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**FREE, ITERATIVELY CLOSED CATEGORIES OF  
 COMPLETE LATTICES**

by Mitchell WAND

Let  $CL$  denote the cartesian closed category of complete lattices, with morphisms continuous over directed chains. It is well-known [5] that there is a morphism  $Y \in CL(M^M, M)$  whose underlying map takes each continuous function  $f \in CL(M, M)$  to the least  $x \in M$  such that  $f(x) = x$ . Let  $T$  be an algebraic theory and  $A: T \rightarrow CL$  a faithful product-preserving functor. We say that  $A$  is *iteratively closed* iff for each  $t \in T(n+m, m)$  there exists a (unique)  $\mu(t) \in T(n, m)$  such that  $A(\mu(t)) = Y \hat{A} t$  (where  $\hat{\phantom{a}}$  denotes exponentiation). The existence of iteratively closed  $A$  is also well-known, and these structures are important for formal language theory and other theoretical areas in computer science [8,6]. Our object in this Note is to construct free iteratively closed algebras.

**1. Definitions and notations.**

We will regard a theory  $T$  as a category whose set of objects is  $\omega$  and in which the object  $n$  is the  $n$ -fold product of the object  $1$ . Theories form a category  $Tb$  when equipped with product-preserving functors as morphisms.

We denote by  $RS$  («ranked sets») the category  $(Sets, \omega)$  of maps  $r: \Omega \rightarrow \omega$  and rank-preserving maps  $b: \Omega \rightarrow \Omega'$ . If  $r: \Omega \rightarrow \omega$  is a ranked set, we use  $\Omega_n$  to denote  $r^{-1}(n)$ . There is a forgetful functor  $V: Tb \rightarrow RS$  sending  $T$  to  $r: \Omega \rightarrow \omega$  where  $\Omega_n = T(n, 1)$ . As is well-known,  $V$  has a left adjoint, the free-theory functor.

Let  $\mu Tb$  denote the category whose objects are iteratively-closed faithful product-preserving functors, and with morphisms from  $A: T \rightarrow CL$  to  $A': T' \rightarrow CL$  precisely those morphisms  $b: T \rightarrow T'$  such that

$$b(\mu(t)) = \mu(b(t)) \quad \text{for each morphism } t \in T.$$

Let  $V$  also denote the forgetful functor  $V: \mu Tb \rightarrow RS$ . Our main result is

that  $V$  has a left adjoint.

Since  $A$  is faithful, we can enrich  $T$  over the category of posets by setting

$$t \leq t' \text{ in } T(n, m) \quad \text{iff} \quad At \leq At' \text{ in the lattice } [An \rightarrow Am].$$

If  $t \in T(k+n, n)$  then  $Y. \hat{A}t$  is given as follows :

$$\text{let } t_0 = \perp \in T(k, n), \text{ let } t_{p+1} = t.(1, t_p); \text{ then } Y. \hat{A}t = \cup At_p.$$

PROPOSITION 1.1. *If  $A$  is iteration-closed, then  $\mu(t) = \cup t_p$ .*

PROOF. If  $A$  is iteration-closed,  $\perp \in T(k, n)$ , so the  $t_p$  are well-defined.  $A(\mu(t)) = \cup A(t_p)$ , by definition. So  $A(\mu(t)) \geq At_p$  for each  $p$ , whence  $\mu(t) \geq t_p$ . Assume that for all  $p$ ,  $u \geq t_p$ . Then

$$Au \geq At_p \quad \text{and} \quad Au \geq \cup A(t_p).$$

So

$$Au \geq A(\mu(t)), \quad \text{and} \quad u \geq \mu(t).$$

So  $\mu(t)$  is a least upper bound for  $\{t_p\}$ . ■

## 2. Lattice-theoretic preliminaries.

In constructing the free theory over  $\Omega$  the most important step is finding the smallest set  $X$  such that

$$X \cong \coprod \{X^r(s) \mid s \in \Omega\}.$$

In that case, the solution was the set of finite  $\Omega$ -trees.

In our case, we seek a lattice  $L$  such that

$$L \cong \coprod \{L^r(s) \mid s \in \Omega\}.$$

In general, we may obtain solutions to such fixed-point equations as a series [4,2], but in this case we can obtain a tractable representation as a lattice of (possibly infinite) trees. We restate the theorem here for precision and completeness. Recall that for any set  $S$ , we denote by  $S^*$  the (underlying set of the) free monoid generated by  $S$ ; we denote the identity of  $S^*$  by  $e$ .

THEOREM 2.1 [3]. *Let  $r: \Omega \rightarrow \omega$  be a ranked set, and let  $D_\Omega$  denote the lattice  $\coprod \{s \mid s \in \Omega\}$ . Let  $L_\Omega = L$  be the set of functions  $t: \omega^* \rightarrow D_\Omega$  sat-*

isfying the following conditions:

- (Ranking) If  $t(w) = s$  and  $r(s) = n$ , then  $t(wj) = \perp$  for all  $j \geq n$ ;
- (Truncation) If  $t(w) \in \{\perp, \top\}$ , then  $t(wx) = t(w)$  for all  $x \in \omega^*$ .

Let  $t \leq t'$  iff  $t(w) \leq t'(w)$  for all  $w$ . Then  $L$  is a complete lattice and

$$L_\Omega \cong \coprod \{L_\Omega^{r(s)} \mid s \in \Omega\}.$$

PROOF. Figure 2.1 illustrates the lattice  $D_\Omega$ . To see that  $L$  is a complete

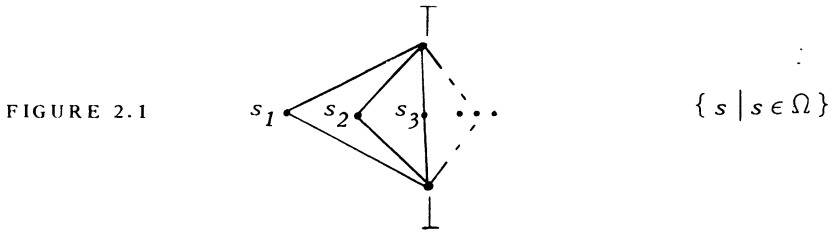


FIGURE 2.1

lattice, let  $\{t_\alpha \mid \alpha \in I\} \subseteq L_\Omega$ . Define  $t(w)$  by induction:

- (i)  $t(e) = \cup \{t_\alpha(e) \mid \alpha \in I\}$ ,
- (ii)  $t(w) = \begin{cases} \top & \text{if } t(w) = \top \\ \perp & \text{if } t(w) = \perp \\ \cup \{t_\alpha(w) \mid \alpha \in I\} & \text{otherwise} \end{cases}$ 
  - (ii-a)
  - (ii-b)
  - (ii-c)

We claim  $t$  is the least upper bound in  $L$  of the  $t_\alpha$ .

First of all,  $t \in L$ ; clauses (ii-a) and (ii-b) force  $t$  to satisfy the truncation axiom, and it is easy to verify that the ranking axiom holds also: if  $t(w) = s \in \Omega_n$ , then it must be that  $t_\alpha(w) \leq s$  for every  $\alpha$ , and hence:

$$t_\alpha(wj) = \perp \text{ for } j \geq n.$$

For each  $\alpha$ ,  $t_\alpha \leq t$ , by an easy induction on  $w$ . To show  $t$  is the least upper bound, assume that

$$t_\alpha \leq u \text{ for all } \alpha.$$

Then  $t(e) \leq u(e)$ . Assume  $t(w) \leq u(w)$ . If  $t(w) = \top$ , then

$$u(w) = \top, \text{ hence } \top = t(w) \leq u(w) = \top.$$

If  $t(w) = \perp$ , then  $t(w) = \perp \leq u(w)$ . Else

$$t(w) = \cup \{t_\alpha(w) \mid \alpha \in I\} \leq u(w).$$

So  $t \leq u$ .

Last, we must show

$$L \cong \coprod \{ L^{\tau(s)} \mid s \in \Omega \}.$$

The lattice on the right-hand side consists of the following components :

- (i) a bottom  $\perp$ ,
- (ii) a top  $\top$ ,
- (iii) for each  $s \in \Omega$ , a copy of  $L^{\tau(s)}$ . (See figure 2.2)

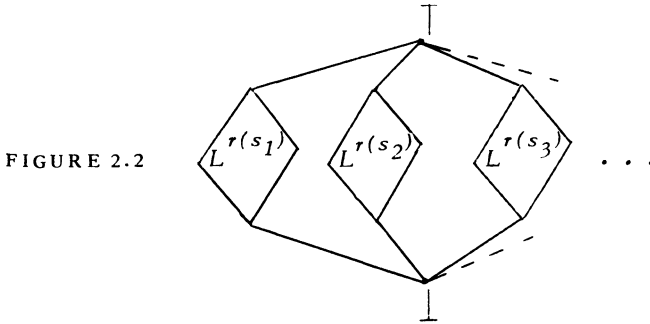


FIGURE 2.2

Define a map  $l: L \rightarrow \coprod L^{\tau(s)}$  as follows :

- (i) if  $t(e) = \perp$ ,  $l(t) = \perp$ ,
- (ii) if  $t(e) = \top$ ,  $l(t) = \top$ ,
- (iii) if  $t(e) = s \in \Omega$  and  $\tau(s) = n$ ,  $l(t) = (t_1, \dots, t_n)$  in the  $s$ -th component of  $\coprod L^{\tau(s)}$ , where  $t_i(w) = t(iw)$ .

This is clearly a bijection and  $l$  is clearly continuous over directed chains. ■

$L_\Omega$  is in fact an initial object in an appropriate category of solutions to [2]

$$D \cong \coprod \{ D^{\tau(s)} \mid s \in \Omega \}.$$

We view  $t$  as defining a countable labelled tree, truncated at nodes labelled  $\perp$  or  $\top$ . We say  $t$  is finite iff  $\{ w \mid t(w) \in \Omega \}$  is finite. We use  $E(\Omega)$  to denote the underlying set of  $L_\Omega$ .

$L_\Omega$  is a subset of  $D_\Omega^{\omega^*}$  but not, in general, a sublattice. However, we have :

LEMMA 2.2. Let  $u_0 \leq u_1 \leq \dots$  be an increasing chain in  $L_\Omega$ . Then

$$(\cup_i) (w) = \cup_i (u_i(w)).$$

PROOF.  $(\cup u_i)(e) = \cup u_i(e)$ ; we calculate  $(\cup u_i)(wn)$  from the definition. If  $(\cup u_i)(w) = \top$ , then  $(\cup u_i)(wn) = \top$ . But

$$(\cup u_i)(w) = \cup u_i(w) = \top$$

implies that  $u_j(w) = \top$  for some  $j$ . Hence

$$u_j(wn) = \top \quad \text{as well, and} \quad \cup u_i(wn) = \top.$$

Similarly, if  $(\cup u_i)(w) = \perp$ , then  $u_i(w) = \perp$  for all  $i$ , and  $u_i(wn) = \perp$  for all  $i$ . So

$$(\cup u_i)(wn) = \perp = \cup u_i(wn). \quad \blacksquare$$

COROLLARY. Let  $\{u_i\}$  be an increasing chain as before. If  $s \in \Omega$ ,

$$(\cup u_i)^{-1}(s) = \cup u_i^{-1}(s).$$

Furthermore, if  $u_i^{-1}(\top) = \emptyset$ , then  $(\cup u_i)^{-1}(\top) = \emptyset$ .  $\blacksquare$

If  $r: \Omega \rightarrow \omega$  is a ranked set, let  $\Omega + X$  denote the ranked set

$$r' = \langle r, X \rightarrow \{0\} \hookrightarrow \omega \rangle : \Omega \amalg X \rightarrow \omega,$$

that is, the ranked set consisting of  $\Omega$  with the members of  $X$  adjoined as new 0-ary elements ( $r'(X) = 0$ ). We will write the new elements of  $\Omega + k$  ( $k = \{0, 1, \dots, k-1\}$ ) as  $x_1, \dots, x_k$ .

Consider the theory  $TE_\Omega$  constructed as follows:

$$TE_\Omega(k, 1) = E(\Omega + k), \quad TE_\Omega(k, n) = [E(\Omega + k)]^n,$$

with the following composition rule:

If  $t \in TE_\Omega(n, 1)$  and  $\langle u_1, \dots, u_n \rangle \in TE_\Omega(k, n)$ , then

$$t \circ \langle u_1, \dots, u_n \rangle (w) = \begin{cases} u_i(v) & \text{if } w = zv \text{ and } t(z) = x_i \\ t(w) & \text{otherwise.} \end{cases}$$

Note that since  $r(x_i) = 0$ ,  $t(zw') = \perp$  for all  $w' \neq e$ , and hence  $z$  is unique if it exists.

LEMMA 2.3. Let  $v = t \circ \langle u_1, \dots, u_n \rangle$  in  $TE_\Omega(k, 1)$ . Then for  $s \in \Omega$

$$v^{-1}(s) = t^{-1}(s) \cup \bigcup_{i=1}^n t^{-1}(x_i)u_i^{-1}(s),$$

$$v^{-1}(x_j) = \bigcup_{i=1}^n t^{-1}(x_i)u_i^{-1}(x_j). \quad \blacksquare$$

Let  $AE_\Omega: TE_\Omega \rightarrow CL$  be the product-preserving functor given by :  
 $AE_\Omega(1) = L_\Omega$  and  $AE_\Omega(t): (u_1, \dots, u_n) \mapsto t. \langle u_1, \dots, u_n \rangle$ .

All of our theories will be subtheories of  $TE_\Omega$  and our algebras will be constructed as composites

$$T \longrightarrow TE_\Omega \xrightarrow{AE_\Omega} CL.$$

We will usually delete the  $AE_\Omega$  and write  $A: T \rightarrow TE_\Omega \rightarrow CL$ .

Now,  $TE_\Omega(k, n)$  has a lattice structure imposed upon it by  $L_{\Omega+k}^n$ . Thus  $AE_\Omega$ , acting on  $TE_\Omega(k, n)$ , is a function whose domain and codomain are both underlying sets of lattices. In fact,  $AE_\Omega$  is (the underlying map of) a morphism of lattices:

LEMMA 2.4. For any  $k, n$ ,

$$AE_\Omega: L_{\Omega+k}^n \longrightarrow [L_\Omega^k \longrightarrow L_\Omega^n]$$

is a continuous morphism of lattices.

PROOF. It will clearly suffice to show the case  $n = 1$ . So let  $u_0 \leq u_1 \leq \dots$  be a chain in  $L_{\Omega+k}$ . We want to show that for any  $t_1, \dots, t_k \in L_\Omega$ ,

$$(\cup u_i). \langle t_1, \dots, t_k \rangle = (\cup u_i. \langle t_1, \dots, t_k \rangle).$$

We refer to these two objects as  $F$  and  $G$ , and we must show that, for all  $w \in \omega^*$ ,  $F(w) = G(w)$ . We can calculate  $F(w)$  from the definition as follows:

$$(\cup u_i). \langle t_1, \dots, t_k \rangle (w) = \begin{cases} t_j(v) & \text{if } (\exists z) ((\cup u_i)(z) = x_j \ \& \ w = zv) \\ u_i(w) & \text{otherwise.} \end{cases}$$

If for some  $z$  and  $j$ ,  $(\cup u_i)(z) = x_j$ , then for every  $i$ ,  $u_i(z) = \perp$  or  $x_j$ . If  $u_i(z) = x_j$ , then  $u_i. \langle t_1, \dots, t_k \rangle = t_j(v)$ . If  $u_i(z) = \perp$  and if  $\alpha$  is a proper initial segment of  $z$ , then  $u_i(\alpha) \leq (\cup u_i)(\alpha) \in \Omega$ . Hence for no  $\alpha$  and no  $j$  is  $u_i(\alpha) = x_j$ . So  $u_i. \langle t_1, \dots, t_k \rangle = u_i(w)$ , which is  $\perp$  by the truncation rule. So  $G(w) = t_j(v) = F(w)$ .

Now assume that for no  $j$  is there a  $z$  such that  $(\cup u_i)(z) = x_j$ . Then

$$G(w) = \cup \{ u_i(w) \mid \sim (\exists z) (\exists j) [ u_i(z) = x_j \ \& \ w = zv ] \} \\ \cup \{ t_j(v) \mid (\exists z) [ w = zv \ \& \ u_i(z) = x_j ] \}.$$

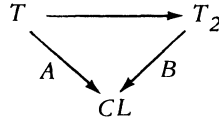
Since the  $u_i$  form a nondecreasing chain and for every  $j$  and initial segment  $z$  of  $w$ ,  $\cup u_i(z) \neq x_j$ , if the second set is nonempty there must be some  $i$  such that  $u_i(z) > x_j$ . Hence  $u_i(z) = \top$ . So  $u_i(w) = \top$ . Hence

$$G(w) = \cup u_i(w) = F(w). \quad \blacksquare$$

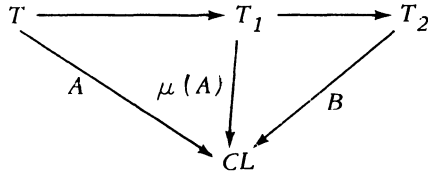
**3. Main Theorems.**

The first theorem asserts that iteration-closures exist.

**THEOREM 3.1.** *If  $A: T \rightarrow CL$  is a product-preserving functor, there exists an object  $\mu(A): T_1 \rightarrow CL$  of  $\mu Th$  and a  $Th$ -morphism  $T \rightarrow T_1$  such that, if  $B: T_2 \rightarrow CL$  is any other object of  $\mu Th$  satisfying*



then  $T_1$  is a subtheory of  $T_2$  and



Furthermore, if  $A$  is faithful, so is  $T \rightarrow T_1$ .

**PROOF.**  $T_1(n, k)$  will be a subset of  $CL(An, Ak)$ , given inductively as follows:

$$\begin{aligned} T_{(0)}(n, k) &= AT(n, k), \\ T_{(i+1)}(n, k) &= AT(n, k) \cup \{ tt' \mid t \in T_{(i)}(m, k), t' \in T_{(i)}(n, m) \} \\ &\quad \cup \{ \langle t_1, \dots, t_k \rangle \mid t_j \in T_{(i)}(m, 1) \} \\ &\quad \cup \{ Y. \hat{t} \mid t \in T_{(i)}(n+k, k) \}. \end{aligned}$$

Then  $T_1(n, k) = \cup T_{(i)}(n, k)$  clearly has the required properties.

**LEMMA.** *For any  $A: T \rightarrow CL$  the constant function  $\perp$  is a morphism of  $\mu(A)$ .*

**PROOF.**  $\perp = \mu(id_1)$ .

Let  $T_\Omega$  be the free theory generated by  $\Omega$ . There is an obvious inclusion  $T_\Omega \rightarrow TE_\Omega$ , since  $T_\Omega$  consists precisely of finite trees in which



$\perp$  and  $\top$  do not appear, except for  $\perp$ 's introduced by truncation. This observation enables us to reach our main theorem.

**THEOREM 3.2.** *The forgetful functor  $V: \mu Th \rightarrow RS$  has a left adjoint.*

**PROOF.** The object function is calculated as follows: Given  $\Omega \in RS$ , construct  $T_\Omega \rightarrow TE_\Omega \rightarrow CL$ . Then the desired object is the closure of this functor. Call this object  $B_\Omega: TB_\Omega \rightarrow CL$ . We must then show that for any  $A \in \mu Th$  any  $RS$ -morphism  $b: \Omega \rightarrow VA$  extends uniquely to a  $\mu Th$ -morphism

$$b^*: B_\Omega \rightarrow A \quad \text{such that} \quad \begin{array}{ccc} \Omega & \longrightarrow & VB_\Omega \\ & \searrow b & \downarrow Vb^* \\ & & VA \end{array}$$

Let  $A: T \rightarrow CL$  be any object of  $\mu Th$  and  $b: \Omega \rightarrow VA = VT$  be an  $RS$ -morphism. Hence  $b$  extends uniquely to a  $Th$ -morphism  $b'$  satisfying:

$$\begin{array}{ccc} \Omega & \longrightarrow & VT_\Omega \\ & \searrow b & \downarrow Vb' \\ & & VT \end{array}$$

Now for  $t \in TB_\Omega(n, k)$  let

$$b^*(t) = \cup \{ b^* t_i(\perp) \mid t_i \in T_\Omega(p, k) \ \& \ (\exists u \in B_\Omega(n, p)) [t = t_i \cdot u] \};$$

$b^*$  is easily shown to be a  $Th$ -morphism, and continuous on the (enriched) morphism sets. To show  $b^*$  preserves iteration-closure, let  $t \in TB_\Omega(k, n)$ .

Recalling Proposition 1.1, set

$$t_0 = \perp, \quad t_{p+1} = t \cdot (I, t_p)$$

and then  $\mu(t) = \cup t_p$ . However, we then have

$$b^*(t_0) = \perp, \quad b^*(t_{p+1}) = b^*(t) \cdot (I, b^*(t_p))$$

and

$$b^*(\mu(t)) = b^*(\cup t_p) = \cup b^*(t_p) = \mu(b^*(t)).$$

Furthermore, by the proof of Theorem 3.1,  $b^*$  is clearly unique. ■

#### 4. Characterization of $TB_\Omega$ .

Our last result is a characterization of  $TB_\Omega$ . Again, let  $S^*$  denote

the free monoid generated by  $S$ . We say  $G \subseteq S^*$  is *recognizable* iff there exists a finite monoid  $M$ , a monoid homomorphism  $Q: S^* \rightarrow M$  and a subset  $F \subseteq M$  such that  $G = Q^{-1}(F)$ . We say  $t \in L_\Omega$  is *rational* iff:

- (i)  $\{s \in \Omega \mid t^{-1}(s) \neq \emptyset\}$  is finite,
- (ii)  $t^{-1}(\epsilon) = \emptyset$ ,
- (iii)  $(\forall s \in \Omega)(t^{-1}(s))$  is a recognizable subset of  $\omega^*$ .

$T B_\Omega$  may now be characterized as follows:

**THEOREM 4.1.**  *$T B_\Omega$  is the subtheory of  $T E_\Omega$  consisting of the rational trees.*

The proof is a tedious but comparatively straightforward exercise in automata theory, relying heavily on Lemma 2.3. Since quite similar results have appeared elsewhere [8,7,1], we will forego reproducing the proof here.

While this characterization of the iteration-closure of

$$T_\Omega \longrightarrow T E_\Omega \longrightarrow CL$$

was known, its freeness was not. One may then apply the triangular identities to find identities in  $\mu T b$ . These, in turn, yield a number of interesting results, primarily in the area of formal languages [8,7] and the semantics of programming languages.

**References.**

1. BEKIC H., Definable operations in general algebras and the Theory of Automata and Flowcharts, *I.B.M. Vienna*, 1969.
2. REYNOLDS J.C., Notes on a lattice-theoretical approach to the Theory of Computation, *Dept. of Systems & Info. Sc.*, Syracuse University, 1972.
3. SCOTT D., The lattice of flow diagrams, *Oxford U. Comp. Lab.*, Rep. PRG-3, 1970.
4. SCOTT D., Data types as lattices, *Lecture Notes, Amsterdam*, 1972.
5. TARSKI A., A lattice-theoretical fixpoint theorem and its applications, *Pacific J. of Math.* 5 (1955), 285-309.
6. WAGNER E.G., An algebraic theory of recursive definitions and recursive languages, *Proc. 3<sup>d</sup> ACM Symp. Tb. Comp.* (1971), 12-23.
7. WAND M., A concrete approach to abstract recursive definitions, *Automata, Languages & Programming* (M. Nivat ed.), North-Holland, 1973, 331-341.
8. WAND M., Mathematical foundations of Formal Language Theory. *Project MAC TR-108*, M.I.T., Cambridge, Mass. 1973.

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