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ON COMPACT TRANSFORMATION GROUPOIDS

by Anthony Karel SEDA

Introduction.

A technique which is often used in the study of G-spaces S, where G is a compact group, is that of averaging with respect to the Haar measure of G. Thus if S is metrizable, one can average any metric e on S to obtain an equivalent metric δ with respect to which each operation of G is an isometry, see [4]. Similarly, when dealing with representations of G, one can obtain invariant inner products for Euclidean spaces S.

In [5] and [6] the concept of a Haar system of measures was introduced for locally compact groupoids G, and in this note it is shown that this concept is adequate to obtain a parallel, in the context of transformation groupoids, of the averaging process for transformation groups.

1. Preliminaries.

Let G be a topological groupoid with object space X, see [3], [5] and [6], and let $\pi : G \to X$ and $\pi' : G \to X$ denote the initial and final maps of G respectively; they are continuous. Let $P : S \to X$ be a a fibre space, that is, P is a continuous surjection, and form

$$G \underset{P}{\times} S = \{ (\alpha, s) \in G \times S \mid \pi(\alpha) = P(s) \}$$

regarded as a subspace of $G \times S$. We recall Ehresmann's well known definition of a *transformation groupoid*, see [3], and say G *acts on* S if there is a continuous function $(\,.\,) : G \underset{P}{\times} S \to S$ with the properties

$$\text{(i)}\quad P(\alpha . s) = \pi'(\alpha); \quad \text{(ii)}\quad \beta.(\alpha . s) = \beta\alpha . s; \quad \text{(iii)}\quad I_{P(s)} . s = s$$

for each identity I_x of G. These relations are understood to hold for all α and β in G and $s \in S$ for which they are defined. Notice that if G acts on S, then each element α of G determines a homeomorphism $P^{-1}(\pi(\alpha)) \to P^{-1}(\pi'(\alpha))$ defined by $s \mapsto \alpha . s$. Notice also that if

$$S \underset{P}{\times} S = \{ (s, t) \in S \times S \mid P(s) = P(t) \},$$

then we have a natural map $S \underset{P}{\times} S \to X$, which we denote by P without causing confusion, and there is an action of G on $S \underset{P}{\times} S$ defined by the relation

$$\alpha . (s, t) = (\alpha . s, \alpha . t).$$

It will be convenient to record our definition of a Haar system of measures. Let G be a locally compact Hausdorff topological groupoid with object space X. A *Haar system of measures* $\{ m, \mu, \mu_x \}$ for G consists of non-trivial Baire measures m defined on G, μ defined on X and μ_x defined on $\pi'^{-1}(x)$, for each $x \in X$, satisfying the following relations for each Baire set E of G:

(i) the function $x \mapsto \mu_x(E_x)$ is μ-measurable;

(ii) $m(E) = \int_X \mu_x(E_x) d\mu(x)$;

(iii) $\mu_x(E_x) = \mu_y(\alpha E_x)$ for each $\alpha \in G(x, y)$ and $x, y \in X$.

In these relations E_x denotes the set $E \cap \pi'^{-1}(x)$.

It is shown in [5] and [6] that such systems always exist and a classification is given there.

2. Invariant Metrics.

If $P : S \to X$ is a fibre space, we say that $\{ \rho_x \}$ is a *family of metrics* for S provided:

(i) for each $x \in X$, ρ_x is a metric on $P^{-1}(x)$ compatible with the subspace topology on $P^{-1}(x)$;

(ii) the function $\rho : S \underset{P}{\times} S \to \mathbf{R}$ defined by $\rho(s, t) = \rho_{P(s)}(s, t)$ is continuous, where $\mathbf{R}$ denotes the real line.

We call such a family $\{ \rho_x \}$ an *invariant family* for an action of a groupoid G on S if the metrics ρ_x are *isometric with respect to the action of* G, that is, for all $x, y \in X$, $\alpha \in G(x, y)$ and $s, t \in P^{-1}(x)$, we have the equality

$$\rho_x(s, t) = \rho_y(\alpha . s, \alpha . t).$$

As usual, we say that a groupoid G is *transitive* if $G(x, y)$ is a non-empty set for all x and y. This condition amounts to requiring that $\pi : \pi'^{-1}(x) \to X$ be surjective for each $x \in X$.

LEMMA 1. *Let G be a transitive compact Hausdorff topological groupoid and let $x \in X$. Then the initial map $\pi : \pi'^{-1}(x) \to X$ is open.*

PROOF. There is a natural action of the group $G(x,x)$ on the left of $\pi'^{-1}(x)$ determined by the composition law in G and the orbit map of this action coincides with π. Since π is a quotient map under our present hypothesis, it is open, see [4].

Using Lemma 1 we can now prove:

LEMMA 2. *Suppose G is a transitive compact Hausdorff topological groupoid with object space X and G acts on the fibre space $P : S \to X$. Suppose ρ_x for each $x \in X$ is a metric on $P^{-1}(x)$ which is compatible with the subspace topology on $P^{-1}(x)$, and the ρ_x are isometric with respect to the action of G. Then $\{\rho_x\}$ is an invariant family of metrics for S.*

PROOF. We must show that $\rho : S \times_P S \to \mathbf{R}$ is continuous.

Let $\mathcal{C}$ be a filter base in $S \times_P S$ converging to (s_0, t_0) and let $P(s_0, t_0) = x_0$. There is no loss of generality involved in supposing that each element $C \in \mathcal{C}$ contains (s_0, t_0). Let $\mathfrak{B}' = P(\mathcal{C})$, then $\mathfrak{B}'$ is a filter base in X converging to x_0 and x_0 belongs to each element of $\mathfrak{B}'$. Now define $\mathfrak{B}$ by $\mathfrak{B} = \{\mathfrak{U} \cap \pi^{-1}(M')\}$, where $\mathfrak{U}$ is a neighbourhood of I_{x_0} in $\pi'^{-1}(x_0)$, where $M' \in \mathfrak{B}'$ and $M' \subset \pi(\mathfrak{U})$. Using lemma 1, we see that $\mathfrak{B}$ is a filter base in $\pi'^{-1}(x_0)$ converging to the identity I_{x_0} and, furthermore, I_{x_0} belongs to each element of $\mathfrak{B}$, see also [2, chap. 1, § 7, No. 6, Proposition 11]. Moreover, since we insist that $M' \subset \pi(\mathfrak{U})$ in defining $\mathfrak{B}$, we have $\pi(\mathfrak{B}) = \mathfrak{B}'$. Evidently

$$(I_{x_0}, (s_0, t_0)) \in B \times C \text{ for each } B \in \mathfrak{B} \text{ and } C \in \mathcal{C}$$
$$\text{and } (I_{x_0}, (s_0, t_0)) \in \pi'^{-1}(x_0) \times_P (S \times_P S).$$

Accordingly, we can take the restriction $\mathfrak{B} \times_P \mathcal{C}$ of the product filter base $\mathfrak{B} \times_P \mathcal{C}$ to the subspace $\pi'^{-1}(x_0) \times_P (S \times_P S)$. Then $\mathfrak{B} \times_P \mathcal{C}$ is a filter base which converges to $(I_{x_0}, (s_0, t_0))$, and so its image $(\,.\,)(\mathcal{B} \underset{P}{\times} \mathcal{C})$ under the action of G is a filter base in $P^{-1}(x_0) \times P^{-1}(x_0)$ converging to $I_{x_0} \cdot (s_0, t_0) = (s_0, t_0)$. Thus

$$\rho_{x_0}((\,.\,)(\mathcal{B} \underset{P}{\times} \mathcal{C})) = \rho((\,.\,)(\mathcal{B} \underset{P}{\times} \mathcal{C}))$$

is a filter base in $\mathbf{R}$ converging to $\rho(s_0, t_0)$.

Let V be any neighbourhood of $\rho(s_0, t_0)$ in $\mathbf{R}$. Then there is a set

$$B \underset{P}{\times} C \in \mathcal{B} \underset{P}{\times} \mathcal{C} \text{ such that } \rho_{x_0}((\,.\,)(B \underset{P}{\times} C)) \subset V,$$

and it is not difficult to see that we can find such a set $B \underset{P}{\times} C$ with the additional property that $\pi(B) = P(C)$. However, for such a set the hypothesis that the metrics ρ_x be isometric with respect to the action of G yields the relation

$$\rho_{x_0}((\,.\,)\ (B \underset{P}{\times} C)) = \rho(C).$$

Thus, $\rho(C) \subset V$ and $C \in \mathcal{C}$. This shows that the filter base $\rho(\mathcal{C})$ converges to $\rho(s_0, t_0)$ in $\mathbf{R}$ and, hence, that ρ is continuous.

This completes the proof of the lemma.

We can now prove our main result:

THEOREM. *Let G be a transitive compact Hausdorff topological groupoid with object space X acting on the fibre space $P: S \to X$, and suppose $\{\rho_x\}$ is a family of metrics for S. Then there exists an invariant family $\{\delta_x\}$ of metrics for S.*

PROOF. Let $\{m, \mu, \mu_x\}$ be a Haar system for G and, for each $x \in X$, define δ_x by

$$\delta_x(s,t) = \int_{\pi'^{-1}(x)} \rho_{\pi(\alpha)}(\alpha^{-1} \cdot s, \alpha^{-1} \cdot t)\, d\mu_x(\alpha)$$

$$= \int_{\pi'^{-1}(x)} \rho(\alpha^{-1} \cdot s, \alpha^{-1} \cdot t)\, d\mu_x(\alpha).$$

Since the integrand is continuous, $\pi'^{-1}(x)$ is compact and μ_x is non-trivial, δ_x is a metric on $P^{-1}(x)$. It is an immediate consequence of the invariance property (iii) of a Haar system that the metrics δ_x are isometric with respect to the action of G. It will follow from the lemma 2 that $\{\delta_x\}$ is an invariant family of metrics for S, as soon as we have shown that ρ_x and δ_x are equivalent for each $x \in X$.

So it remains to show that ρ_x and δ_x are equivalent. To do this, let $\{s_n\}$ be a sequence in $P^{-1}(x)$ such that $\{s_n\}$ converges in δ_x to $s \in P^{-1}(x)$, that is, $\delta_x(s_n, s) \to 0$, and suppose $\rho_x(s_n, s)$ does not converge to zero. Then there exists a positive number ε and a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\rho_x(s_{n_k}, s) > \varepsilon$ for all k. But $\delta_x(s_{n_k}, s) \to 0$ and so by a well-known theorem of F. Riesz [1, Page 69] there is a subsequence $\{s_{n_{k_m}}\}$ of $\{s_{n_k}\}$ such that

$$\rho_x(\alpha^{-1} . s_{n_{k_m}}, \alpha^{-1} . s) \to 0$$

for μ_x almost all α. Since each α acts as a homeomorphism, we can take $\alpha = I_{x_0}$ to obtain a contradiction to the inequality

$$\rho_x(s_{n_{k_m}}, s) > \varepsilon.$$

Conversely, suppose $\rho_x(s_n, s) \to 0$, then

$$\rho_x(\alpha^{-1} . s_n, \alpha^{-1} . s) \to 0 \text{ for each } \alpha .$$

Since $\{s_n\} \cup \{s\}$ is a compact subset of $P^{-1}(x)$ and ρ is continuous, the function $\rho(\alpha^{-1} . t, \alpha^{-1} . s)$ is bounded for $t \in \{s_n\} \cup \{s\}$ and $\alpha \in \pi'^{-1}(x)$. Therefore $\delta_x(s_n, s) \to 0$ by the dominated convergence theorem of Lebesgue.

Thus, δ_x and ρ_x are equivalent metrics for each $x \in X$ and the proof is complete.

We remark that the metrics here can be replaced by inner products (on Euclidean fibre spaces) and entirely analogous results obtained. We leave details to the reader.

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