respectively, there exists a sequence $u_n : \omega_1 \times \omega_2 \rightarrow [\alpha, \beta]$ that gives rise to the Young measure ν and the sequence defined by $v_n := \int_{\omega_1} u_n(x, y) dx$ gives rise to the Young measure μ .

The author presented a particular case of the above result (the Young measure $\nu = (\nu_{x,y})_{(x,y) \in \omega_1 \times \omega_2}$ did not depend on x : $\nu_{x,y} = \nu_y$) at the Equadiff99 conference held in Berlin.

Proof. For the direct implication let $t \in [0, 1]$ arbitrarily fixed. Then by Lemma 4.5 there exists a subsequence v_{n_k} of v_n and a sequence of characteristic functions $\chi_k^v : \omega_2 \rightarrow \{0, 1\}$, $\chi_k^v \rightharpoonup t$ such that

$$\chi_k^v v_{n_k} \rightharpoonup \int_0^t g(y, s) ds.$$

On the other hand, applying Lemma 4.5 with the corresponding subsequence (u_{n_k}) and with the same number t , we obtain that for any sequence of characteristic functions $\chi_k : \omega_1 \times \omega_2 \rightarrow \{0, 1\}$ such that $\chi_k \rightarrow t$, up to a subsequence, we have that

$$\text{weak } \lim_k \chi_k u_{n_k} \leq \int_0^t f(x, y, s) ds, \tag{3.2}$$

and in particular the above inequality holds for χ_k^v . Then

$$\begin{aligned} \int_0^t g(y, s) ds &= \text{weak } \lim_k \chi_k^v v_{n_k} = \text{weak } \lim_k \chi_k^v \int_{\omega_1} u_{n_k}(x, y) dx \\ &= \text{weak } \lim_k \int_{\omega_1} \chi_k^v(y) u_{n_k}(x, y) dx \leq \int_{\omega_1} \int_0^t f(x, y, s) ds dx, \end{aligned}$$

and the direct implication turns out, for $t \in [0, 1[$. For $t = 1$, $\chi_k^v \rightarrow 1$ strongly and (3.2) holds with equality which yields equality in (3.1).

Conversely, suppose that f and g satisfy (3.1) for $t \in [0, 1)$ and the corresponding equality for $t = 1$. We construct a sequence u_n giving rise to the Young measure ν and such that the sequence $v_n := \int_{\omega_1} u_n(x, y) dx$ gives rise to the Young measure μ as follows: by property (2.1), (3.1) is equivalent to $g(y, \cdot) \prec \int_{\omega_1} f(x, y, \cdot) dx$. Applying Lemma 4.4 it turns out that there exists a positive measure θ on $[0, 1] \times [0, 1]$ whose both projections

are the Lebesgue measure and such that

$$g(y, t) = \int_0^1 \int_{\omega_1} f(x, y, s) dx d\theta_t(s).$$

Here θ_t is the disintegration of θ with respect to the Lebesgue measure. Define $u : \omega_1 \times \omega_2 \times [0, 1] \times [0, 1] \rightarrow [\alpha, \beta]$ by $u(x, y, s, t) := f(x, y, \Phi_t(s))$, where $\Phi_t : [0, 1] \rightarrow [0, 1]$ has the distribution measure θ_t . The existence of Φ_t is ensured by Remark 4.1. Then g writes

$$g(y, t) = \int_0^1 \int_{\omega_1} f(x, y, \Phi_t(s)) dx ds = \int_0^1 \int_{\omega_1} u(x, y, s, t) dx ds. \tag{3.3}$$

One takes $u_n(x, y) := u(x, y, [nx_1], [ny_1])$ (here $[z]$, with $z \in \mathbb{R}$, represents the fractional part of z and, by x_1 , we represent the first component of x). By Riemann-Lebesgue theorem it turns out that u_n gives rise to the Young measure ν . By a similar argument, the sequence w_n defined by $w_n(y) := g(y, [ny_1])$ gives rise to the Young measure μ . Having in mind the definition of the sequence $v_n(y) := \int_{\omega_1} u_n(x, y) dx$, it turns out that $v_n - w_n$ converges uniformly to 0, and since w_n gives rise to the Young measure μ , so does the sequence v_n , and the proof is complete. \square

4. AUXILIARY RESULTS

Remark 4.1. Given a finite positive measure μ on $[\alpha, \beta]$, $\mu : \mathcal{B}([\alpha, \beta]) \rightarrow \mathbb{R}_+$, there exists a measurable function $f : [0, \mu([\alpha, \beta])] \rightarrow [\alpha, \beta]$ that transports the Lebesgue measure on $[0, \mu([\alpha, \beta])]$ into μ , i.e. $\mu(B) = \mathcal{L}_1(f^{-1}(B))$ for all B in $\mathcal{B}([\alpha, \beta])$. We can take for instance the nonincreasing function that is given by

$$f(x) := \sup\{t \in [\alpha, \beta] : \mu([t, \beta]) > x\}.$$

The following two properties are to be used in the proof of Lemma 4.4.

Property 4.2. Consider a finite positive measure θ on $\Omega \times \Omega'$ with $\text{proj}_{\Omega'} \theta = \nu$. If the measure ν is absolutely continuous with respect to a positive measure l , $d\nu(y) = \tau(y)dl(y)$ for a function $\tau \in L^1_l(\Omega')$, then θ may be disintegrated with respect to l according to the formula

$$d\theta(x, y) = \theta_y^l(x)dl(y),$$

where $\theta_y^l(x) = \tau(y)\theta_y(x)$ and $(\theta_y)_{y \in \Omega'}$ is the disintegration of θ with respect to ν .

Let (Ω, m) and (Ω', l) be two measure spaces with the same total mass ($m(\Omega) = l(\Omega') < +\infty$), m and l being positive measures.

Property 4.3. If $f : \Omega \rightarrow \mathbb{R}$ and $g : \Omega' \rightarrow \mathbb{R}$ are integrable functions such that $f(x) < g(y)$ for all $x \in \Omega$ and $y \in \Omega'$, then $\int_{\Omega} f(x)dm(x) < \int_{\Omega'} g(y)dl(y)$.

Lemma 4.4. Given two measurable functions $u : \Omega \rightarrow [\alpha, \beta]$ and $v : \Omega' \rightarrow [\alpha, \beta]$ such that for all continuous and convex functions $\Phi : [\alpha, \beta] \rightarrow \mathbb{R}$, $\int_{\Omega'} \Phi(v(y))dl(y) \leq \int_{\Omega} \Phi(u(x))dm(x)$ holds. There exists then a positive measure θ , on $\Omega \times \Omega'$, whose projections on Ω and Ω' are m and l , respectively, and such that l -almost everywhere in Ω'

$$v(y) = \int_{\Omega} u(x)d\theta_y(x),$$

where $(\theta_y)_{y \in \Omega'}$ is the disintegration of θ with respect to l .

This result is a consequence of the results obtained by Ryll in [7] and [8] regarding equivalent characterizations, in terms of doubly stochastic operators, for the preorder relation \prec between functions. Lemma 4.4 is also very close to Cartier’s theorems stated in terms of dilatations (see Ths. 1 and 2 in [4] and Ths. 35 and 36 in [6], p. 288). We shall give a direct (sketched) proof.

The proof is to be done in 3 steps.

Step 1.

Denote by Θ the set of positive measures θ on $\Omega \times \Omega'$ such that the projections of θ on Ω and Ω' are m and l , respectively. Θ is a subset of the linear space of measures on $\Omega \times \Omega'$. The set Θ is compact with respect to the weak topology of measures.

Let us define for each $\theta \in \Theta$ the function $w(y) = \int_{\Omega} u(x)d\theta_y(x)$ where $(\theta_y)_{y \in \Omega'}$ is the disintegration of θ with respect to l . Note that the dependency of w on θ is linear. Let us introduce the functional $J : \Theta \rightarrow \mathbb{R}_+$,

$$J(\theta) = \int_{\Omega'} (v - w)^2 dl(y).$$

J depends on θ through w , and, due to the lower semicontinuity of the norm in $L^2_l(\Omega')$, it turns out that J is lower semicontinuous. Since Θ is compact, J attains its minimum, *i.e.* there exists $\theta^0 \in \Theta$ such that $J(\theta^0) = \inf_{\theta \in \Theta} J(\theta)$. Denote by $w_0(y) := \int_{\Omega} u(x)d\theta^0_y(x)$ the function corresponding to θ^0 . We shall prove that $w_0 = v$.

Step 2.

θ^0 has the following property: if A_1 and A_2 are subsets of Ω such that

$$u(x_1) < u(x_2) \text{ for all } x_1 \in A_1, x_2 \in A_2$$

and if B_1 and B_2 are subsets of Ω' such that

$$v(y_1) - w_0(y_1) < v(y_2) - w_0(y_2) \text{ for all } y_1 \in B_1, y_2 \in B_2,$$

then, either $\theta^0(A_1 \times B_2) = 0$ or $\theta^0(A_2 \times B_1) = 0$. Otherwise, if $\theta^0(A_1 \times B_2) > 0$ and $\theta^0(A_2 \times B_1) > 0$ one can increase θ^0 in the sets $A_1 \times B_1$ and $A_2 \times B_2$ and decrease θ^0 in the sets $A_1 \times B_2$ and $A_2 \times B_1$ by maintaining the same projections m and l on Ω and Ω' , respectively, and obtain a smaller value for J . Indeed, assuming that $\theta^0(A_1 \times B_2) > 0$ and $\theta^0(A_2 \times B_1) > 0$ we can choose two measures on $\Omega \times \Omega'$, θ^{12} and θ^{21} such that $\text{supp}\theta^{12} \subset A_1 \times B_2$, $\theta^{12} \leq \theta^0 \upharpoonright_{A_1 \times B_2}$, $\text{supp}\theta^{21} \subset A_2 \times B_1$, $\theta^{21} \leq \theta^0 \upharpoonright_{A_2 \times B_1}$ and $\theta^{12}(A_1 \times B_2) = \theta^{21}(A_2 \times B_1) = \gamma > 0$.

Denote $\mu_1 := \text{proj}_{\Omega} \theta^{12}$ ($\text{supp}\mu_1 \subset A_1$), $\nu_2 := \text{proj}_{\Omega'} \theta^{12}$ ($\text{supp}\nu_2 \subset B_2$), $\mu_2 := \text{proj}_{\Omega} \theta^{21}$ ($\text{supp}\mu_2 \subset A_2$), $\nu_1 := \text{proj}_{\Omega'} \theta^{21}$ ($\text{supp}\nu_1 \subset B_1$). Note that $\mu_1(A_1) = \nu_2(B_2) = \mu_2(A_2) = \nu_1(B_1) = \gamma$.

Let us define $\theta^{11} := \frac{1}{\gamma} \mu_1 \otimes \nu_1$, $\theta^{22} := \frac{1}{\gamma} \mu_2 \otimes \nu_2$ and note that $\text{supp}\theta^{11} \subset A_1 \times B_1$ and $\text{supp}\theta^{22} \subset A_2 \times B_2$.

Consider $\theta^t := \theta^0 + t(\theta^{11} - \theta^{12} - \theta^{21} + \theta^{22})$. For $t \in [0, 1]$ the measure θ^t is positive since $\theta^{12} \leq \theta^0$ and $\theta^{21} \leq \theta^0$ on the disjoint sets $A_1 \times B_2$ and, respectively, $A_2 \times B_1$. Calculating the projection of θ^t on Ω we have for every $A \in \mathcal{B}(\Omega)$ that $\theta^t(A \times \Omega') = \theta^0(A \times \Omega') + t(\theta^{11}(A \times \Omega') - \theta^{12}(A \times \Omega') - \theta^{21}(A \times \Omega') + \theta^{22}(A \times \Omega')) = m(A) + t\left(\frac{1}{\gamma} \mu_1(A) \nu_1(\Omega') - \theta^{12}(A \times B_2) - \theta^{21}(A \times B_1) + \frac{1}{\gamma} \mu_2(A) \nu_2(\Omega')\right) = m(A) + t\left(\frac{1}{\gamma} \mu_1(A) \nu_1(B_1) - \mu_1(A) - \mu_2(A) + \frac{1}{\gamma} \mu_2(A) \nu_2(B_2)\right) = m(A)$, that is $\text{proj}_{\Omega} \theta^t = m$ and by similar arguments we obtain also that $\text{proj}_{\Omega'} \theta^t = l$. Therefore $\theta^t \in \Theta$ for $t \in [0, 1]$.

Let us calculate $J(\theta^t)$ in order to evaluate the derivative $\frac{d}{dt} J(\theta^t) \big|_{t=0}$. Let us make first some remarks useful to the computations. $\theta^{21} \leq \theta^0$ on $A_2 \times B_1$, consequently, their projections satisfy $\nu_1 \leq l$ and therefore ν_1 is absolutely continuous with respect to l . According to Property 4.2 we can disintegrate θ^{11} and θ^{21} with respect to l . Similar arguments employed with the measure ν_2 permit us to conclude that θ^{12} and θ^{22} may be disintegrated with respect to l . In the following, by $(\bar{\theta}_y^{11})$, $(\bar{\theta}_y^{12})$, $(\bar{\theta}_y^{21})$ and $(\bar{\theta}_y^{22})$, we denote the disintegrations

with respect to l of θ^{11} , θ^{12} , θ^{21} and θ^{22} , respectively.

$$\begin{aligned} J(\theta^t) &= \int_{\Omega'} (v(y) - \int_{\Omega} u(x)d\theta_y^t(x))^2 dl(y) \\ &= \int_{\Omega'} \left(v(y) - w_0(y) - t \left(\int_{\Omega} u(x)d\bar{\theta}_y^{11}(x) - \int_{\Omega} u(x)d\bar{\theta}_y^{12}(x) \right. \right. \\ &\quad \left. \left. - \int_{\Omega} u(x)d\bar{\theta}_y^{21}(x) + \int_{\Omega} u(x)d\bar{\theta}_y^{22}(x) \right) \right)^2 dl(y). \end{aligned}$$

Since in the above expression the integrand is a polynomial of degree 2 in t , we obtain that

$$\begin{aligned} \frac{d}{dt}J(\theta^t) |_{t=0} &= 2 \int_{\Omega'} (v(y) - w_0(y)) \left(\int_{\Omega} u(x)d\bar{\theta}_y^{11}(x) - \int_{\Omega} u(x)d\bar{\theta}_y^{12}(x) \right. \\ &\quad \left. - \int_{\Omega} u(x)d\bar{\theta}_y^{21}(x) + \int_{\Omega} u(x)d\bar{\theta}_y^{22}(x) \right) dl(y) \\ &= 2 \left(\int_{\Omega \times \Omega'} u(x)\delta v(y)d\bar{\theta}^{11}(x, y) - \int_{\Omega \times \Omega'} u(x)\delta v(y)d\bar{\theta}^{12}(x, y) \right. \\ &\quad \left. - \int_{\Omega \times \Omega'} u(x)\delta v(y)d\bar{\theta}^{21}(x, y) + \int_{\Omega \times \Omega'} u(x)\delta v(y)d\bar{\theta}^{22}(x, y) \right), \end{aligned}$$

where $\delta v(y) := v(y) - w_0(y)$ and we had in mind the disintegration formulae for θ^{11} , θ^{12} , θ^{21} and θ^{22} with respect to l .

Let (θ_y^{11}) and (θ_y^{21}) be the disintegrations of θ^{11} and θ^{21} with respect to ν_1 , which is the projection of both on Ω' . Similarly, let (θ_y^{12}) and (θ_y^{22}) be the disintegrations of θ^{12} and θ^{22} with respect to ν_2 . Then

$$\begin{aligned} \frac{d}{dt}J(\theta^t) |_{t=0} &= 2 \int_{B_1} \delta v(y) \left(\int_{A_1} \frac{1}{\gamma} u(x)d\mu_1(x) - \int_{A_2} u(x)d\theta_y^{21}(x) \right) d\nu_1(y) \\ &\quad + 2 \int_{B_2} \delta v(y) \left(\int_{A_2} \frac{1}{\gamma} u(x)d\mu_2(x) - \int_{A_1} u(x)d\theta_y^{12}(x) \right) d\nu_2(y). \end{aligned}$$

Applying Property 4.3 in each point $y \in B_1$ with the function u on A_1 and A_2 , and the probability measures $\frac{1}{\gamma}\mu_1$ on A_1 and θ_y^{21} on A_2 , we obtain that $\alpha(y) := \int_{A_1} \frac{1}{\gamma} u(x)d\mu_1(x) - \int_{A_2} u(x)d\theta_y^{21}(x) < 0$. Similar arguments lead to $\beta(y) := \int_{A_2} \frac{1}{\gamma} u(x)d\mu_2(x) - \int_{A_1} u(x)d\theta_y^{12}(x) > 0$. With the above notations we have

$$\frac{d}{dt}J(\theta^t) |_{t=0} = 2 \left(\int_{B_1} \delta v(y)\alpha(y)d\nu_1(y) + \int_{B_2} \delta v(y)\beta(y)d\nu_2(y) \right). \tag{4.1}$$

Note that the measures $-\alpha(y)d\nu_1$ and $\beta(y)d\nu_2$ have the same mass. Indeed

$$\begin{aligned} - \int_{B_1} \alpha(y)d\nu_1(y) &= \int_{B_1} \int_{A_2} u(x)d\theta_y^{21}(x)d\nu_1(y) - \int_{B_1} \int_{A_1} \frac{1}{\gamma} u(x)d\mu_1(x)d\nu_1(y) \\ &= \int_{A_2 \times B_1} u(x)d\theta^{21}(x, y) - \int_{A_1 \times B_1} u(x)d\theta^{11}(x, y) = \int_{A_2} u(x)d\mu_2(x) - \int_{A_1} u(x)d\mu_1(x) \end{aligned}$$

and

$$\begin{aligned} \int_{B_2} \beta(y) d\nu_2(y) &= \int_{A_2 \times B_2} \frac{1}{\gamma} u(x) d\mu_2(x) d\nu_2(y) - \int_{A_1 \times B_2} u(x) d\theta^{12}(x, y) \\ &= \int_{A_2} u(x) d\mu_2(x) - \int_{A_1} u(x) d\mu_1(x). \end{aligned}$$

Now we can apply Property 4.3 with the function δv on B_1 and B_2 , and with the measures $-\alpha(y)d\nu_1$ on B_1 and $\beta(y)d\nu_2$ on B_2 , respectively, and from (4.1) we obtain that

$$\frac{d}{dt} J(\theta^t) |_{t=0} < 0$$

which contradicts the fact that θ^0 is a minimum for J .

Step 3.

Suppose that $v(y) - w_0(y) \neq 0$ on a set of positive l measure. We show then that there exists a real number r that satisfies:

$$\int_{\Omega'} (v(y) - r)_+ dl(y) > \int_{\Omega} (u(x) - r)_+ dm(x),$$

which is in contradiction with the hypothesis since $\Phi(\lambda) := (\lambda - r)_+$ is a continuous convex function (by f_+ we represent the positive part of the function f). The number r is constructed from the following considerations:

Define for all $p \in \mathbb{R}$ the set $C_p := \{x \in \Omega \mid u(x) > p\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}$. Note that if for some $p_1 \in \mathbb{R}$, $\theta^0(C_{p_1}) = 0$ then $\theta^0(C_p) = 0$ for every $p > p_1$, since $C_p \subset C_{p_1}$. Let $p_0 := \inf\{p \in \mathbb{R} \mid \theta^0(C_p) = 0\}$. Then $\theta^0(C_{p_0}) = 0$ since C_{p_0} may be written as $C_{p_0} := \bigcup_n C_{p_n}$ for some sequence $p_n \searrow p_0$ and having in mind that $\theta^0(C_{p_n}) = 0$ for each n . For all $p < p_0$, $\theta^0(C_p) > 0$ and by Step 2 it turns out that $\theta^0(\{x \in \Omega \mid u(x) \leq p\} \times \{y \in \Omega' \mid v(y) - w_0(y) \geq 0\}) = 0$. Taking a sequence $p_n \nearrow p_0$ we get by employing the same arguments as above, that $\theta^0(\{x \in \Omega \mid u(x) < p_0, v(y) - w_0(y) \geq 0\}) = 0$.

Analogously, define for all $q \in \mathbb{R}$ the set $D_q := \{x \in \Omega \mid u(x) < q\} \times \{y \in \Omega' \mid v(y) - w_0(y) > 0\}$ and note that if for some $q_1 \in \mathbb{R}$, $\theta^0(D_{q_1}) = 0$, then $\theta^0(D_q) = 0$ for every $q < q_1$ since $D_q \subset D_{q_1}$. Let $q_0 := \sup\{q \in \mathbb{R} \mid \theta^0(D_q) = 0\}$. By similar arguments to the ones used above with C_{p_0} , $\theta^0(D_{q_0}) = 0$. Moreover, using a sequence $q_n \searrow q_0$ we obtain also that $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}) = 0$.

Note that $p_0 \leq q_0$ otherwise, consider s such that $q_0 < s < p_0$. Then $\theta^0(C_s) > 0$ and $\theta^0(D_s) > 0$ that is, $\theta^0(\{x \in \Omega \mid u(x) > s\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}) > 0$ and $\theta^0(\{x \in \Omega \mid u(x) < s\} \times \{y \in \Omega' \mid v(y) - w_0(y) > 0\}) > 0$ which contradicts Step 2.

Consider now $r = q_0$ (any number between p_0 and q_0 may be taken). Let us evaluate

$$\int_{\Omega} (u(x) - q_0)_+ dm(x) = \int_{\Omega \times \Omega'} (u(x) - q_0)_+ d\theta^0(x, y) = \int_{\substack{\{x \mid u > q_0\} \times \\ \{y \mid v - w_0 > 0\}}} (u(x) - q_0) d\theta^0(x, y), \tag{4.2}$$

where for the last equality we had in mind the above deduced relation $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}) = 0$.

In order to evaluate $\int_{\Omega'} (v(y) - q_0)_+ dl(y)$ let us first make the following analysis:

In the set $\{y \in \Omega' \mid v(y) - w_0(y) < 0\}$, since $\theta^0(C_{p_0}) = 0$ i.e. $\theta^0(\{x \in \Omega \mid u(x) > p_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}) = 0$, and having in mind the definition $w_0 := \int_{\Omega} u(x) d\theta^0(x)$, it turns out that $w_0(y) \leq p_0$ and therefore $v(y) < w_0(y) \leq p_0 \leq q_0$.

In the set $\{y \in \Omega' \mid v(y) - w_0(y) = 0\}$, since $\theta^0(\{x \in \Omega \mid u(x) < p_0\} \times \{y \in \Omega' \mid v(y) = w_0(y)\}) = 0$ and $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) = w_0(y)\}) = 0$ as subsets of $\{x \in \Omega \mid u(x) < p_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \geq 0\}$ and respectively, $\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}$, we have that $w_0 \in [p_0, q_0]$ and in particular $v(y) \leq q_0$.

In the set $\{y \in \Omega' \mid v(y) - w_0(y) > 0\}$, since $\theta^0(D_{q_0}) = 0$ we obtain that $w_0(y) \geq q_0$ and hence $v(y) > q_0$.

So the only points where $(v - q_0)_+$ does not vanish are $\{y \in \Omega' \mid v(y) - w_0(y) > 0\}$ and in this set we have $v(y) > w_0(y) \geq q_0$.

Now we can evaluate $\int_{\Omega'} (v(y) - q_0)_+ dl(y)$ as follows, where the first inequality occurs applying Property 4.3 with the functions v and w_0 on Ω' and with the same measure l :

$$\begin{aligned} \int_{\Omega'} (v(y) - q_0)_+ dl(y) &= \int_{\{y \mid v - w_0 > 0\}} (v(y) - q_0) dl(y) > \int_{\{y \mid v - w_0 > 0\}} (w_0(y) - q_0) dl(y) \\ &= \int_{\{y \mid v - w_0 > 0\}} \left(\int_{\Omega} u(x) d\theta_y^0(x) - q_0 \right) dl(y) \\ &= \int_{\Omega \times \{y \mid v - w_0 > 0\}} (u(x) - q_0) d\theta^0(x, y) \geq \int_{\substack{\{x \mid u > q_0\} \times \\ \{y \mid v - w_0 > 0\}}} (u(x) - q_0) d\theta^0(x, y). \end{aligned} \tag{4.3}$$

Then (4.2) and (4.3) yield the contradiction. □

The following lemma is the most important ingredient in the proof of Theorem 3.1. Tartar suggested the statement below and the main idea of its proof as well.

The domain ω_2 in the sequel is as in Theorem 3.1.

Lemma 4.5. *Given a sequence $v_n : \omega_2 \rightarrow [\alpha, \beta]$, assume that it gives rise to the Young measure $\mu = (\mu_y)$. One defines a function $g : \omega_2 \times (0, 1) \rightarrow [\alpha, \beta]$ such that $g(y, \cdot)$ is nonincreasing and its distribution measure is μ_y . Then, given $\theta \in [0, 1]$, for all sequences $\chi_n : \omega_2 \rightarrow \{0, 1\}$ such that $\chi_n \rightharpoonup \theta$, up to a subsequence, we have that*

$$\text{weak } \lim_n \chi_n v_n \leq \int_0^\theta g(y, s) ds. \tag{4.4}$$

Moreover, there exists a sequence $\bar{\chi}_n : \omega_2 \rightarrow \{0, 1\}$, $\bar{\chi}_n \rightharpoonup \theta$ such that, up to a subsequence

$$\bar{\chi}_n v_n \rightharpoonup \int_0^\theta g(y, s) ds. \tag{4.5}$$

Proof. Consider a real number $\theta \in [0, 1]$ arbitrarily fixed and consider a sequence of characteristic functions $\chi_n \rightharpoonup \theta$. Denote by π the Young measure associated to the pair (v_n, χ_n) . So $\pi = (\pi_y)_{y \in \omega_2}$ is a measure on $[\alpha, \beta] \times \{0, 1\}$, where $\pi_y = \pi_y^1 \otimes \delta_{\chi=0} + \pi_y^2 \otimes \delta_{\chi=1}$. The projection of π_y on $\{0, 1\}$ is $(1 - \theta)\delta_0 + \theta\delta_1$ and the projection of π_y on $[\alpha, \beta]$ is μ_y . Therefore $\mu_y = \pi_y^1 + \pi_y^2$ and since π_y^1 and π_y^2 are positive measures, it turns out that π_y^2 is absolutely continuous with respect to μ_y . Then by Radon-Nikodym theorem we have that $d\pi_y^2(v) = \eta_y d\mu_y(v)$ where η_y is a positive function in $L^1_{\mu_y}([\alpha, \beta])$ such that $0 \leq \eta_y \leq 1$ μ_y -almost everywhere in $[\alpha, \beta]$.

The weak limit of χ_n is θ so $\pi_y^2([\alpha, \beta]) = \theta$. The weak limit of $\chi_n v_n$ is calculated as follows:

$$\chi_n v_n \rightharpoonup \int \chi v d\pi_y(\chi, v) = \int \chi v d(\pi_y^1 \otimes \delta_{\chi=0}) + \int \chi v d(\pi_y^2 \otimes \delta_{\chi=1}) = \int v d\pi_y^2(v) = \int_0^\theta h(y, s) ds, \tag{4.6}$$

where $h : \omega_2 \times (0, \theta) \rightarrow [\alpha, \beta]$ is a nonincreasing function whose distribution measure is π_y^2 . Consider h , for instance, given by

$$h(y, s) := \sup\{x \in [\alpha, \beta] : \pi_y^2([x, \beta]) > s\}.$$

Since for g one can take the following similar definition

$$g(y, s) := \sup\{x \in [\alpha, \beta] : \mu_y([x, \beta]) > s\},$$

having in mind that $\pi_y^2 \leq \mu_y$, one obtains that for all $y \in \omega_2$, $h(y, s) \leq g(y, s)$, for \mathcal{L}_1 almost every $s \in]0, \theta[$. Thus $\int_0^\theta h(y, s)ds \leq \int_0^\theta g(y, s)ds$ and then (4.6) implies (4.4).

We prove in the sequel that the equality is reached, up to a subsequence, that is, there exists a sequence $\bar{\chi}_k \rightarrow \theta$ such that $\bar{\chi}_k v_{n_k} \rightarrow \int_0^\theta g(y, s)ds$, for a subsequence $(v_{n_k})_k$ of $(v_n)_n$. Note that there exists $c \in]\alpha, \beta]$ such that $\mu_y(]c, \beta]) \leq \theta \leq \mu_y([c, \beta])$. Indeed, having in mind the definition of g , it is sufficient to take $c = g(y, \theta)$. Let us define

$$\bar{\eta}_y(v) = \begin{cases} 1, & \text{if } v > c, \\ \gamma, & \text{if } v = c, \\ 0, & \text{if } v < c, \end{cases}$$

where $\gamma = \frac{\theta - \mu_y(]c, \beta])}{\mu_y(\{c\})}$ if $\mu_y(\{c\}) \neq 0$ and γ may be any number between 0 and 1 if $\mu_y(\{c\}) = 0$. Then the following expression writes:

$$\int v \bar{\eta}_y(v) d\mu_y(v) = \int_{]c, \beta]} v d\mu_y(v) + c\gamma \mu_y(\{c\}).$$

Denote by I the first term and by II the second term in the above sum. Let us calculate the \mathcal{L}_1 -measure of the set $\{s : g(y, s) > c\}$:

$$\mathcal{L}_1(\{s : g(y, s) > c\}) = \int_0^1 \chi_{\{g(y, s) > c\}}(s) ds = \int_{[\alpha, \beta]} \chi_{\{v > c\}}(v) d\mu_y(v) = \mu_y(]c, \beta]).$$

Since $g(y, \cdot)$ is nonincreasing it turns out that $\{s : g(y, s) > c\}$ is the interval with extremities 0 and $\mu_y(]c, \beta])$. Hence for all $s \in]\mu_y(]c, \beta]), \theta]$, we have that $g(y, s) = c$ and then the first term I yields

$$I = \int_{]c, \beta]} v d\mu_y(v) = \int_0^1 g(y, s) \chi_{\{s: g(y, s) > c\}}(s) ds = \int_0^{\mu_y(]c, \beta])} g(y, s) ds.$$

If $\mu_y(\{c\}) \neq 0$, the second term gives:

$$II = c(\theta - \mu_y(]c, \beta])) = c \int_{\mu_y(]c, \beta])}^\theta ds = \int_{\mu_y(]c, \beta])}^\theta g(y, s) ds,$$

and consequently

$$I + II = \int_0^\theta g(y, s) ds.$$

If $\mu_y(\{c\}) = 0$ then $\mu_y(]c, \beta]) = \theta = \mu_y([c, \beta])$ and $II = 0$ while

$$I = \int_{]c, \beta]} v d\mu_y(v) = \int_0^{\mu_y(]c, \beta])} g(y, s) ds = \int_0^\theta g(y, s) ds.$$

It remains now to show that there exists a sequence $\bar{\chi}_n$ such that the pair $(v_n, \bar{\chi}_n)$ gives rise to the Young measure $\pi_y = \pi_y^1 \otimes \delta_0 + \pi_y^2 \otimes \delta_1$ with $d\pi_y^2 = \bar{\eta}_y d\mu_y$ and $d\pi_y^1 = (1 - \bar{\eta}_y) d\mu_y$. This existence is ensured by the following lemma applied with $K = \{0, 1\}$. □

Let Ω be a bounded open set in \mathbb{R}^N .

Lemma 4.6. *Given v_n a sequence $v_n : \Omega \rightarrow [\alpha, \beta]$ giving rise to the Young measure μ and given a family of probability measures $(\pi_{x,v})_{(x,v) \in \Omega \times [\alpha, \beta]}$ with support in \bar{K} (K is a bounded subset of \mathbb{R}^p), there exists a sequence*

$(\chi_n), \chi_n : \Omega \rightarrow K$ such that for all continuous functions $F : [\alpha, \beta] \times \overline{K} \rightarrow \mathbb{R}$ we have, up to a subsequence:

$$F(v_n, \chi_n) \rightarrow \int_{[\alpha, \beta] \times \overline{K}} F(v, u) d\pi_{x,v}(u) d\mu_x(v),$$

that is the sequence (v_n, χ_n) gives rise to the Young measure π .

The above result generalizes Theorem 5 in [9] (p. 147), and the proof uses analogous arguments.

Proof. Consider the following two sets:

$$M_1 = \{ \pi \text{ measure on } \Omega \times [\alpha, \beta] \times \overline{K} \text{ such that it is a narrow limit of Young measures associated to } (v_n, \chi) \text{ where } \chi : \Omega \rightarrow K \text{ is some measurable function} \}$$

and

$$M_2 = \{ \pi \text{ measure on } \Omega \times [\alpha, \beta] \times \overline{K}, \pi \geq 0, \text{supp}\pi \subset \Omega \times [\alpha, \beta] \times \overline{K}, \text{proj}_{\Omega \times [\alpha, \beta]} \pi = \mu \}.$$

We prove that equality $\overline{M}_1 = M_2$ holds.

First step. We prove that \overline{M}_1 is convex.

Consider $\pi_1, \pi_2, \dots, \pi_q \in M_1$. Then each π_i is the narrow limit of Young measures associated to pairs (v_n, χ_i) that is, for all bounded Carathéodory integrands ϕ ,

$$\int_{[\alpha, \beta] \times \overline{K}} \phi(x, v, \lambda) d\pi_i(v, \lambda) = \lim_m \phi(x, v_m(x), \chi_i(x)).$$

We shall show that $\pi := \sum_{i=1}^q \pi_i \theta_i$ belongs to \overline{M}_1 , where θ_i are real nonnegative numbers such that $\sum_{i=1}^q \theta_i = 1$. By Theorem 3 in [9], there exist the following sequences of characteristic functions $\psi_{i,n} : \Omega \rightarrow \{0, 1\}$ such that $\sum_{i=1}^q \psi_{i,n} = 1$ and $\psi_{i,n} \rightarrow \theta_i$. Consider $\chi_n := \sum_{i=1}^q \psi_{i,n}(x) \chi_i(x)$. Then $\phi(x, v_m(x), \chi_n(x)) = \sum_{i=1}^q \phi(x, v_m(x), \chi_i(x)) \psi_{i,n}(x)$ and passing to the limit first in m and then in n , one obtains that, for all bounded Carathéodory integrands ϕ ,

$$\lim_n \lim_m \phi(x, v_m(x), \chi_n(x)) = \int_{[\alpha, \beta] \times \overline{K}} \phi(x, v, \lambda) d\pi(v, \lambda)$$

and consequently $\pi \in \overline{M}_1$.

Second step. We prove that $\overline{\text{conv}}(\overline{M}_1) = M_2$. We know that in order to find the closed convex hull of M_1 we need only to consider the affine continuous functions which are positive on M_1 (by Hahn-Banach theorem). But an affine continuous function on the space of measures on $\Omega \times [\alpha, \beta] \times \overline{K}$ has the following form

$$\pi \rightarrow \langle \pi, \Phi_0(x, v, \lambda) \rangle + \delta$$

where Φ_0 is continuous bounded and $\delta \in \mathbb{R}$. Such affine continuous function is positive on M_1 if

$$\langle \pi, \Phi_0(x, v, \lambda) \rangle + \delta \geq 0$$

for all $\pi \in M_1$. This is equivalent to

$$\lim_m \int_{\Omega} \Phi_0(x, v_m(x), \chi(x)) dx + \delta \geq 0 \tag{4.7}$$

for all $\chi : \Omega \rightarrow K$. Define $\Psi_0(x, v) := \inf_{\lambda \in \overline{K}} \{\Phi_0(x, v, \lambda)\}$. Then Ψ_0 is continuous bounded and therefore (4.7) is equivalent to

$$\lim_m \int_{\Omega} \Psi_0(x, v_m(x)) dx + \delta \geq 0.$$

Let $\Phi_0(x, v, \lambda) = \Psi_0(x, v) + \chi_0(x, v, \lambda)$, with $\chi_0(x, v, \lambda) \geq 0$. So, we have that

$$\overline{\text{conv}}(\overline{M}_1) = \{\pi \text{ measure that satisfies } \langle \pi, \Phi_0 \rangle + \delta \geq 0 \text{ for all } \Phi_0 \text{ and } \delta \text{ satisfying} \\ \Phi_0 = \Psi_0 + \chi_0, \text{ for some function } \chi_0 \geq 0, \text{ and } \langle \Psi_0, \mu \rangle + \delta \geq 0\}.$$

Now it remains to prove that the above conditions characterize M_2 .

- 1) Take $\Psi_0 = 0$ and $\delta = 0$. Then $\langle \pi, \chi_0 \rangle \geq 0$ for all $\chi_0 \geq 0$. Therefore $\pi \geq 0$.
- 2) If $\chi_0 = 0$ in $\Omega \times [\alpha, \beta] \times \overline{K}$ then $\chi_0 \geq 0$ and $-\chi_0 \geq 0$. Then $\langle \pi, \chi_0 \rangle \geq 0$ and $\langle \pi, -\chi_0 \rangle \geq 0$ and hence $\langle \pi, \chi_0 \rangle = 0$, therefore $\text{supp } \pi \subset \Omega \times [\alpha, \beta] \times \overline{K}$.
- 3) Consider $\delta = -\langle \Psi_0, \mu \rangle$ and $\chi_0 = 0$. Then $\langle \pi, \Psi_0 \rangle + \delta \geq 0$ for all continuous bounded function Ψ_0 such that $\delta = -\langle \Psi_0, \mu \rangle$. So, $\langle \pi, \Psi_0 \rangle \geq \langle \mu, \Psi_0 \rangle$. On the other hand $\langle \pi, -\Psi_0 \rangle - \delta \geq 0$. Then $\langle \pi, \Psi_0 \rangle \geq \langle \mu, \Psi_0 \rangle$. Hence $\langle \pi, \Phi_0 \rangle = \langle \mu, \Psi_0 \rangle$, and consequently $\text{proj}_{\Omega \times [\alpha, \beta]} \pi = \mu$.

From 1), 2) and 3) above it turns out that $\overline{\text{conv}}(\overline{M}_1) \subset M_2$.

Conversely, let us prove that $M_2 \subset \overline{\text{conv}}(\overline{M}_1)$. Consider $\pi \in M_2$. Then $\langle \pi, \Phi_0 \rangle + \delta \geq 0$ since $\langle \pi, \Phi_0 \rangle = \langle \pi, \chi_0 \rangle + \langle \pi, \Psi_0 \rangle$, $\langle \pi, \chi_0 \rangle \geq 0$ and $\langle \pi, \Psi_0 \rangle = \langle \mu, \Psi_0 \rangle \geq 0$. Hence $\pi \in \overline{\text{conv}}(\overline{M}_1)$.

Thus we obtain the existence of a sequence $\bar{\chi}_m : \Omega \rightarrow K$ that satisfies

$$\text{weak } \lim_m \text{ weak } \lim_n \int_{\Omega} \phi(x, v_n(x), \bar{\chi}_m(x)) dx = \int_{[\alpha, \beta] \times \overline{K}} \phi(x, v, \lambda) d\pi_x(v, \lambda),$$

for all bounded Carathéodory integrands ϕ . One can extract then a diagonal subsequence n_k such that:

$$\text{weak } \lim_k \int_{\Omega} \phi(x, v_{n_k}(x), \bar{\chi}_{n_k}(x)) dx = \int_{[\alpha, \beta] \times \overline{K}} \phi(x, v, \lambda) d\pi_x(v, \lambda).$$

□

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