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GERMS OF QUASI-CONTINUOUS FUNCTIONS

by YUH-CHING CHEN

Introduction.

The notion of quasi-topological spaces was first introduced by Kowalsky [12] under the German name «Limesräume». Since then it has been applied to various branches of mathematics such as differential Geometry [1], [8], functional analysis [1], [2], [4], [6], [7], theory of differentiations [1], [3], [10], [14], [15], and algebraic topology [16]. It was Bastiani [1] who first applied this notion to differentiable manifolds and introduced the French term -quasi-topologie- which is not related to the quasi-topology defined by Spanier [18]. Since this work is inspired by some works of Ehresmann's school [8], [14], [15], [16], the term *quasi-topology* here is a translation of the French «quasi-topologie».

In this paper, we try to generalize the notions of germs of functions and sheaves in topological sense to that of π-germs of functions and π-sheaves in quasi-topological sense and to study the relations between these notions. We begin with a brief review of some basic definitions and properties on quasi-topologies and the introduction of the notion of germs and π-germs of functions using inductive limits. Then we generalize the notions of pre-sheaves and sheaves over a topological space to that of π-presheaves and π-sheaves over a quasi-topological space and show that every π-sheaf E is reflected by the sheaf of germs of quasi-continuous local sections of E. In fact the category of π-sheaves over a quasi-topological space (X, π) contains a reflective full subcategory isomorphic to the category of abelian sheaves over the underlying topological space (X, T_π) of (X, π). Finally, we show that the canonical injective structure of this reflective subcategory determines an effacement structure [13] on the category of π-sheaves. The homological Algebra of this effacement structure appears more complicated than the relative homological Algebra of Eilenberg-Moore [9] and Maranda [17]. We shall defer this pending further investigations.

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1. Germs and π-germs of functions.

π will always stand for a quasi-topology on a set X. It is a function that associates to each $x \in X$ a family $\pi(x)$ of filters of subsets of X satisfying the conditions:

(1) $F_1, F_2 \in \pi(x)$ implies $F_1 \cap F_2 \in \pi(x)$,

(2) $F_1 \in \pi(x)$ and $F_2 \supset F_1$ implies $F_2 \in \pi(x)$,

(3) the filter x^ϵ of all subsets of X containing x is in $\pi(x)$.

If $F \in \pi(x)$, we say that F *converges to* x in π. The pair (X, π) is called a *quasi-topological space.*

Let (X, π) and (E, τ) be quasi-topological spaces. A function $f:(X,\pi) \to (E,\tau)$, often written $f: X \to E$, is (π,τ)*-continuous* (called *quasi-continuous* in [1], [15]) if, for every $x \in X$ and every $F \in \pi(x)$, the images of the sets in F under f generate a filter $f(F) \in \tau(f(x))$. f is *quasi-open* if for every $x \in X$ and every $G \in \tau(f(x))$, there is $F \in \pi(x)$ with $f(F) \subset G$. If A is a subset of X, the quasi-topology induced on A by π is denoted $\pi | A$ and we say that $(A, \pi | A)$ is a (quasi-topological) *subspace* of (X, π).

A topology T on X is identified with the quasi-topology π_T on X in which the filter of neighborhoods of $x \in X$ is the smallest filter in $\pi_T(x)$. The category $\mathfrak{T}$ of topological spaces and continuous functions is identified with a full subcategory of the category $\mathfrak{Q}\mathfrak{T}$ of quasi-topological spaces and quasi-continuous functions (see e.g. [15]).

The *underlying topology* T_π of π is defined as follows: A set $U \subset X$ is open (in T_π) if and only if, for every $x \in U$, $F \in \pi(x)$ implies

$U \in F$. Thus every (π, τ)-continuous function $f: X \to E$ is continuous in the underlying topologies T_π and T_τ. Note that $1:(X, \pi) \to (X, T_\pi)$ is (π, T_π)-continuous. In fact, the underlying topology functor $T: \mathfrak{Q}\mathfrak{T} \to \mathfrak{T}$ is left adjoint to the inclusion functor $\mathfrak{T} \subset \mathfrak{Q}\mathfrak{T}$, i.e., $\mathfrak{T}$ is identified with a coreflective subcategory of $\mathfrak{Q}\mathfrak{T}$ (in French term, T is a projector functor). For further definitions and properties concerning quasi-topologies we refer the readers to [1], [16].

Convention. Let A be a subset of X. A (π, τ)-continuous function $f: A \to E$ is the restriction to A of a $(\pi | U, \tau)$-continuous function from an open neighborhood U of A to E. Thus if α is the directed set (by inclusion) of open neighborhoods of A and if $\mathfrak{Q}\mathfrak{T}(U, E)$ denotes the set of (π, τ)-continuous functions from $U \in \alpha$ to E, then the *restriction map*

$$(1.1) \qquad r: \lim_{\to \alpha} \mathfrak{Q}\mathfrak{T}(U, E) \to \mathfrak{Q}\mathfrak{T}(A, E)$$

is a surjection, where $\{\mathfrak{Q}\mathfrak{T}(U, E) \,|\, U \in \alpha\}$ forms a direct system of sets of (π, τ)-continuous functions with genuine restriction maps.

Let $Q(x) = \bigcap \{F_i \,|\, F_i \in \pi(x)\}$ be the filter that is the intersecsection of all filters F_i in $\pi(x)$. Then each $A_x \in Q(x)$ is the union of a family of subsets of X one from each filter $F_i \in \pi(x)$. A set $A_x \in Q(x)$ is called a *π-neighborhood* of x (thus every neighborhood is a π-neighborhood). Notice that: (1) each $A_x \in Q(x)$ contains x, but $Q(x)$ may not converge to x in π, and (2) A_x may not be a π-neighborhood of another point $y \in A_x$.

We proceed now to define germs and π-germs of functions using inductive limits. Let $O(x)$ be the set of open neighborhoods of x (open in the underlying topology T_π of π). Then $O(x) \subset Q(x)$. Order both sets $O(x)$ and $Q(x)$ by inclusion. Then $O(x)$ is a directed subset of $Q(x)$. The inductive limit

$$\lim_{\to} \mathfrak{Q}\mathfrak{T}(A, E),\ A \in Q(x) \ \text{(resp. } \lim_{\to} \mathfrak{Q}\mathfrak{T}(U, E),\ U \in O(x)),$$

is the set of *π-germs* (resp. *germs*) *of (π, τ)-continuous functions at x*. Each $f: A \to E$ in $\mathfrak{Q}\mathfrak{T}(A, E)$ (resp. $f: U \to E$ in $\mathfrak{Q}\mathfrak{T}(U, E)$) determines a π-germ (resp. germ) f_x of a (π, τ)-continuous function at x. Often, we simply call f_x the *limit of* f *at* x. Since $O(x) \subset Q(x)$ as directed sets, there is a map

$$(1.2) \quad \gamma_x : \varinjlim \mathfrak{Q}\mathfrak{T}(U, E) \to \varinjlim \mathfrak{Q}\mathfrak{T}(A, E)$$

that associates to each germ f_x at x a π-germ $f'_x = \gamma_x(f_x)$ at x. It follows from (1.1) that γ_x is surjective. If $O(x)$ is cofinal in $Q(x)$, then the notions of germs and π-germs coincide. In particular, this is the case when π is topological.

2. π-sheaves.

A map $p: E \to X$ of quasi-topological spaces (E, τ) and (X, π) defines a *π-sheaf* E if the following conditions are satisfied (see [5]):

(S1) For every point $f_x \in E$ with $p(f_x) = x$, there exists a subset $U_f \subset E$ containing f_x such that the map $p \mid U_f$ is a (τ, π)-homeomorphism of $(U_f, \tau \mid U_f)$ onto an open neighborhood U_x of x;

(S2) τ is the final quasi-topology determined by all $\tau \mid U_f$ via inclusion maps;

(S3) For every $x \in X$, the *stalk* $E_x = p^{-1}(x)$ is an abelian group and the group operations are quasi-continuous in τ.

In particular, if all U_f in (S1) can be choosen open in the underlying topology T_τ of τ, then we say that p *spreads* E over X and E is a *π-spreading space*. It is easy to see that:

PROPOSITION 2.1. *If E is a π-sheaf, then p is (τ, π)-continuous and quasi-open* (cf. proposition 1.2.16 of [15]).

COROLLARY 2.2. *If (E, τ) is a π-spreading space, then (E, T_τ) is an abelian sheaf over the topological space (X, T_π).* (We assume that the readers are familiar with the general theory of abelian sheaves).

Let $p: (E, \tau) \to (X, \pi)$ be a π-sheaf. A *section of* E *over a subset* A of X is a function $s: A \to E$ which is the restriction to A of a section s' of E over an open neigboorhood U of A (i.e. $s': U \to E$ is a $(\pi \mid U, \tau)$-continuous function such that ps' is the identity of U); in particular, s is (π, τ)-continuous on A. The set $\Gamma(A, E)$ of sections of E

over A is an abelian group (the addition is pointwise). It follows that the restriction map

$$(2.1) \qquad r : \lim_{\to \alpha} \Gamma(U, E) \to \Gamma(A, E)$$

is an epimorphism for every π-neighborhood A of $x \in X$. This will be referred to as property (PS) in the definition of π-presheaves in the next section.

A map $\phi : E \to F$ of π-sheaves (E, τ) and (F, σ) is called a *π-homomorphism* if: (1) ϕ is (τ, σ)-continuous, and (2) for every $x \in X$ the map $\phi_x = \phi \mid E_x$ is a group homomorphism of E_x into F_x. We write $\phi = \{ \phi_x \mid x \in X \}$. The classes of π-sheaves and π-homomorphisms form a category $\mathfrak{L}_\pi$ called the *category of π-sheaves.*

PROPOSITION 2.3. *The class of π-spreading spaces form a full subcategory $\mathfrak{L}_X$ of $\mathfrak{L}_\pi$. If $\mathfrak{L}_T$ denotes the category of abelian sheaves over the topological space (X, T_π), then the underlying topology functor $T : \mathfrak{Q}\mathfrak{T} \to \mathfrak{T}$ induces an isomorphism $T_X : \mathfrak{L}_X \to \mathfrak{L}_T$ of categories.*

Indeed, T_X sends a π-spreading space $p : (E, \tau) \to (X, \pi)$ to an abelian sheaf $p : (E, T_\tau) \to (X, T_\pi)$. The inverse of T_X is defined as follows. Let $\pi : (E, \mathfrak{U}) \to (X, T_\pi)$ be an abelian sheaf, where $\mathfrak{U}$ is a topology on E. Then, by definition, for every point $f_x \in E$ with $p(f_x) = x$, there is a $U_f \in \mathfrak{U}$ such that $p \mid U_f$ is a homeomorphism of U_f onto an open neighborhood U_x of x. Endow each U_f with a quasi-topology τ_f that makes $p \mid U_f$ a (τ, π)-homeomorphism and let τ be the final quasi-topology on E determined by all τ_f via inclusion maps. Then $p : (E, \tau) \to (X, \pi)$ is a π-spreading space. T_X^{-1} carries $p : (E, \mathfrak{U}) \to (X, T_\pi)$ to $p : (E, \tau) \to (X, \pi)$.

3. Construction of π-sheaves.

Let Q_π be the category whose class of objects is the set

$$\{\emptyset\} \cup \bigcup_{x \in X} Q(x) = \{ A_x \in Q(X) \mid x \in X \} \cup \{\emptyset\}$$

of π-neighborhoods of points of X and whose morphisms are inclusion maps, and let Ab be the category of abelian groups and homomorphisms. A π-presheave is a contravariant functor $P : Q_\pi \to Ab$ satisfying the condition:

(PS) For every $A \in Q(x)$, the restriction map $r : \lim_{\to\alpha} P(U) \to P(A)$ is an epimorphism, where α is the set of all open neighborhoods U of A directed by inclusion. A *homomorphism* of π-presheaves is a natural transformation of functors. π-presheaves and their homomorphisms form a category $\mathcal{P}_\pi$ of functors.

A typical example of a π-presheaf is the π-presheaf ΓE of local sections of a π-sheaf E defined as follows. For every inclusion map $i : B \to A$ in Q_π, the map $(\Gamma E)(i) : \Gamma(A, E) \to \Gamma(B, E)$ is the restriction map of sections of E over A to that of E over B. The property (PS) is verified by (2.1). In fact, there is a functor $\Gamma : \mathcal{L}_\pi \to \mathcal{P}_\pi$ called a *local section functor.*

Let P be a π-presheaf. We shall construct the *associated π-sheaf* SP of P as follows. For every $x \in X$ let

$$(3.1) \qquad (SP)_x = \lim_{\to} P(A_x), \quad A_x \in Q(x)$$

be the set of limits f_x of $f \in P(A_x)$ at x, and let

$$(3.2) \qquad SP = \bigcup_{x \in X} (SP)_x .$$

We shall endow SP with a quasi-topology τ so that the projection $p : SP \to X$ defined by $p(f_x) = x$ is a π-sheaf: For each open set U of X and for each $f \in P(U)$ let

$$(3.3) \qquad U_f = \{ f_x \in (SP)_x \mid x \in U \text{ and } f_x = \textit{limit of } f \textit{ at } x \}$$

be the set of points of SP which are the limits of f at points of U. Endow each U_f with a quasi-topology that makes $p \mid U_f : U_f \to U$ a (τ_f, π)-homeomorphism. Then we have

$$SP = \bigcup \{ U_f \mid U \text{ open in } X,\ f \in P(U) \},$$

and τ_f and τ_g agree on $U_f \cap V_g$ for any two sets U_f and V_g defined by (3.3). τ is the final quasi-topology on SP determined by all τ_f via inclusion maps. Then

PROPOSITION 3.1. *$p : (SP, \tau) \to (X, \pi)$ is a π-sheaf called the associated π-sheaf of the π-presheaf P.*

In practice, most of π-sheaves are constructed in this way from π-

presheaves of π-germs of functions satisfying some prescribed properties such as quasi-continuous, quasi-holomorphic [15], etc... In fact the introduction of the notion of π-sheaves is motivated by this sort of examples. We should point out that in the construction above the sets U_f are not open in T_τ in general. Since the limit f_x of $f \in P(U)$ is taken on the directed set $Q(x)$, not on $O(x)$, there may exist f, $g \in P(U)$ with $U_f \cap U_g$ not open in U_f or U_g.

Finally, we shall see that there is a functor $S: \mathcal{P}_\pi \to \mathcal{L}_\pi$ that sends a homomorphism $\rho: P \to P'$ of π-presheaves to a π-homomorphism $\phi: SP \to SP'$ defined as follows. Since ρ is a natural transformation of functors, it consists of a family of group homomorphisms $\rho_A: P(A) \to P'(A)$ indexed by the objects A of Q_π. For a point $f_x \in SP$, that is the limit of $f \in P(A)$ at x, let $\phi(f_x)$ be the limit of $\rho_A(f) \in P'(A)$ at x, i.e.

$$(3.4) \qquad \phi(f_x) = g_x, \quad \text{where } g = \rho_A(f) \in P'(A).$$

It is obvious that ϕ is a π-homomorphism and that S is a functor.

4. The functors S and Γ.

THEOREM 4.1. *The functor $S: \mathcal{P}_\pi \to \mathcal{L}_\pi$ is left adjoint to the functor $\Gamma: \mathcal{L}_\pi \to \mathcal{P}_\pi$.*

The proof will follow two lemmas.

LEMMA 1. *There is a natural transformation from the composite functor $S\Gamma$ of S and Γ to the identity functor of $\mathcal{L}_\pi$.*

PROOF. Let E be a π-sheaf. Then for every $x \in X$,

$$(4.1) \qquad (S\Gamma E)_x = \varinjlim\, (\Gamma E)(A) = \varinjlim\, \Gamma(A, E), \quad A \in Q(x).$$

That is, $(S\Gamma E)_x$ is the group of π-germs at x of local sections of E; a point in $(S\Gamma E)_x$ is the π-germ s_x at x represented by a section $s \in \Gamma(A, E)$. Let $\theta(s_x) = s(x)$. Then $\theta: S\Gamma E \to E$ so defined is a π-homomorphism. The class of θ (indexed by the objects E of $\mathcal{L}_\pi$) form a natural transformation from $S\Gamma$ to the identity functor of $\mathcal{L}_\pi$. Moreover, it is easily shown that:

COROLLARY. *The quasi-topology of E is the final quasi-topology deter-*

mined by that of $S\Gamma E$ *via* θ, *i.e.*, θ *is a* π-*epimorphism.*

LEMMA 2. *There is a natural transformation from the identity functor of* $\mathcal{P}_\pi$ *to the composite functor* ΓS *of* Γ *and* S.

PROOF. Let P be a π-presheaf. Then $(\Gamma SP)(A)=\Gamma(A,SP)$ for every $A\in|Q_\pi|$. Define a map $h_A:P(A)\to\Gamma(A,SP)$ by $h_A(f)=s$ with $s(x)=f_x$ for every $x\in A$. This is well defined since, for every $f\in P(A)$, the family $\{f_x\,|\,x\in A\}$ do define a section s of SP over A. It is easily checked that h_A is a homomorphism and that

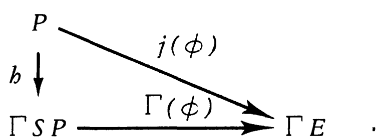
$$(4.2)\qquad h=\{h_A:P(A)\to(\Gamma SP)(A)\,|\,A\in|Q_\pi|\}$$

is a π-homomorphism from P to ΓSP. The family of such h (indexed by the objects P of $\mathcal{P}_\pi$) form a natural transformation from the identity functor of $\mathcal{P}_\pi$ to ΓS.

PROOF OF THEOREM 4.1. We want to show that there is a natural equivalence

$$(4.2)\qquad j:\mathcal{L}_\pi(SP,E)\to\mathcal{P}_\pi(P,\Gamma E)$$

of the set of π-homomorphisms from SP to E to the set of homomorphisms from P to ΓE. For any $\phi\in\mathcal{L}_\pi(SP,E)$, define $j(\phi)=\Gamma(\phi)h$ as in the diagram



Then $j(\phi)$ consists of a set of homomorphisms $j(\phi)_A:P(A)\to(\Gamma E)(A)$ defined by

$$(4.4)\qquad j(\phi)_A(f)=(\Gamma(\phi)_A h_A)(f)=\Gamma(\phi)_A(s)=\phi s,$$

where $s\in\Gamma(A,E)$ is the section $s(x)=f_x$. We claim that j is a bijection with inverse k defined by $k(\rho)=\theta S(\rho)$ for every $\rho\in\mathcal{P}_\pi(P,\Gamma E)$. Indeed,

$$(4.5)\qquad (jk(\rho))_A(f)(x)=\theta S(\rho)(s(x))=\theta S(\rho)(f_x)=$$
$$\theta(\rho_A(f))_x=\rho_A(f)(x)$$

for every $f \in P(A)$ shows that $jk(\rho)=\rho$. On the other hand,

$$(4.6) \qquad kj(\phi)(f_x)=\theta S(j(\phi))(f_x)=\theta(j(\phi)_A(f))_x = (j(\phi)_A(f))(x)=\phi(f_x)$$

shows that $kj(\phi)=\phi$. Since j and k are defined by functors and natural transformations, the bijection j is natural.

REMARK. (1) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are additive categories and j is indeed a natural isomorphism of groups.

(2) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are not abelian categories. For example, $\mathcal{P}_\pi$ is not closed under the formation of kernels since the property (PS) which is defined by colimits is not preserved by kernels.

5. The subcategory $\mathcal{L}_X$ of $\mathcal{L}_\pi$.

Recall that T_π is the underlying topology of π. Regard T_π as a category with morphisms inclusion maps; then it is a full subcategory of Q_π. Let $\mathcal{P}_X$ be the category of presheaves over (X, T_π), i.e., the category of contravariant functors from T_π to Ab. Then there is a functor $R': \mathcal{P}_\pi \to \mathcal{P}_X$ defined by $R'(P)=P\,|\,T_\pi$. On the other hand, we define a functor $J': \mathcal{P}_X \to \mathcal{P}_\pi$ as follows. For every presheaf $G: T_\pi \to Ab$, let $J'G$ be a mapping on Q_π to Ab with

$$(5.1) \qquad (J'G)(A)=\varinjlim_{\alpha} G(U), \quad \forall A \in |Q_\pi|,$$

where α is the set of open neighborhoods of A directed by inclusion. Then $J'G$ verifies the property (PS) and thus defines a π-presheaf. By a routine limit argument in category theory, one shows that the correspondence $G \to J'G$ defines a functor J' from $\mathcal{P}_X$ to $\mathcal{P}_\pi$ and that

PROPOSITION 5.1. *J' is left adjoint to R'. Moreover, the composite functor $R'J'$ of R' and J' is naturally equivalent to the identity functor of $\mathcal{P}_X$ (and therefore J' is a full embedding).*

Recall that $\mathcal{L}_X$ is a full subcategory of $\mathcal{L}_\pi$ (see proposition 2.3). $\Gamma\,|\,\mathcal{L}_X$ defines a functor Γ' from $\mathcal{L}_X$ to $\mathcal{P}_X$ that can be identified with the composite functor $R'\Gamma J$, where J is the inclusion functor of $\mathcal{L}_X$ in $\mathcal{L}_\pi$. On the other hand, a functor $S': \mathcal{P}_X \to \mathcal{L}_X$ with

(5.2) $\qquad S'G = \bigcup_{x \in X} (S'G)_x$, where $(S'G)_x = \varinjlim G(U)$, $U \in O(x)$,

can be defined by replacing Q_π by T_π in the construction of S in section 3. Notice that here the limit f_x of f at x is taken on $O(x)$ instead of $Q(x)$; contrary to the remark of section 3, $U_f \cap V_g$ is always open in τ_f and τ_g. Therefore, the quasi-topology τ on $S'G$ is the only quasi-topology that renders each U_f an open subset of $S'G$ (cf. [16], p.28). In fact, the family of all subsets U_f form a basis for the underlying topology T_τ on $S'G$. Similar to theorem 4.1 we have

PROPOSITION 5.2. *S' is left adjoint to Γ'. Moreover, the composite functor $S'\Gamma'$ is naturally equivalent to the identity functor of $\mathcal{L}_X$.*

If $\mathcal{L}_X$ is identified with the category $\mathcal{L}_T$ of abelian sheaves over (X, T_π) by the functor T_X of proposition 2.3, then the functors S' and Γ' are identified with the associated sheaf functor and the local section functor, respectively, of the theory of sheaves.

Like $\mathcal{L}_T$, $\mathcal{L}_X$ is an abelian category with enough injectives; it is AB5 (see [11]). The injective structure on $\mathcal{L}_X$ is called *the canonical injective structure on* $\mathcal{L}_X$.

6. Germs of local sections of a π-sheaf.

In the diagram

$$
\begin{array}{ccc}
\mathcal{L}_X & \underset{R}{\overset{J}{\rightleftarrows}} & \mathcal{L}_\pi \\
\Gamma' \downarrow \uparrow S' & & \Gamma \downarrow \uparrow S \\
\mathcal{P}_X & \underset{J'}{\overset{R'}{\leftrightarrows}} & \mathcal{P}_\pi
\end{array}
$$

of categories and functors, let $R = S'R'\Gamma$. Then for any π-sheaf E,

(6.1) $\qquad (RE)_x = (S'R'\Gamma E)_x = \varinjlim \Gamma(U, E)$, $U \in O(x)$,

since $(R'\Gamma E)(U) = \Gamma(U, E)$ for every U in T_π. Thus $RE = \bigcup_{x \in X} (RE)_x$ is the π-sheaf (indeed a sheaf) of germs of local sections of E. Since

$$(6.2) \qquad S\Gamma E = \bigcup_{x\in X} \varinjlim \Gamma(A,E),\ A\in Q(x)$$

is the π-sheaf of π-germs of local sections of E, formula (1.2) and property (PS) show that there is a surjective π-homomorphism $\zeta : RE \to S\Gamma E$ defined by the set of group epimorphisms:

$$(6.3) \qquad \zeta_x : \varinjlim \Gamma(U,E) \to \varinjlim \Gamma(A,E),\ x\in X.$$

We claim that the quasi-topology σ' on $S\Gamma E$ is the final quasi-topology determined by the quasi-topology σ on RE via ζ and therefore ζ is a π-epimorphism. Indeed, since σ (resp. σ') is the final quasi-topology determined by the family $\sigma_f = \sigma \mid U_f$ (resp. $\sigma'_f = \sigma' \mid U_f$) via inclusion maps $U_f \subset RE$ (resp. $U_f \subset S\Gamma E$), every $p \mid U_f$ is a (σ,π)-homeomorphism (resp. (σ',π)-homeomorphism) of U_f onto U. Now, let (Y,σ'') be a quasi-topological space and let $\phi : S\Gamma E \to Y$ be a map such that $\phi\zeta$ is (σ,σ'')-continuous. Then, since each $\phi \mid U_f = (\phi\zeta \mid U_f)(\zeta \mid U_f)^{-1}$ is (σ',σ'')-continuous, so is ϕ (cf. [16]). This shows that σ' is the final quasi-topology determined by σ via ζ.

Recall (corollary of lemma 1 of section 4) that $\theta : S\Gamma E \to E$ is a π-epimorphism; so is

$$(6.4) \qquad \psi : RE \to E,\ \psi = \theta\zeta.$$

We identify $\mathfrak{L}_X$ with $\mathfrak{L}_T$ and see that every π-sheaf E is a quotient of the sheaf of germs of local sections of E. More generally, we shall prove that $\mathfrak{L}_X$ is a reflective subcategory of $\mathfrak{L}_\pi$ and that ψ of (6.4) is a reflection. Thus every π-sheaf is *reflected* by the sheaf of germs of its local sections.

THEOREM 6.1. *$R : \mathfrak{L}_\pi \to \mathfrak{L}_X$ is right adjoint to the inclusion functor $J : \mathfrak{L}_X \to \mathfrak{L}_\pi$, i.e., R is a reflector.*

PROOF. We want to show that for E' in $\mathfrak{L}_X$ and E in $\mathfrak{L}_\pi$, there is a natural bijection

$$(6.5) \qquad \mathfrak{L}_\pi(JE',E) \to \mathfrak{L}_X(E',RE).$$

We observe that $J = SJ'\Gamma'$. Indeed, since

$$(J'\Gamma'E')(A) = \varinjlim_{\alpha} \Gamma(U,E'), \qquad \forall A\in |Q_\pi|,$$

we have

$$(SJ'\Gamma'E')_x = \varinjlim (J'\Gamma'E')(A) = \varinjlim \Gamma(U, E') = E'_x .$$

Therefore $SJ'\Gamma'E' = E'$. Now, by theorem 4.1 and proposition 5.1,

$\mathfrak{L}_\pi(JE', E) = \mathfrak{L}_\pi(SJ'\Gamma'E', E) \approx \mathfrak{P}_\pi(J'\Gamma'E', \Gamma E) \approx \mathfrak{P}_X(\Gamma'E', R'\Gamma E)$.

Since $S' \mid \Gamma'(\mathfrak{L}_X)$ is a full embedding, proposition 5.2 shows that

$$\mathfrak{P}_X(\Gamma'E', R'\Gamma E) \approx \mathfrak{L}_X(S'\Gamma'E', S'R'\Gamma E) \approx \mathfrak{L}_X(E', RE).$$

Thus, $\mathfrak{L}_\pi(JE', E)$ is naturally isomorphic to $\mathfrak{L}_X(E', RE)$.

COROLLARY 6.2. *$R \mid \mathfrak{L}_X$ is the identity functor of $\mathfrak{L}_X$.*

7. An effacement structure on $\mathfrak{L}_\pi$.

Let the canonical injective structure of $\mathfrak{L}_X$ be denoted $(\mathfrak{M}, \varepsilon)$ and identify $\mathfrak{L}_X$ with $\mathfrak{L}_T$. Then $\mathfrak{M}$ is the class of sheaf monomorphisms and ε is the class of injective sheaves [11]. $(\mathfrak{M}, \varepsilon)$ induces an effacement structure $(\mathfrak{F}', \mathfrak{H}')$ on $\mathfrak{L}_X$, where $\mathfrak{F}' = \mathfrak{M}$ and $\mathfrak{H}'$ is the class of π-homomorphisms that factor through an injective object of $\mathfrak{L}_X$ (see proposition 2.13 of [13]).

The notion of an *effacement structure* on a category C was first defined by Zimmermann [19] under the German name «Erweiterungspaare». It consists of a pair $(\mathfrak{F}, \mathfrak{H})$ of two classes of morphisms of C satisfying the following three conditions:

(1) $\mathfrak{H}$ is the class of all morphisms $h: A \to B$ such that, for every $f: X \to Y$ in $\mathfrak{F}$ and for every given $u: X \to A$ in C, there is a morphism $v: Y \to B$ such that $vf = hu$;

(2) $\mathfrak{F}$ is the class of all morphisms $f: X \to Y$ such that, for every $h: A \to B$ in $\mathfrak{H}$ and for every given $u: X \to A$ in C, there is a morphism $v: Y \to B$ such that $vf = hu$;

(3) for every object A in C, there is a morphism $f \in \mathfrak{F} \cap \mathfrak{H}$ with domain A.

We need also the following definition from [13]. $\mathfrak{F}' \subset \mathfrak{F}$ is a *subbasis* for $\mathfrak{F}$ if $\mathfrak{F}$ is the class of morphisms $f: X \to Y$ such that there exists a push-out

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Z \end{array}$$

with $f' \in \mathfrak{F}'$ and a morphism $k: Y \to Z$ with $kf = \phi$.

Let $(\mathfrak{F}', \mathfrak{H}')$ be the effacement structure on $\mathfrak{L}_X$ induced by $(\mathfrak{M}, \varepsilon)$ as mentioned in the first paragraph. We have

THEOREM 7.1. *There is an effacement structure* $(\mathfrak{F}, \mathfrak{H})$ *on* $\mathfrak{L}_\pi$ *in which* $\mathfrak{H} = R^{-1}(\mathfrak{H}')$ *and* $\mathfrak{F}$ *has* $\mathfrak{F}'$ *as a subbasis.*

In view of theorem 4.6 of [13], $(\mathfrak{F}, \mathfrak{H})$ is the inverse transfer of $(\mathfrak{F}', \mathfrak{H}')$ by the pair of adjoint functors J and R provided that push-outs exist in $\mathfrak{L}_\pi$.

Given π-homomorphisms $\phi: E \to F$ and $\psi: E \to G$, we shall construct the push-out K of ϕ by ψ. For every $x \in X$, let K_x be the push-out of $\phi_x: E_x \to F_x$ by $\psi_x: E_x \to G_x$ (since push-outs exist in Ab) and let $K = \bigcup_{x \in X} K_x$. Then we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle i} \\ G & \xrightarrow{j} & K \end{array}$$

Endow K with the final quasi-topology τ determined by the quasi-topologies of F and G via i and j. Then (K, τ) is a π-sheaf which is the push-out of ϕ by ψ.

Finally, we shall generalize the notion of injective resolutions in sheaf theory to that of π-injective resolutions. Recall that for every sheaf E' (e.g. $E' = RE$ of a π-sheaf E) there is an injective resolution

$$0 \to E' \to Q'_*$$

defined as follows: Choose a map $i': E' \to Q'_0$ in $\mathfrak{F}' \cap \mathfrak{H}'$ (i.e. i' is a monomorphism with Q'_0 injective) and let G_0 be the cokernel of i'. Then we obtain an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{j'} G_0 \longrightarrow 0$$

Choose a map $k' : G_0 \to Q'_1$ in $\mathfrak{F}' \cap \mathfrak{H}'$ and let G_1 be the cokernel of k'. Then an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \longrightarrow G_1 \longrightarrow 0, \quad d'_0 = k'j',$$

is obtained. Continuing in this way, we get an injective resolution

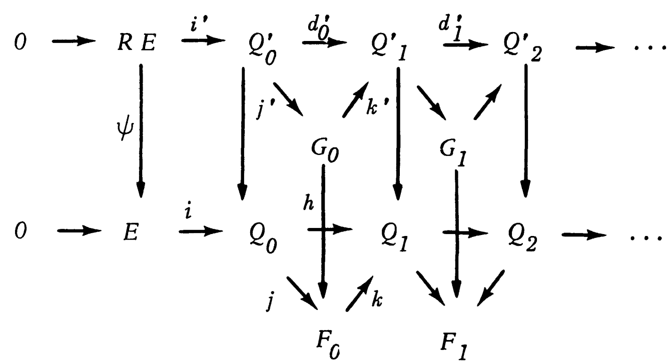
$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \xrightarrow{d'_1} Q'_2 \longrightarrow \cdots$$

We shall generalize this construction to one for a π-sheaf E. Let Q_0 be the push-out of $i' : RE \to Q'_0$ by the reflection $\psi : RE \to E$

$$\begin{array}{ccc} RE & \xrightarrow{\quad i' \quad} & Q'_0 \\ \psi \downarrow & & \downarrow \psi_0 \\ E & \xrightarrow{\quad i \quad} & Q_0 \end{array} .$$

Then it is easy to show that, since $i' \in \mathfrak{F}' \cap \mathfrak{H}'$, $i \in \mathfrak{F} \cap \mathfrak{H}$. Moreover, i is a π-monomorphism. In the diagram

(7.1)



let F_0 be the cokernel of i, Q_1 be the push-out of k' by h, F_1 be the cokernel of $d_0 = kj$ and Q_2 be a push-out again. Then by repeating this construction we obtain an exact sequence

$$(7.2) \qquad 0 \xrightarrow{i} E \xrightarrow{d_0} Q_0 \xrightarrow{d_1} Q_1 \longrightarrow Q_2 \longrightarrow \cdots$$

called a *π-injective resolution* of E in $\mathfrak{L}_\pi$. We remark that, in general, neither Q_i is injective in $\mathfrak{L}_\pi$, nor is RQ_i in $\mathfrak{L}_X$. We call (7.2) a «π-injective» resolution just because it is constructed out of an *injective* resolution of RE. Nevertheless, when $Q'*$ is replaced by another injective resolution of RE, the corresponding π-injective resolution of E is chain homotopic to (7.2). Therefore, there is a cohomology theory on $\mathfrak{L}_\pi$ defined by resolutions that generalizes the sheaf cohomology. We shall study this further under a more general topic.

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