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Localization and cohomology of nilpotent groups

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LOCALIZATION AND COHOMOLOGY OF NILPOTENT GROUPS*

by Peter HILTON

1. Introduction.

The theory of the P-localization of groups, where P is a family of primes, appears to have been first discussed by Malcev and Lazard [9, 10]. In their work emphasis was placed on the explicit construction of the localization and properties of the localization G_P of the nilpotent group G were deduced from the construction, utilizing nilpotent group theory. Baumslag [1] has given a comprehensive treatment of the main properties of nilpotent groups as they relate to the problem of localization, and has himself [2] explicitly shown how to construct G_P in the case of an arbitrary nilpotent group G and an arbitrary family of primes P, thus extending the generality of Malcev's original construction. Bousfield-Kan [3] exploit this general Malcev construction in their study of completion and localization. There is a very readable account of the theory of nilpotent groups and a description of the localization construction in the notes of Warfield [11].

In this paper we adopt a completely different approach. Starting with the completely elementary theory of localization of abelian groups, we show that a localizing functor $L : N_c \to N_c$ may be built up inductively with respect to c, where N_c is the category of nilpotent groups of nilpotency class $\leqslant c$. Our tool in demonstrating this is the cohomology theory of groups and, in particular, the interpretation of $H^2(Q;A)$, where A is an abelian group (trivial Q-module), as the collection of equivalence classes of central extensions of Q by the abelian central kernel A. We incorporate into the inductive hypothesis the key fact that the natural (localizing) homomorphism $e : G \to G_P$ (where $G_P = LG$) is a P-isomorphism, meaning that $ker\, e$ consists of elements of torsion prime to P and that, for any $y \in G_P$, there exists n prime to P with $y^n \in im\, e$.

★ Conférence donnée au Colloque d'Amiens.

In our approach we do not have to construct G_P explicitly; we simply prove that it exists and that the functor L and the natural transformation e have certain important properties. By using this approach we are able to deduce various properties of nilpotent groups, some of which make no explicit mention of localization.

The plan of the paper is as follows. In Section 2 we document the facts we need about the localization of abelian groups — essentially to start the induction. In Section 3 we describe the (co)homology theory of groups, especially with regard to the second cohomology group, and we recall the Lyndon-Hochschild-Serre spectral sequence. In Section 4 we prove the main results, showing that the localization exists and has properties which we demand of it; as indicated, a key property is already built into the inductive hypothesis. In Section 5 we make applications to the theory of nilpotent groups; we emphasize that no sophisticated theory has gone into these applications; we simply exploit the fact that localization exists and has some nice properties.

Among the facts which immediately emerge is that if $\phi : G \to H$ is a P-isomorphism of nilpotent groups, then $\phi_P : G_P \to H_P$ is an isomorphism. To prove the converse of this appears to require deeper properties of nilpotent groups than those exploited in Sections 2-5 — indeed, in those sections we use nothing deeper than a group of nilpotency class c is naturally expressible as a central extension of a group of nilpotency class $c - 1$. However, in Section 6, we refer to Hall's theory of basic commutators [5] in order to prove a lemma which turns out to be crucial in establishing the converse. Once established, the converse then enables us to develop a theory of P-isomorphisms of nilpotent groups which parallels standard results on nilpotent groups.

In Section 7 we prove various results on the localization of nilpotent groups which are particularly relevant to the study of nilpotent spaces [6,7].

2. Localization of abelian groups.

In this section we collect together the results we will need on the P-localization $e : A \to A_P$ of an abelian group, with respect to a family of primes P. We recall that $A_P = A \otimes \mathbf{Z}_P$, where $\mathbf{Z}_P$ is the ring of integers localized at P, that is, the subring of the rationals $\mathbf{Q}$, consisting of those rationals k / l such that l is prime to P; and e is the natural homomorphism $a \to a \otimes 1$. Note that a P-local group is just a $\mathbf{Z}_P$-module.

PROPOSITION 2.1. *Localization is exact.*

PROPOSITION 2.2. *Localization commutes with direct limits.*

PROPOSITION 2.3. *Given*

$$
\begin{array}{ccccc}
A' & \rightarrowtail & A & \twoheadrightarrow & A'' \\
\downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} \\
B' & \rightarrowtail & B & \twoheadrightarrow & B''
\end{array}
$$

with exact rows, then if any two of ϕ', ϕ, ϕ'' P-localize, so does the third.

PROPOSITION 2.4.

$$e \otimes 1 : A \otimes B \to A_P \otimes B, \qquad e \otimes e : A \otimes B \to A_P \otimes B_P ,$$

$$Tor(e, 1) : Tor(A, B) \to Tor(A_P, B),$$

$$Tor(e, e) : Tor(A, B) \to Tor(A_P, B_P)$$

all P-localize.

PROOF. The first two assertions are obvious; we will be content to prove the third. Let $R \rightarrowtail F \twoheadrightarrow A$ be a free abelian presentation of A. Then $R_P \rightarrowtail F_P \twoheadrightarrow A_P$ is exact with R_P, F_P flat. Thus we have

$$
\begin{array}{ccccccc}
Tor(A, B) & \rightarrowtail & R \otimes B & \rightarrow & F \otimes B & \twoheadrightarrow & A \otimes B \\
\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
Tor(A_P, B) & \rightarrowtail & R_P \otimes B & \rightarrow & F_P \otimes B & \twoheadrightarrow & A_P \otimes B
\end{array}
$$

Two applications of Proposition 2.3 now show that $Tor(e, 1)$ P-localizes.

PROPOSITION 2.5. $\widetilde{H}_*(e) : \widetilde{H}_*(A) \to \widetilde{H}_*(A_P)$ P-localizes, where $\widetilde{H}_*$ is

reduced homology with integer coefficients.

PROOF. Let $\mathcal{F}$ be the family of abelian groups for which the assertion holds. First, we claim that $\mathbf{Z}/p^k \in \mathcal{F}$. For if $A = \mathbf{Z}/p^k$, $p \in P$, then $A_P = A$, $e = 1$, and $\widetilde{H}_*(A)$ is P-local, so the assertion is clear. If $A = \mathbf{Z}/p^k$, $p \in P'$, the complementary set of primes, then $A_P = 0$, $\widetilde{H}_*(A)$ is P'-torsion, and the assertion is again clear.

Second, we claim that $\mathbf{Z} \in \mathcal{F}$. For $\mathbf{Z}_P$ is torsion-free of rank 1, so that

$$H_n(\mathbf{Z}_P) = \mathbf{Z}_P, \quad n = 1, \quad H_n(\mathbf{Z}_P) = 0, \quad n \geqslant 2.$$

Thus $\widetilde{H}_*(e) = H_1(e)$ and $H_1(e)$ plainly P-localizes.

We next argue that, if A, $B \in \mathcal{F}$, so does $A \oplus B$. This follows immediately from the natural Künneth formula for $H_*(A \oplus B)$, together with Propositions 2.3 and 2.4. Thus we have proved that if A is finitely generated, then $A \in \mathcal{F}$.

Finally, we use a direct limit argument to show that every $A \in \mathcal{F}$. For A is the (directed) union of its finitely generated subgroups, H_* commutes with direct limits, and Proposition 2.2 then enables us to complete the argument.

DEFINITION. We say that $\phi: A \to B$ is a P-*isomorphism* if $\ker\phi$, $\operatorname{coker}\phi$ are both P'-torsion groups.

PROPOSITION 2.6. $\phi: A \to B$ P-*localizes if and only if* B *is* P-*local and* ϕ *is a* P-*isomorphism.*

PROOF. We first show that $e: A \to A_P$ is a P-isomorphism. We may embed e in the exact sequence

$$Tor(A, \mathbf{Z}_P/\mathbf{Z}) \rightarrowtail A \xrightarrow{\ e\ } A_P \longrightarrow A \otimes \mathbf{Z}_P/\mathbf{Z}.$$

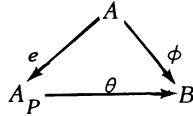Thus we must show that $Tor(A, \mathbf{Z}_P/\mathbf{Z})$ and $A \otimes \mathbf{Z}_P/\mathbf{Z}$ are P'-torsion groups. Now

$$\mathbf{Z}_P/\mathbf{Z} = \bigoplus_{p \in P'} \mathbf{Z}/p^\infty,$$

where $\mathbf{Z}/p^\infty$ is the p-Prüfer group. It is thus plain that $\mathbf{Z}_P/\mathbf{Z}$ is a P'-torsion group, from which it readily follows that $A \otimes \mathbf{Z}_P/\mathbf{Z}$ and

$Tor(A, \mathbf{Z}_P / \mathbf{Z})$ are P'-torsion groups.

Conservely suppose that B is P-local and $\phi: A \to B$ is a P-isomorphism. We thus have a commutative diagram

$$
\begin{array}{ccc}
 & A & \\
e \swarrow & & \searrow \phi \\
A_P & \xrightarrow{\quad \theta \quad} & B
\end{array}
$$

and the proof of Proposition 2.6 is completed by means of the following two lemmas.

LEMMA 2.7. *If* $\beta\alpha$ *and* α *are* P-*isomorphisms, so is* β.

LEMMA 2.8. *If* $\theta: C \to D$ *is a* P-*isomorphism and* C, D *are* P-*local, then* θ *is an isomorphism.*

PROOF of 2.7. It is trivial that if $coker\,\beta\alpha$ is P'-torsion, so is $coker\,\beta$. From the exact sequence

$$ker\,\alpha \rightarrowtail ker\,\beta\alpha \longrightarrow ker\,\beta \longrightarrow coker\,\alpha$$

we deduce that if $ker\,\beta\alpha$ and $coker\,\alpha$ are P'-torsion, so is $ker\,\beta$.

PROOF of 2.8. It is trivial that if $ker\,\theta$ is P'-torsion and C is P-local, then $ker\,\theta = 0$. Now suppose $coker\,\theta$ is P'-torsion and let $d \in D$. Then $nd = \theta c$ for some $c \in C$, $n \in P'$. Since C is P-local, $c = nc_1$, $c_1 \in C$, so $n(d - \theta c_1) = 0$. Since D is P-local, $d - \theta c_1 = 0$, so $coker\,\theta = 0$.

PROPOSITION 2.9. *If* B *is* P-*local, then* $e^*: Hom(A_P, B) \cong Hom(A, B)$, $e^*: Ext(A_P, B) \cong Ext(A, B)$.

PROOF. The first isomorphism simply expresses the universal property of e. As to the second, let $B \rightarrowtail I \twoheadrightarrow J$ be an injective presentation of B as $\mathbf{Z}_P$-module. Since $\mathbf{Z}_P$ is flat, it follows that $B \rightarrowtail I \twoheadrightarrow J$ is also an injective presentation of B as abelian group. Thus we have a commutative diagram

$$
\begin{array}{ccc}
Hom(A_P, I) \longrightarrow Hom(A_P, J) \longrightarrow Ext(A_P, B) \\
\downarrow e^* \qquad\qquad \downarrow e^* \qquad\qquad \downarrow e^* \\
Hom(A, I) \longrightarrow Hom(A, J) \longrightarrow Ext(A, B).
\end{array}
$$

Since the first two vertical arrows are isomorphisms, so is the third.

3. Cohomology and central extensions.

In this section we recall the results we will need on the homology and cohomology of groups. We will be concerned exclusively with cohomology groups $H^k(Q;A)$, where A is a *trivial* Q-module.

PROPOSITION 3.1. *There is a natural universal coefficient sequence*

$$Ext(H_{k-1}Q, A) \rightarrowtail H^k(Q;A) \longrightarrow Hom(H_k Q, A).$$

We now study the interpretation of $H^2(Q;A)$ as the group of equivalence classes of central extensions of the quotient group Q by the abelian kernel A. Let $\xi \in H^2(Q;A)$ be represented by

(3.1) $$A \xrightarrowtail{\mu} G \xrightarrow{\varepsilon} Q,$$

and let $\phi : A \rightarrow A_1$. Form the product $A_1 \times G$ and embed A by $^{(*)}$
$a \rightarrow (-\phi a, \mu a)$. Then A is central in $A_1 \times G$ and we may form the quotient group G_1; write an element of G_1 as $\{a_1, g\}$, the coset containing (a_1, g). There are evident maps $\mu_1 : A_1 \rightarrow G_1$, $\psi : G \rightarrow G_1$, given by $\mu_1(a_1) = \{a_1, 1\}, \psi(g) = \{0, g\}$ and

(3.2)
$$\begin{array}{ccc} A & \xrightarrow{\ \mu\ } & G \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A_1 & \xrightarrow{\ \mu_1\ } & G_1 \end{array}$$

commutes. Moreover

PROPOSITION 3.2. *In (3.2), μ_1 maps A_1 injectively to a central subgroup of G_1 and $coker\,\mu_1 = Q$.*

PROOF. It is trivial that μ_1 is injective and that $\mu_1 A_1$ is central in G_1. Let $\varepsilon_1 : G_1 \rightarrow Q$ be given by $\varepsilon_1 \{a_1, g\} = \varepsilon g$. It is plain that ε_1 is well-defined and surjective. Moreover $\varepsilon_1 \mu_1 = 0$, and, if $\varepsilon_1 \{a_1, g\} = 1$, then $g = \mu a$, so

$$\{a_1, g\} = \{a_1, \mu a\} = \{a_1 + \phi a, 1\} = \mu_1(a_1 + \phi a).$$

Thus (3.2) enlarges to a map of central extensions

(*) Note that A is written additively, while G and Q are written multiplicatively.

(3.3)

$$\xi:\quad A \xrightarrow{\ \mu\ } G \xrightarrow{\quad \varepsilon \quad} Q$$

$$\xi_1:\quad A_1 \xrightarrow{\ \mu_1\ } G_1 \xrightarrow{\quad \varepsilon_1 \quad} Q$$

with vertical maps ϕ, ψ, $\|$

and we know $[8]$:

PROPOSITION 3.3. *Let* $\phi: A \to A_1$ *induce* $\phi_*: H^2(Q;A) \to H^2(Q;A_1)$. *Then* $\phi_*(\xi) = \xi_1$.

Now let $\rho: Q' \to Q$. Form the pull-back of ρ and ε (3.1); it is then known that we obtain a map of central extensions

(3.4)

$$\xi':\quad A \xrightarrow{\ \mu''\ } G' \xrightarrow{\quad \varepsilon' \quad} Q'$$

$$\xi:\quad A \xrightarrow{\ \mu\ } G \xrightarrow{\quad \varepsilon \quad} Q$$

with vertical maps $\|$, σ, ρ

and we know $[8]$:

PROPOSITION 3.4. *Let* $\rho: Q' \to Q$ *induce* $\rho^*: H^2(Q;A) \to H^2(Q';A)$. *Then* $\rho^*(\xi) = \xi'$. *Moreover*

(3.5) $$\phi_* \rho^* = \rho^* \phi_*.$$

PROPOSITION 3.5. *Suppose given two central extensions*

$$\xi_1: A_1 \xrightarrow{\ \mu_1\ } G_1 \xrightarrow{\quad \varepsilon_1 \quad} Q_1, \qquad \xi_2: A_2 \xrightarrow{\ \mu_2\ } G_2 \xrightarrow{\quad \varepsilon_2 \quad} Q_2,$$

and maps $\phi: A_1 \to A_2$, $\rho: Q_1 \to Q_2$. *Then one may find* $\tau: G_1 \to G_2$ *yielding a commutative diagram*

(3.6)

$$\xi_1:\quad A_1 \xrightarrow{\ \mu_1\ } G_1 \xrightarrow{\quad \varepsilon_1 \quad} Q_1$$

$$\xi_2:\quad A_2 \xrightarrow{\ \mu_2\ } G_2 \xrightarrow{\quad \varepsilon_2 \quad} Q_2$$

with vertical maps ϕ, τ, ρ

if and only if $\phi_*(\xi_1) = \rho^*(\xi_2)$. *Moreover, if* τ *exists, then* τ' *also yields a commutative diagram (3.6) if and only if*

(3.7) $\tau'(x) = \tau(x)\mu_2 \kappa \varepsilon_1(x)$, $x \in G_1$, *for some* $\kappa: Q_1 \to A_2$.

PROOF. The key observation is that *any* diagram of the type (3.3), with common quotient group, must, in principle, have been constructed as (3.3)

was constructed, so that the existence of the diagram implies that $\phi_*(\xi) = \xi_1$; and, similarly, any diagram of the type (3.4), with common kernel, must, in principle, have been constructed as (3.4) was constructed, so that the existence of the diagram implies that $\rho^*(\xi) = \xi'$.

Now suppose that τ exists in (3.6). Pulling back by means of ρ, we find a commutative diagram

$$
\begin{array}{ccccccc}
\xi_1: & A_1 & \rightarrowtail & G_1 & \longrightarrow & Q_1 \\
 & \downarrow & & \phi \downarrow & & \downarrow \tau_1 & \| \\
\rho^*(\xi_2): & A_2 & \rightarrowtail & G & \longrightarrow & Q_1 \\
 & \downarrow & & \| & & \downarrow \tau_2 & \downarrow \rho \\
\xi_2: & A_2 & \rightarrowtail & G_2 & \longrightarrow & Q_2
\end{array}
$$

with $\tau_2 \tau_1 = \tau$. Thus, by the observation above, $\rho^*(\xi_2) = \phi_*(\xi_1)$.

Conversely, suppose that $\rho^*(\xi_2) = \phi_*(\xi_1)$. Then we have a diagram

$$
\begin{array}{ccccccc}
\xi_1: & A_1 & \rightarrowtail & G_1 & \longrightarrow & Q_1 \\
 & \downarrow & & \phi \downarrow & & \downarrow \tau_1 & \| \\
\phi_*(\xi_1): & A_2 & \rightarrowtail & G & \longrightarrow & Q_1 \\
 & \downarrow & & \| & & \downarrow \omega & \| \\
\rho^*(\xi_2): & A_2 & \rightarrowtail & G' & \longrightarrow & Q_1 \\
 & \downarrow & & \| & & \downarrow \tau_2 & \downarrow \rho \\
\xi_2: & A_2 & \rightarrowtail & G_2 & \longrightarrow & Q_2
\end{array}
$$

and we may take $\tau = \tau_2 \omega \tau_1$.

The final statement of the proposition is almost immediate. Since $\varepsilon_2 \tau = \varepsilon_2 \tau'$, we find a function $\theta : G_1 \to A_2$, given by

$$
\mu_2 \theta(x) = \tau(x)^{-1} \tau'(x), \quad x \in G_1.
$$

Moreover, θ is a homomorphism since ξ_2 is a central extension. Since

$$
\tau \mu_1 = \tau' \mu_1, \quad \theta \mu_1 = 0,
$$

so θ induces $\kappa : Q_1 \to A_2$ with $\kappa \varepsilon_1 = \theta$. Conversely, if τ' is given by

(3.7), then τ' is a homomorphism and preserves the commutativity of (3.6) when used to replace τ.

PROPOSITION 3.6 (LYNDON-HOCHSCHILD-SERRE spectral sequence). *Given any extension of groups*

$$M \rightarrowtail G \twoheadrightarrow Q,$$

there exists a spectral sequence $\{E_r^{pq}\}$, *with* $E_2^{pq} = H_p(Q; H_q M)$, *converging finitely to the graded group associated with* $H_* G$, *suitably filtered.*

4. Localization of nilpotent groups.

Let G be a group and let

$$\ldots \subseteq \Gamma^{i+1}(G) \subseteq \Gamma^i(G) \subseteq \ldots$$

be the lower central series of G; thus

$$\Gamma^1(G) = G \quad \text{and} \quad \Gamma^{i+1}(G) = [G, \Gamma^i(G)].$$

We say that G is *nilpotent of class* c if

$$\Gamma^c(G) \neq \{1\}, \qquad \Gamma^{c+1}(G) = \{1\}.$$

We then write $nil\, G = c$.

DEFINITION. Let G be nilpotent. We say that G is P-*local* if $x \mapsto x^n$, $x \in G$, is bijective for all $n \in P'$. A homomorphism $e: G \to G_P$, with G_P nilpotent, is said to P-*localize* G if G_P is P-local and e has the universal property for homomorphisms of G into P-local nilpotent groups: that is, if H is P-local, then $e^*: Hom(G_P, H) \cong Hom(G, H)$.

DEFINITION. Let G, H be nilpotent groups. We say that $\phi: G \to H$ is a P-*isomorphism* if:

(i) every element in $ker\, \phi$ is a P'-torsion element;

(ii) for every $y \in H$, there exist $n \in P'$, $x \in G$, with $\phi(x) = y^n$.

It is immediately clear that this definition generalizes that given in

Section 2 for abelian groups (nilpotent groups G with $nil\, G \leqslant 1$). If ϕ satisfies (i) we call it *P-injective*, if ϕ satisfies (ii) we call it *P-surjective*.

Our main object in this section is to prove the following theorems.

THEOREM 4.1. *With each nilpotent group G we may associate its P-localization $e : G \to G_P$. Moreover $nil\, G_P \leqslant nil\, G$.*

THEOREM 4.2. *A homomorphism $\phi : G \to H$ of nilpotent groups P-localizes G if and only if H is P-local and ϕ is a P-isomorphism.*

Let N be the category of nilpotent groups, and let N_c be the full subcategory of N consisting of groups G with $nil\, G \leqslant c$. We will achieve our objective by proving the following theorem by induction on c.

THEOREM 4.3. *For each $c \geqslant 1$ we may find a functor $L_c : N_c \to N_c$ and a natural transformation $e_c : 1 \to L_c$ such that $L_c(G)$ is P-local and $e_c(G)$, $G \in N_c$, has the universal property in N_c. We may choose L_c, e_c so that*
$$L_c \,|\, N_{c-1} = L_{c-1}, \qquad e_c \,|\, N_{c-1} = e_{c-1}.$$
Further $\phi : G \to H$ P-localizes G in N_c if and only if H is P-local and ϕ is a P-isomorphism.

PROOF. The assertion is true for $c = 1$, for we constructed L_1 in Section 2 and proved Proposition 2.6. We now assume the assertion holds for $c - 1$, $c \geqslant 2$, and prove it for c. We write L for L_{c-1}, e for e_{c-1}, and G_P for $L\, G$. Thus we have defined

(4.1) $L : N_{c-1} \to N_{c-1}, \qquad e : 1 \to L,$

and wish to extend L to $L : N_c \to N_c$, and to extend e correspondingly, to have the universal property in N_c. We draw some consequences from the truth of Theorem 4.3 in the case $c - 1$. First however we need to enunciate two propositions about N.

PROPOSITION 4.4. *Let $G' \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} G''$ be a central extension in N. Then if G', G'' are P-local, so is G.*

PROOF. Let $x \in G$, $n \in P'$. Then $\varepsilon x = y''^n = \varepsilon\, y^n$, for some $y'' \in G''$, $y \in G$.

Thus $x = y^n \mu(y')$, $y' \in G'$. But $y' = x'^m$ for some $x' \in G'$, so

$$x = y^n \mu(x')^n = (y \mu(x'))^n,$$

since $\mu G'$ is central in G.

Suppose now that $x^n = y^n$, x, $y \in G$, $n \in P'$. Then $\varepsilon x^n = \varepsilon y^n$, so

$$\varepsilon x = \varepsilon y, \quad x = y \mu(x'), \quad x' \in G'.$$

Then $x^n = y^n \mu(x'^m)$, since $\mu G'$ is central in G, so $x'^m = 1$, $x' = 1$, $x = y$.

PROPOSITION 4.5. *Let*

$$
\begin{array}{ccccc}
G' & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\longrightarrow} & G'' \\
\downarrow{\phi'} & & \downarrow{\phi} & & \downarrow{\phi''} \\
H' & \underset{\overline{\mu}}{\rightarrowtail} & H & \underset{\overline{\varepsilon}}{\longrightarrow} & H''
\end{array}
$$

be a map of central extensions in N. *Then if* ϕ', ϕ'' *are* P-*isomorphisms, so is* ϕ.

PROOF. Let $y \in H$. Since ϕ'' is P-surjective, there exist $x'' \in G''$, $n \in P'$, with $\overline{\varepsilon}(y^n) = \phi''(x'')$. Let $x'' = \varepsilon x_0$, $x_0 \in G$. Then

$$\overline{\varepsilon}(y^n) = \overline{\varepsilon} \phi(x_0), \quad \text{so} \quad y^n = \phi(x_0) \overline{\mu}(y'), \quad y' \in H'.$$

Since ϕ' is P-surjective, there exist $x' \in G'$, $m \in P'$ with $y'^m = \phi'(x')$. Then, since $\overline{\mu}H'$ is central in H,

$$y^{mn} = \phi(x_0^m) \overline{\mu}(y'^m) = \phi(x_0^m \mu(x')), \quad mn \in P',$$

so ϕ is P-surjective.

Let $x \in G$ with $\phi x = 1$. Then $\phi'' \varepsilon(x) = 1$ so, ϕ'' being P-injective, there exists $n \in P'$ with $\varepsilon(x^n) = 1$. Thus $x^n = \mu x'$, $x' \in G'$, and $\phi' x' = 1$. Since ϕ' is P-injective, $x'^m = 1$ for some $m \in P'$, so $x^{mn} = 1$, $mn \in P'$, and ϕ is P-injective.

We are now ready to exploit the inductive hypothesis to prove a series of propositions which will enable us to establish Theorem 4.3.

PROPOSITION 4.6. $L : N_{c-1} \rightarrow N_{c-1}$ *is exact.*

PROOF. We have the diagram, in N_{c-1},

$$G' \xrightarrow{\ \mu\ } G \xrightarrow{\ \varepsilon\ } G''$$

(commutative diagram with vertical maps e, bottom row $G'_P \xrightarrow{\mu_P} G_P \xrightarrow{\varepsilon_P} G''_P$)
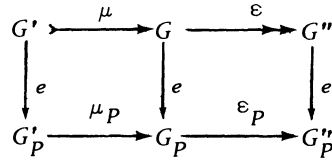
with the top row short exact, and wish to prove the bottom row short exact. We use the fact that e is a P-isomorphism.

First, ε_P is surjective. For let $y'' \in G''_P$. Then, for some $n \in P'$, $y''^n = e\, x''$, $x'' \in G''$. Let $x'' = \varepsilon x$. Then $y''^n = e\, \varepsilon x = \varepsilon_P\, e\, x$. Thus ε_P is P-surjective, and one proves, just as in the proof of lemma 2.8, that a P-surjection of P-local groups is a surjection.

Second, μ_P is injective. For let $\mu_P\, y' = 1$, $y' \in G'_P$. Then, for some $n \in P'$, $y'^n = e\, x'$, $x' \in G'$, so

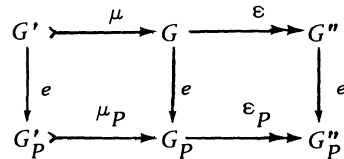$$e\, \mu\, x' = \mu_P\, e\, x' = \mu_P\, y'^n = 1\,.$$

Thus $\mu\, x'^m = 1$ for some $m \in P'$. Thus $x'^m = 1$, $y'^{mn} = 1$, and $mn \in P'$, so $y' = 1$ since G'_P is P-local.

Third, μ_P is the kernel of ε_P. Of course $\varepsilon_P\, \mu_P = 0$ so we must prove that $ker\, \varepsilon_P \subset im\, \mu_P$. Let $\varepsilon_P\, y = 1$, $y \in G$. Then, for some $n \in P'$, $y^n = e\, x$, $x \in G$, so

$$e\, \varepsilon\, x = \varepsilon_P\, e\, x = \varepsilon_P\, y^n = 1\,.$$

Thus $\varepsilon\, x^m = 1$ for some $m \in P'$. Thus $x^m = \mu\, x'$, $x' \in G'$, so that $y^{mn} = {} = \mu_P\, e\, x'$, and $mn \in P'$. Again one argues just as in the proof of Lemma 2.8 that, since $y^{mn} \in im\, \mu_P$ and G'_P, G_P are P-local, therefore $y \in im\, \mu_P$.

PROPOSITION 4.7. *If, in the commutative diagram, in N_{c-1},*

$$G' \xrightarrow{\ \mu\ } G \xrightarrow{\ \varepsilon\ } G''$$

(commutative diagram with vertical maps e, bottom row $G'_P \xrightarrow{\mu_P} G_P \xrightarrow{\varepsilon_P} G''_P$)

the top extension is central, so is the bottom.

PROOF. Let $x' \in G'$, $y \in G_P$. Then $y^n = e\, x$, $x \in G$, for some $n \in P'$. Thus

$$(\mu x')^{-1} x (\mu x') = x\,, \quad \text{so} \quad (\mu_P\, e\, x')^{-1} y^n (\mu_P\, e\, x') = y^n\,.$$

Since G_P has unique n^{th} roots, $(\mu_P\, e\, x')^{-1} y (\mu_P\, e\, x') = y$, so $\mu_P\, e\, x'$

belongs to the center of G_P .

Now let $y' \in G_P$. Then $y'^m = e x'$, $x' \in G'$, for some $m \in P'$. Thus, for any $y \in G_P$,

$$y^{-1} (\mu_P y')^m y = (\mu_P y')^m .$$

Since G_P has unique m^{th} roots, $y^{-1} (\mu_P y') y = \mu_P y'$, so $\mu_P y'$ belongs to the center of G_P . Thus $\mu_P G'_P$ is central in G_P .

THEOREM 4.8. *Let* $G \in N_i$, $i \leqslant c - 1$. *Then* $\widetilde{H}_*(e) : \widetilde{H}_*(G) \to \widetilde{H}_*(G_P)$ *P -localizes.*

PROOF. We argue by induction on i , the theorem being true for $i = 1$ (Proposition 2.5). Suppose the theorem true for all groups K with $nil K \leqslant i - 1$, $i \geqslant 2$, and let $nil G \leqslant i$. Let Z be the center of G . Then $nil Z = 1$, $nil G / Z \leqslant i - 1$, and, by Proposition 4.6 and 4.7, we have a map of central extensions

$$(4.3) \quad \begin{array}{ccccc} Z & \rightarrowtail & G & \twoheadrightarrow & G / Z \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ Z_P & \rightarrowtail & G_P & \twoheadrightarrow & (G / Z)_P \end{array}$$

Then (4.3) induces a map of spectral sequences $\{ E_r^{pq} \} \to \{ \widetilde{E}_r^{pq} \}$, where (Proposition 3.6)

$$E_2^{pq} = H_p (G / Z ; H_q Z) , \qquad \widetilde{E}_2^{pq} = H_p ((G / Z)_P ; H_q Z_P) ,$$

the coefficients being trivial in both cases. It now follows from the inductive hypothesis, together with Propositions 2.3, 2.4, taken in conjonction with the natural universal coefficient sequence in homology, that (4.3) induces $e_2 : E_2^{pq} \to \widetilde{E}_2^{pq}$ which is P -localization unless $(p, q) = (0, 0)$. By Proposition 2.1 we now readily infer that $e_\infty : E_\infty^{pq} \to \widetilde{E}_\infty^{pq}$ is also P -localization unless $(p, q) = (0, 0)$.

Since, for any n , $H_n G$ $(H_n G_P)$ has a finite filtration whose associated graded group is

$$E_\infty^{pq} \qquad (\widetilde{E}_\infty^{pq}), \quad \text{with} \quad p + q = n ,$$

it follows from Proposition 2.3 that $H_n(e) : H_n G \to H_n G_P$ P -localizes for $n \geqslant 1$.

COROLLARY 4.9. *Let* $G \in N_{c-1}$ *and let* A *be a* P-*local abelian group.* *Then* $e : G \to G_P$ *induces* $e^* : H^*(G_P ; A) \cong H^*(G ; A)$.

PROOF. The case $n = 0$ being trivial, we show that

$$e^* : H^n(G_P ; A) \cong H^n(G ; A), \qquad n \geqslant 1.$$

Consider the diagram

$$\begin{array}{ccc}
Ext(H_{n-1} G_P, A) \rightarrowtail H^n(G_P ; A) \twoheadrightarrow Hom(H_n G_P, A) \\
\downarrow{e'} \qquad\qquad \downarrow{e^*} \qquad\qquad \downarrow{e''} \\
Ext(H_{n-1} G, A) \rightarrowtail H^n(G ; A) \twoheadrightarrow Hom(H_n G, A)
\end{array}$$

induced by e. It follows immediately from Theorem 4.8 and Proposition 2.9 that e'' is an isomorphism. If $n = 1$ this completes the argument; if $n \geqslant 2$ it follows immediately from Theorem 4.8 and Proposition 2.9 that e' is an isomorphism. Thus e^* is an isomorphism.

We are now ready to carry out the inductive step establishing Theorem 4.3 and hence Theorems 4.1, 4.2.

PROOF OF THEOREM 4.3. Assume G nilpotent with $nil\, G \leqslant c$. We then have a central extension

$$(4.4) \qquad \Gamma^c \rightarrowtail G \twoheadrightarrow G/\Gamma^c, \quad \Gamma^c = \Gamma^c(G),$$

with $nil\, \Gamma^c \leqslant 1$, $nil\, G/\Gamma^c \leqslant c-1$. Let (4.4) represent $\xi \in H^2(G/\Gamma^c ; \Gamma^c)$. Then $e_* \xi \in H^2(G/\Gamma^c ; \Gamma^c_P)$ and, by Corollary 4.9, there exists a unique element $\xi_P \in H^2((G/\Gamma^c)_P ; \Gamma^c_P)$ such that

$$(4.5) \qquad e^* \xi_P = e_* \xi.$$

Let the central extension

$$(4.6) \qquad \Gamma^c_P \rightarrowtail G_P \twoheadrightarrow (G/\Gamma^c)_P$$

represent ξ_P. Then by Proposition 3.5, we have a commutative diagram

$$(4.7) \qquad \begin{array}{ccc}
\Gamma^c & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & G/\Gamma^c \\
\downarrow{e} & & \downarrow{e} & & \downarrow{e} \\
\Gamma^c_P & \underset{\mu_P}{\rightarrowtail} & G_P & \underset{\varepsilon_P}{\twoheadrightarrow} & (G/\Gamma^c)_P
\end{array}$$

We first remark that, since (4.6) is a central extension and

$nil(G/\Gamma^c)_P \leqslant c-1$, then G_P is nilpotent and $nil\,G_P \leqslant c$. We next remark that, by Proposition 4.4, G_P is P-local. We also remark that if, in fact, $nil\,G \leqslant c-1$, then

$$\Gamma^c = \{\,1\,\}, \quad \Gamma_P^c = \{\,1\,\}, \quad G/\Gamma^c = G,$$

and we naturally take $G_P = (G/\Gamma^c)_P$, preserving the same P-localization if we define $L\,G = G_P$, as we propose to do.

Now, by Proposition 4.5, $e : G \to G_P$ in (4.7) is a P-isomorphism. It follows, by a trivial modification of the argument of Lemma 2.8 (we restate the result in Lemma 4.11) that $e : G \to G_P$ is thus an isomorphism if G is P-local. Then the universal property of e will follow directly from the fact, still to be proved, that e is a natural transformation of functors. For we then readily infer that

$$e^* : Hom\,(\,G_P\,,H\,) \to Hom\,(\,G\,,H\,)$$

is surjective if H is P-local in N_c, and the fact that e^* is injective follows immediately from the fact that e is P-surjective and H is P-local.

Thus it remains to define L on morphisms of N_c as a functor, and to prove the naturality of e. Let $\phi : G \to \overline{G}$ in N_c, and let $\overline{\Gamma}^c = \Gamma^c(\overline{G})$. Then we have

(4.8)

and our object is to define $\phi_P : G_P \to \overline{G}_P$ to make (4.8) commutative. It is clear that any ϕ_P yielding $\phi_P\,e = e_\phi$ is uniquely determined so the functoriality of L is automatic once a suitable ϕ_P is defined.

In (4.8) ϕ', ϕ'' are induced by ϕ, and

$$e_*\,\xi = e^*\,\xi_P\,, \qquad e_*\,\overline{\xi} = e^*\,\overline{\xi}_P\,.$$

Thus

$$e^* \phi_P^{\prime\prime *} \bar{\xi}_P = \phi^{\prime\prime *} e^* \bar{\xi}_P = \phi^{\prime\prime *} e_* \bar{\xi} = e_* \phi^{\prime\prime *} \bar{\xi} \quad (3.5)$$

$$= e_* \phi_*^! \xi \quad (\text{Prop. } 3.5) = \phi_{P*}^{\prime} e_* \xi = \phi_{P*}^{\prime} e^* \xi_P = e^* \phi_{P*}^{\prime} \xi_P \, .$$

But

$$e^* : H^2 ((G / \Gamma^c)_P ; \bar{\Gamma}_P^c) \cong H^2 (G / \Gamma^c ; \bar{\Gamma}_P^c) ,$$

by Corollary 4.9, so $\phi_P^{\prime\prime *} \bar{\xi}_P = \phi_{P*}^{\prime} \xi_P$. Thus (Prop. 3.5) we may find $\tau : G_P \to \bar{G}_P$ so that

(4.9) $\qquad\qquad\qquad \tau \mu_P = \bar{\mu}_P \phi_P^{\prime} , \qquad \bar{\varepsilon}_P \tau = \phi_P^{\prime\prime} \varepsilon_P \, .$

Consider the diagram

(4.10)

Here

$$\psi^{\prime} = e \phi^{\prime} = \phi_P^{\prime} e , \qquad \psi^{\prime\prime} = e \phi^{\prime\prime} = \phi_P^{\prime\prime} e \, .$$

It is thus clear that (4.10) commutes if $\psi = e \phi$ or $\psi = \tau e$, so that Proposition 3.5 implies that there is $\theta : G / \Gamma^c \to \bar{\Gamma}_P^c$, such that

$$e \phi (x) = \tau e (x) . \bar{\mu}_P \theta \varepsilon (x) , \qquad x \in G \, .$$

Let $\theta_P : (G / \Gamma^c)_P \to \bar{\Gamma}_P^c$ be given by $\theta_P e = \theta$, and define

$$\phi_P (y) = \tau (y) . \bar{\mu}_P \theta_P \varepsilon_P (y) , \qquad y \in G_P \, .$$

Again by Proposition 3.5 we have, from (4.9),

$$\phi_P \mu_P = \bar{\mu}_P \phi_P^{\prime} , \qquad \bar{\varepsilon}_P \phi_P = \phi_P^{\prime\prime} \varepsilon_P \, ,$$

and also

$$\phi_P e (x) = \tau e (x) . \bar{\mu}_P \theta_P \varepsilon_P e (x) = \tau e (x) . \bar{\mu}_P \theta \varepsilon (x) = e \phi (x) , \quad x \in G \, .$$

We are now left merely with the final assertion of Theorem 4.3. We know that $e : G \to G_P$ is a P-isomorphism and G_P is P-local. We prove the converse just as for Proposition 2.6. We state the appropriate lemmas again, explicitly.

LEMMA 4.10. *If, in* N, $\beta\alpha$ *and* α *are* P-*isomorphisms, so is* β.

PROOF. It is trivial that $\beta\alpha$ is P-surjective, so is β. Now consider

$$G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \quad \text{and let } x_2 \in G_2, \ \beta x_2 = 1.$$

Since α is P-surjective, there exists n prime to P with $x_2^n = \alpha x_1$, $x_1 \in G_1$. Then $\beta\alpha x_1 = 1$, so, $\beta\alpha$ being P-injective, there exists m prime to P with $x_1^m = 1$. Then $x_2^{mn} = 1$, mn prime to P, so β is P-injective.

LEMMA 4.11. *In* N *a* P-*isomorphism between* P-*local groups is an isomorphism.*

PROOF. We make a trivial modification of the proof of Lemma 2.8.

Now that the proof of Theorems 4.1, 4.2 is complete, we may, of course, restate Propositions 4.6, 4.7, Theorem 4.8 and Corollary 4.9 without any reference to the inductive parameter c. We ask the reader to assume those restatements made.

5. Applications.

In this section we show how the existence of a P-localization $e : G \rightarrow G_P$, together with the characterization of e as a P-isomorphism to a P-local group, may be used to infer facts about nilpotent groups.

THEOREM 5.1. *Let* Π *be a family of primes. Then if* G *is nilpotent it has a* Π-*torsion subgroup* T_Π.

PROOF. Let $P = \Pi'$. Localize at P by $e : G \rightarrow G_P$. Since e is a P-isomorphism, every element in $ker\,e$ is a Π-torsion element. Since G_P is P-local, every Π-torsion element is in $ker\,e$. Thus $ker\,e = T_\Pi$.

THEOREM 5.2. *Suppose* G *is nilpotent and has no* Π-*torsion. Then if* $x^n = y^n$, x, $y \in \Pi$, *it follows that* $x = y$.

PROOF. Localize at $P = \Pi'$. Then $e : G \rightarrow G_P$ is injective and $e(x)^n = e(y)^n$. Since G_P is P-local, $e(x) = e(y)$, so $x = y$.

COROLLARY 5.3. *The nilpotent group G is P-local if and only if it has no P'-torsion and $x \mapsto x^n$, $x \in G$, is surjective for all $n \in P'$.*

THEOREM 5.4. *Let $G' \rightarrowtail G \twoheadrightarrow G''$ be a short exact sequence of nilpotent groups. Then if any two of G', G, G'' are P-local, so is the third.*

PROOF. Since P-localization is exact, we obtain a map of short exact sequences

$$
\begin{array}{ccccc}
G' & \rightarrowtail & G & \longrightarrow\!\!\!\!\!\rightarrow & G'' \\
\downarrow & & \downarrow & & \downarrow \\
G'_P & \rightarrowtail & G_P & \longrightarrow\!\!\!\!\!\rightarrow & G''_P
\end{array}
$$

The hypothesis implies that two of the vertical arrows are isomorphisms. So therefore is the third, and the theorem is proved.

We now turn to results which make explicit mention of P-localization.

THEOREM 5.5. *Let ,*

$$
\begin{array}{ccccc}
G' & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\longrightarrow\!\!\!\!\!\rightarrow} & G'' \\
\downarrow{\scriptstyle\phi'} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi''} \\
H' & \underset{\bar{\mu}}{\rightarrowtail} & H & \underset{\bar{\varepsilon}}{\longrightarrow\!\!\!\!\!\rightarrow} & H''
\end{array}
$$

be a map of short exact sequences of nilpotent groups. Then if any two of ϕ', ϕ, ϕ'' P-localize, so does the third.

PROOF. By Theorem 5.4, we infer that H', H, H'' are all P-local. By P-localizing the top row, we obtain

$$
\begin{array}{ccccc}
G' & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\longrightarrow\!\!\!\!\!\rightarrow} & G'' \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
G'_P & \overset{\mu_P}{\rightarrowtail} & G_P & \overset{\varepsilon_P}{\longrightarrow\!\!\!\!\!\rightarrow} & G''_P \\
\downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi''} \\
H' & \underset{\bar{\mu}}{\rightarrowtail} & H & \underset{\bar{\varepsilon}}{\longrightarrow\!\!\!\!\!\rightarrow} & H''
\end{array}
$$

where $\psi' e = \phi'$, $\psi e = \phi$, $\psi'' e = \phi''$. Moreover, the commutativity relations

$\overline{\mu}\psi' = \psi\mu_P$, $\overline{\varepsilon}\psi = \psi''\varepsilon_P$ follow from the universal property of e. But the hypothesis implies that two of ψ', ψ, ψ'' are isomorphisms. So therefore is the third, and the theorem is proved.

THEOREM 5.6. *Let G be nilpotent and let $\phi: G \to K$ P-localize G. Then $\Gamma^i \phi : \Gamma^i G \to \Gamma^i K$ P-localizes $\Gamma^i G$, for all $i \geqslant 1$.*

PROOF. It follows from Theorem 5.5 that it is sufficient to prove that the homomorphism $\phi_i : G/\Gamma^i G \to K/\Gamma^i K$, induced by ϕ, P-localizes. We argue by induction on i, the assertion being trivial for $i = 1$ and following from Theorem 4.8 for $i = 2$. Thus we assume that ϕ_i P-localizes, $i \geqslant 2$, and prove that ϕ_{i+1} P-localizes. A second application of Theorem 5.5 shows that it is sufficient to prove that the homomorphism

$$\overline{\phi} : \Gamma^i G / \Gamma^{i+1} G \to \Gamma^i K / \Gamma^{i+1} K,$$

induced by ϕ, P-localizes. We apply the 5-term exact sequence in the homology of groups to the diagram

$$\begin{array}{ccc} G & \longrightarrow & G/\Gamma^i G \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi_i} \\ K & \longrightarrow & K/\Gamma^i K \end{array}$$

to obtain

$$\begin{array}{ccccccccc} H_2 G & \longrightarrow & H_2(G/\Gamma^i G) & \longrightarrow & \Gamma^i G/\Gamma^{i+1} G & \longrightarrow & G_{ab} & \longrightarrow & (G/\Gamma^i G)_{ab} & \longrightarrow & 0 \\ \downarrow{\scriptstyle\phi_*} & & \downarrow{\scriptstyle\phi_{i*}} & & \downarrow{\scriptstyle\overline{\phi}} & & \downarrow{\scriptstyle\phi_{ab}} & & \downarrow{\scriptstyle\phi_{iab}} & & \\ H_2 K & \longrightarrow & H_2(K/\Gamma^i K) & \longrightarrow & \Gamma^i K/\Gamma^{i+1} K & \longrightarrow & K_{ab} & \longrightarrow & (K/\Gamma^i K)_{ab} & \longrightarrow & 0 \end{array}$$

Then ϕ_*, ϕ_{ab} P-localize by Theorem 4.8 and ϕ_{i*}, ϕ_{iab} P-localize by the inductive hypothesis and Theorem 4.8. It is now evident, by a mild extension of Proposition 2.3, that $\overline{\phi}$ P-localizes.

Our next theorem appeared explicitly in [6]; we include it here for completeness.

THEOREM 5.7. *Let $\phi: G \to K$ be a homomorphism of nilpotent groups. Then ϕ P-localizes if and only if $\widetilde{H}_*(\phi) : \widetilde{H}_*(G) \to \widetilde{H}_*(K)$ P-localizes.*

PROOF. Theorem 4.8 asserts that $\widetilde{H}_*(\phi)$ P-localizes if ϕ P-localizes. We next prove that if $\widetilde{H}_*(K)$ is P-local, then K is P-local. For let $e : K \to K_p$ P-localize. Then $\widetilde{H}_*(e) : \widetilde{H}_*(K) \to \widetilde{H}_*(K_p)$ P-localizes; but $\widetilde{H}_*(K)$ is P-local, so $\widetilde{H}_*(e)$ is an isomorphism. It follows from the Stallings-Stammbach Theorem that e is an isomorphism.

Now let $\widetilde{H}_*(\phi)$ P-localize. Then $H_*(K)$ is P-local, so K is P-local. Thus ϕ factors as $G \xrightarrow{e} G_p \xrightarrow{\psi} K$ and

$$\widetilde{H}_*(G) \xrightarrow{\widetilde{H}_*(\phi)} \widetilde{H}_*(K)$$

with $\widetilde{H}_*(e)$ and $\widetilde{H}_*(\psi)$ through $\widetilde{H}_*(G_p)$

But since $\widetilde{H}_*(\phi)$, $\widetilde{H}_*(e)$ both P-localize, $\widetilde{H}_*(\psi)$ is an isomorphism. Thus the Stallings-Stammbach Theorem again implies that ψ is an isomorphism, so that ϕ P-localizes.

We turn now to a consideration of the upper central series of a nilpotent group:

$$Z(G) = Z^1(G) \subseteq Z^2(G) \subseteq \ldots \subseteq Z^c(G) = G,$$

where $Z(G)$ is the center of G. We then easily prove

PROPOSITION 5.8. *If* G *is* P-*local, so is* $Z^i(G)$, $i \geqslant 1$. *Moreover,* $e : G \to G_p$ *sends* $Z^i(G)$ *to* $Z^i(G_p)$.

PROOF. To prove $Z(G)$ P-local, we only have to show that if $x \in G$ and $x^n \in Z(G)$, $n \in P'$, then $x \in Z(G)$. But if $x^n \in Z(G)$, then $y^{-1} x^n y = x^n$ for all $y \in G$. Taking n^{th} roots, $y^{-1} x y = x$, so $x \in Z(G)$. An easy induction, involving two applications of Theorem 5.4, now shows that $Z^i(G)$ is P-local, $i \geqslant 1$.

It was proved in Proposition 4.7 that $e : G \to G_p$ sends $Z(G)$ to $Z(G_p)$. Thus, again, an easy induction shows that e sends $Z^i(G)$ to $Z^i(G_p)$, $i \geqslant 1$.

Let us write $Z^i(e) : Z^i(G) \to Z^i(G_p)$ for the restriction of $e : G \to G_p$. Then we may prove

THEOREM 5.9. *If G is finitely generated nilpotent and $e: G \to G_P$ P-localizes G, then the restriction $Z^i(e): Z^i(G) \to Z^i(G_P)$ P-localizes $Z^i(G)$.*

PROOF. This was proved in $[6]$ for $i = 1$, but we will repeat the proof in order to demonstrate that only facts about nilpotent groups explicitly mentioned in this paper are required. It is plain, in view of Proposition 5.8 and Theorem 4.2, that we only have to prove that $Z(e)$ is P-surjective. We proceed by a series of lemmas.

LEMMA 5.10. *Let G be a group, x, $y \in G$ and the commutator $[x, y] \in \Gamma^i$. Then $[x, y^n] \equiv [x, y]^n \mod \Gamma^{i+1}$.*

PROOF. This is immediately evident from the commutator identity

$$[x, y_1 y_2] = [x, y_1][y_1, [x, y_2]][x, y_2],$$

where $[x, y] = xyx^{-1}y^{-1}$.

LEMMA 5.11. *Let G be nilpotent and $[x, y] \in T_\Pi$, the Π-torsion subgroup of G (Theorem 5.1). Then $[x, y^n] = 1$ for some $n \in \Pi$.*

PROOF. First observe (by passing to the quotient group G/T_Π) that if $[x, y] \in T_\Pi$, then $[x, y^r] \in T_\Pi$ for all r. Now, since G is nilpotent, it suffices to show that, for each i, there exists $n_i \in \Pi$ with $[x, y^{n_i}] \in \Gamma^i$. For $i = 2$ take $n_i = 1$, and proceed by induction on i. For if $[x, y^{n_i}] \in \Gamma^i$, $n_i \in \Pi$, then, since $[x, y^{n_i}] \in T_\Pi$, there exists $m_i \in \Pi$ with $[x, y^{n_i}]^{m_i} = 1$, and, by Lemma 5.10,

$$[x, y^{n_i m_i}] \in \Gamma^{i+1}, \qquad n_i m_i \in \Pi.$$

LEMMA 5.12. *Let G be finitely generated nilpotent, and let $e(y) \in Z(G_P)$. Then there exists $n \in P'$ with $y^n \in Z(G)$.*

PROOF. Let $G = (x_1, x_2, \ldots, x_k)$. For each x_j we have $e[x_j, y] = 1$, so that $[x_j, y] \in T_{P'}$. Thus there exists $n_j \in P'$, with $[x_j, y^{n_j}] = 1$, by Lemma 5.11. Set $n = n_1 n_2 \ldots n_k$. Then

$$n \in P' \qquad \text{and} \qquad [x_j, y^n] = 1, \qquad j = 1, 2, \ldots, k,$$

so that $y^n \in Z(G)$.

We now prove that $Z(e):Z(G) \to Z(G_P)$ is P-surjective. Let $b \in Z(G_P)$. Then there exists $m \in P'$ with $b^m = e(y)$, $y \in G$. By Lemma 5.12, there exists $n \in P'$ with $y^n \in Z(G)$ and $e(y^n) = b^{mn}$, $mn \in P'$. Thus $Z(e)$ is P-surjective and Theorem 5.9 is proved for $i = 1$.

The argument is now completed by an easy induction. Assume $i \geqslant 2$, and that $Z^{i-1}(e)$ P-localizes. Then the induced map

$$e_i : G / Z^{i-1}(G) \to G_P / Z^{i-1}(G_P)$$

also P-localizes, by Theorem 5.5. Moreover, $G / Z^{i-1}(G)$ is finitely generated, so that

$$Z(e_i): Z^i(G) / Z^{i-1}(G) \to Z^i(G_P) / Z^{i-1}(G_P)$$

P-localizes. Finally, a second application of Theorem 5.5 establishes that $Z^i(e)$ P-localizes.

6. P-isomorphisms.

It is trivial to prove that a composite of P-isomorphisms is a P-isomorphism. It thus readily follows from Lemmas 4.10, 4.11 that if, in the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & H \\
\downarrow{\scriptstyle e} & \xrightarrow{\phi_P} & \downarrow{\scriptstyle e} \\
G_P & \longrightarrow & H_P
\end{array}
$$

(6.1)

in N, ϕ is a P-isomorphism, then ϕ_P is an isomorphism. Our main objective in this section is to prove the converse of this, and thereby to be able to obtain consequences about the properties of P-isomorphisms of nilpotent groups. However, the converse appears to require deeper properties of nilpotent groups than those so far exploited. We will use Philip Hall's theory of basic commutators [5] to prove the following result, which we have not succeeded in finding in the literature.

THEOREM 6.1. *Let G be a group, x, $y \in G$ with $y^n = 1$. Then*

$$(xy)^{n^i} \equiv x^{n^i} \quad mod \, \Gamma^{i+1}(G).$$

PROOF. We may assume without loss of generality that G is generated by x, y. Then $\Gamma^i(G)/\Gamma^{i+1}(G)$ is generated by the cosets of certain i-fold basic commutators. Moreover, an i-fold commutator is linear in each of its arguments, modulo $\Gamma^{i+1}(G)$. Since an i-fold basic commutator, $i \geqslant 2$, must involve y as an argument, it follows that, for any such commutator c, $c^n \in \Gamma^{i+1}(G)$, so that, for any element

$$q \in \Gamma^i(G), \quad i \geqslant 2, \quad q^n \in \Gamma^{i+1}(G).$$

We now proceed to prove the theorem by induction on i. If $i = 1$, then

$$(xy)^n \equiv x^n y^n \mod \Gamma^2 = x^n.$$

Now suppose that $(xy)^{n^i} = x^{n^i} q$, for some $i \geqslant 1$, with $q \in \Gamma^{i+1}$. Then $q^n \in \Gamma^{i+2}$, so

$$(xy)^{n^{i+1}} = (x^{n^i} q)^n \equiv x^{n^{i+1}} q^n \mod \Gamma^{i+2} \equiv x^{n^{i+1}} \mod \Gamma^{i+2},$$

and the theorem is proved.

COROLLARY 6.2. *Let G be a nilpotent group with $nil\, G \leqslant c$. Then if x, $y \in G$, with $y^n = 1$,*

$$(xy)^{n^c} = x^{n^c}.$$

PROPOSITION 6.3. *Given $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3$ in N, then if $\beta\alpha$ and β are P-isomorphisms, so is α.*

PROOF. It is trivial that α is P-injective if $\beta\alpha$ is P-injective. Now let $y \in G_2$. Since $\beta\alpha$ is P-surjective, $\beta(y^m) = \beta\alpha(x)$, for some $x \in G_1$, $m \in P'$. Then, since β is P-injective, $y^m = \alpha(x)b$, with $b \in G_2$ and $b^n = 1$, $n \in P'$. By Corollary 6.2,

$$y^{mn^c} = \alpha(x^{n^c}) \quad \text{where} \quad nil\, G_2 \leqslant c.$$

Since $mn^c \in P'$, it follows that α is P-surjective.

COROLLARY 6.4. *In the diagram (6.1) in N, ϕ is a P-isomorphism if and only if ϕ_P is an isomorphism.*

It now follows (see [4]) that the P-isomorphisms of N are pre-

cisely those morphisms of N which are rendered invertible by the locali-
zation functor; we are thus led to the general theory of completions as in
[3,4], but we do not take that direction here. Instead, we draw some
immediate consequences.

THEOREM 6.5. *Let*

$$
\begin{array}{ccc}
G' \rightarrowtail G \longrightarrow\!\!\!\!\!\rightarrow G'' \\
\Big\downarrow \phi' \qquad \Big\downarrow \phi \qquad \Big\downarrow \phi'' \\
H' \rightarrowtail H \longrightarrow\!\!\!\!\!\rightarrow H''
\end{array}
$$

be a map of short exact sequences of nilpotent groups. Then if any two of
ϕ', ϕ, ϕ'' *are P-isomorphisms, so is the third.*

PROOF. We P-localize. Since P-localization is exact and two of ϕ'_P,
ϕ_P, ϕ''_P are isomorphisms, so is the third. The theorem thus follows from
Corollary 6.4.

THEOREM 6.6. *Let* $\phi : G \rightarrow K$ *be a P-isomorphism of nilpotent groups.*
Then $\Gamma^i \phi : \Gamma^i G \rightarrow \Gamma^i K$ *is a P-isomorphism.*

PROOF. We P-localize. Since ϕ_P is an isomorphism $\Gamma^i (\phi_P)$ is an iso-
morphism. But, by Theorem 5.6, $\Gamma^i (\phi_P) = (\Gamma^i \phi)_P$. Thus, by Corollary
6.4, $\Gamma^i \phi$ is a P-isomorphism.

THEOREM 6.7. *Let* $\phi : G \rightarrow K$ *be a homomorphism of nilpotent groups. Then*
ϕ *is a P-isomorphism if and only if* $\widetilde{H}_* (\phi) : \widetilde{H}_* (G) \rightarrow \widetilde{H}_* (K)$ *is a P-iso-*
morphism.

PROOF. We P-localize. Then:

   ϕ is a P-isomorphism $\Longleftrightarrow$ ϕ_P is an isomorphism $\Longleftrightarrow$ $\widetilde{H}_* (\phi_P)$ is
   an isomorphism $\Longleftrightarrow$ $\widetilde{H}_* (\phi)_P$ is an isomorphism $\Longleftrightarrow$ $\widetilde{H}_* (\phi)$ is a
   P-isomorphism.

Here the first equivalence uses Corollary 6.4; the second uses the Stallings-
Stammbach Theorem; the third uses Theorem 5.7; and the fourth again uses
Corollary 6.4.

THEOREM 6.8. *The following assertions about the nilpotent groups* G , K

are equivalent (see $[4]$ *):*

   (i) G_P and K_P are isomorphic;

   (ii) there exists a nilpotent group L and P-isomorphisms $\alpha: G \to L$,
$\beta \cdot K \to L$ *;*

   (iii) there exists a nilpotent group M and P-isomorphisms $\gamma: M \to G$,
$\delta: M \to K$ *.*

PROOF. It is plain from Corollary 6.4 that (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i).
Also, since e is a P-isomorphism, the diagram

$$G \xrightarrow{\ \omega e\ } K_P \xleftarrow{\ e\ } K, \qquad \text{where} \qquad \omega: G_P \cong K_P,$$

shows that (i) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) now follows from

PROPOSITION 6.9. *Let*

$$
\begin{array}{ccc}
M & \xrightarrow{\ \gamma\ } & G \\
\delta \downarrow & & \downarrow \alpha \\
K & \xrightarrow{\ \beta\ } & L
\end{array}
$$

*be a pull-back in the category of groups, and let G, K, L be nilpotent.
Then M is nilpotent and δ is a P-isomorphism if α is a P-isomorphism.*

PROOF. M is a subgroup of $G \times K$ and thus certainly nilpotent. Since
$\ker \alpha \cong \ker \delta$ it is plain that δ is P-injective if and only if α is P-injec-
tive. Now let α be P-surjective and let $x \in K$. Then there exists $y \in G$,
$n \in P'$ with $\alpha(y) = \beta(x^n)$. But then

$$(y, x^n) \in M \qquad \text{and} \qquad \delta(y, x^n) = x^n,$$

so δ is P-surjective. This proves Proposition 6.9 and, with it, Theorem 6.8.

   Let us call G, K *P-isomorphic* if any of conditions (i), (ii),
(iii) of Theorem 6.8 hold.

THEOREM 6.10. *Let G, K be finitely generated P-isomorphic nilpotent
groups. Then $Z^i(G)$, $Z^i(K)$ are P-isomorphic for all $i \geqslant 1$.*

PROOF. Apply Theorem 5.9.

REMARK. In fact, in the category of nilpotent groups, the four implications

$$\phi \ P\text{-inj} \iff \phi_P \ \text{inj}, \quad \phi \ P\text{-surj} \iff \phi_P \ \text{surj}$$

are all valid, refining Corollary 6.4. Of these, the implication

$$\phi_P \ \text{surj} \implies \phi \ P\text{-surj}$$

alone requires Corollary 6.2, the others being elementary. Using these implications, one may obtain the following variant of the Stallings-Stammbach Theorem:

THEOREM 6.11. *Let* $\phi : G \to K$ *be a homomorphism of nilpotent groups. Then* ϕ *is a* P*-isomorphism if and only if* ϕ_{ab} *is a* P*-isomorphism and* $H_2(\phi)$ *is a* P*-surjection.*

7. Localization of finitely generated nilpotent groups.

In this section we prove three theorems which are useful in applications of localization to homotopy theory. We begin with some lemmas.

LEMMA 7.1. [5] *A subgroup of a finitely generated nilpotent group is finitely generated.*

PROOF. From the Hall theory of basic commutators and an easy induction we readily infer that if G is finitely generated nilpotent, so are the terms $\Gamma^i G$ of the lower central series. We now prove the lemma by induction on $nil \, G$, since it is true if $nil \, G = 1$.

Let H be a subgroup of G. Then $H \cap \Gamma^2 G$ is a subgroup of $\Gamma^2 G$ and $nil \, \Gamma^2 G < nil \, G$, so, by the inductive hypothesis, $H \cap \Gamma^2 G$ is finitely generated. We have the short exact sequence

$$H \cap \Gamma^2 G \rightarrowtail H \twoheadrightarrow H\Gamma^2(G)/\Gamma^2(G);$$

but $H\Gamma^2(G)/\Gamma^2(G) \subseteq G_{ab}$, and hence is finitely generated. Thus H is itself finitely generated.

Let P_l, $l \geqslant 0$, be the complement of the set $\{p_1, p_2, \ldots, p_l\}$ consisting of the first l primes.

LEMMA 7.2. *If G is finitely generated nilpotent, then there exists l such that $G_{P_l} \to G_0$ is injective, where G_0 is the rationalization of G.*

PROOF. Since G is finitely generated, so is its torsion subgroup T. Since T is then easily seen to be finite, T has p-torsion for only finitely many primes p. Thus, for l sufficiently large, G_{P_l} is torsion free and $G_{P_l} \to G_0$ is injective.

THEOREM 7.3. *Let G, K be finitely generated, nilpotent groups and let $\phi : G \to K_0$ be a homomorphism. Then there exists l such that ϕ has a unique lift into K_{P_l}.*

PROOF. We first choose l so that $K_{P_l} \to K_0$ is injective. Let $(x_1, x_2, \ldots, x_n)$ generate G, let $e : K \to K_0$ rationalize K, and then find $y_i \in K$, $m_i \in \mathbf{N}$, such that

$$e(y_i) = \phi(x_i)^{m_i}, \qquad i = 1, 2, \ldots, n.$$

Now subject l to the further condition that

$$m_i \in P_l', \qquad i = 1, 2, \ldots, n,$$

and factorize e as

$$K \xrightarrow{\ e_1\ } K_{P_l} \xrightarrow{\ e_2\ } K_0, \qquad e = e_2 e_1.$$

Since K_{P_l} is P_l-local, we have

$$e_1(y_i) = z_i^{m_i}, \qquad \text{so} \qquad \phi(x_i)^{m_i} = e_2(z_i)^{m_i},$$

$$\phi(x_i) = e_2(z_i), \qquad i = 1, 2, \ldots, n,$$

and ϕ lifts uniquely into K_{P_l}.

Let us say that the nilpotent group G has *property* A if G is the pull-back of its localizations G_P over its rationalization G_0. Then it has been observed (e.g. in $[6]$) that a finitely generated abelian group has property A.

PROPOSITION 7.4. *Let $G' \xrightarrow{\ \mu\ } G \xrightarrow{\ \varepsilon\ } G''$ be a short exact sequence of nilpotent groups. Then if G', G'' have property A, so has G.*

PROOF. Write $e_p : G \to G_p$ for the localization, $r_p : G_p \to G_0$ for the rationa-

lization. We want to prove that, given $g_p \in G_p$, with $r_p g_p = g_0$, for all p , there exists a unique $g \in G$ with $e_p g = g_p$. Now

$$\varepsilon_p g_p \in G_p'' \qquad \text{and} \qquad r_p'' \varepsilon_p g_p = \varepsilon_0 g_0$$

for all p . Since G'' has property A, there exists a unique $g'' \in G''$ with $e_p'' g'' = \varepsilon_p g_p$. Let $g'' = \varepsilon \overline{g}$. Then $\varepsilon_p g_p = \varepsilon_p e_p \overline{g}$, so that $g_p = (e_p \overline{g}) g_p'$, where $g_p' \in G_p'$, and we regard μ , μ_p , μ_0 as inclusions. Moreover $g_0 = (r \overline{g})(r_p' g_p')$ for all p , where $r : G \to G_0$ is the rationalization, so that the elements g_p' have a common rationalization. Since G' has property A, there exists a unique $g' \in G'$ with $e_p' g' = g_p'$. Then $g_p = e_p (\overline{g} g')$.

It remains to prove uniqueness. We suppose $g \in G$ with $e_p g = 1$, for all p . Then $e_p'' \varepsilon g = 1$, for all p , so $\varepsilon g = 1$, by uniqueness, and $g \in G'$. But then $e_p' g = 1$, for all g , so $g = 1$ by uniqueness.

THEOREM 7.5. *A finitely generated nilpotent group has property* A.

PROOF. We argue by induction on the nilpotency class of the group G , since the theorem is true if $nil\, G = 1$. We suppose $nil\, G = c \geqslant 2$, and that the theorem has been proved for finitely generated nilpotent groups of class $\leqslant c - 1$. But then (see Lemma 7.1)

$$\Gamma^2(G) \rightarrowtail G \twoheadrightarrow G / \Gamma^2(G)$$

is a short exact sequence of finitely generated nilpotent groups with

$$nil\, \Gamma^2 G < nil\, G , \qquad nil\, G / \Gamma^2 G = 1 .$$

Thus $\Gamma^2 G$ and $G / \Gamma^2 G$ have property A and so, by Proposition 7.4, has G .

THEOREM 7.6. *Let* $\phi : G \to H$ *be a homomorphism of nilpotent groups. Then* ϕ *is an isomorphism if and only if* ϕ_p *is an isomorphism for all* p .

PROOF. We assume ϕ_p an isomorphism for all p , that is, ϕ is a p-isomorphism for all p . Since $ker\, \phi$ is a torsion group, and all primes are forbidden, $ker\, \phi = \{ 1 \}$. Now let $y \in H$. Then, for each p , we have $x_{(p)} \in G$, n_p prime to p , and $y^{n_p} = \phi x_{(p)}$. Since $gcd(n_p) = 1$, we may find integers a_p , almost all zero, such that $\sum a_p n_p = 1$. Set $x = \prod x_{(p)}^{a_p}$. It is then plain that $y = \phi x$.

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