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INTERNAL PRESHEAVES TOPOSES

by Marta BUNGE

ABSTRACT.

Let $\underline{E}$ be an elementary topos. The main theorem of this paper is an elementary (first-order) characterization of categories of internal presheaves in $\underline{E}$, i.e., of categories of the form $\underline{E}^{\underline{C}^o}$ for some internal category $\underline{C}$ in $\underline{E}$.

In order to state the theorem we need some notation and definitions. Let $\underline{F} \xrightarrow{f} \underline{E}$ be a geometric morphism of toposes, with $\epsilon : f^* f_* \to 1$ the counit of $f^* \dashv f_*$. For any map γ in $\underline{F}$, $\gamma^* \dashv \Pi_\gamma$; let $\epsilon^\gamma : \gamma^* . \Pi_\gamma \to 1$ be the counit of this adjunction.

An internal family $A \xrightarrow{\alpha} I$ in $\underline{F}$ is said to be $\underline{E}$-generating iff the canonical map

$$\pi_1 . \epsilon^{\alpha} . (\epsilon \times 1) : f^* f_* \Pi_\alpha (X \times A) \underset{I}{\times} A \to X$$

is an epimorphism, and it is said to be $\underline{E}$-atomic provided the two following conditions hold :

(1) for each $e : E \to f_* I$ in $\underline{E}$, the canonical map

$$\chi_E : E \underset{f_* I}{\times} f_* \Pi_{\alpha \times I} (A \times A) \to f_* \Pi_\alpha [(f^* E \underset{I}{\times} A) \times A]$$

whose twice exponential adjoint is

$$1 \times [\epsilon^{\alpha \times I} (\epsilon \times 1)] : f^* E \underset{f_* I}{\times} f^* f_* \Pi_{\alpha \times I} (A \times A) \to (f^* E \underset{I}{\times} A) \times A$$

is an isomorphism ;

(2) for each morphism $g : X \to Y$ of $\underline{F}$, if g is an epimorphism so is the induced map

$$f_* \Pi_\alpha (g \times A) : f_* \Pi_\alpha (X \times A) \to f_* \Pi_\alpha (Y \times A) \text{ in } \underline{E}.$$

THEOREM. *Let* $\underline{F} \xrightarrow{f} \underline{E}$ *be a geometric morphism of topoi. Then the following conditions are equivalent:*

(a) there exists an internal category $\underline{C}$ *in* $\underline{E}$ *and a factorization of geometric morphisms*

$$\begin{tikzcd} \underline{F} \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \underline{E}^{\underline{C}^o} \arrow[dl] \\ & \underline{E} & \end{tikzcd}$$

such that $\Phi = (\Phi^*, \Phi_*)$ *is an adjoint equivalence of categories;*

(b) there exists an internal family $A \xrightarrow{a} I$ *of* $\underline{F}$ *which is* $\underline{E}$*-generating and* $\underline{E}$*-atomic.*

This theorem generalizes one due to the author (Dissertation, Univ. of Pennsylvania, 1966) in which categories of presheaves $\underline{S}^{\underline{C}^o}$, for $\underline{S}$ the category of sets, are characterized. However, neither the statement nor any of its subsequent proofs were elementary (first order). It is by exploiting the powerful internal structure of elementary topoi (as defined and developed by F.W. Lawvere and M. Tierney in 1970) that it is possible to achieve a first order formulation and proof of a more general form of this theorem, the one given above.

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INTRODUCTION.

A locally small category $\underline{X}$ is equivalent to a category of presheaves $\underline{S}^{\underline{C}^o}$ if and only if $\underline{X}$ satisfies the following conditions :

(1) $\underline{X}$ is regular (in the sense of having finite limits and colimits and satisfying that every congruence relation is a kernel pair and every epimorphism a coequalizer);

(2) there exists a family $(A_i)_{i \in I}$ of objects of $\underline{X}$, indexed by a set I, such that:

(a) $(A_i)_{i \in I}$ is generating;

(b) each A_i is projective, i.e., $hom_{\underline{X}}(A_i, -): \underline{X} \to \underline{S}$ takes epis into onto functions ;

(c) in $\underline{X}$ arbitrary small coproducts of members of $(A_i)_{i \in I}$ exist;

(d) for each A_i, $hom_{\underline{X}}(A_i, -): \underline{X} \to \underline{S}$ preserves coproducts.

The above theorem was proved in [4]; therein, objects A satisfying (2b) and (2d) above were called atoms, by analogy with a similar situation for Boolean algebras and fields of sets. In [4], descent techniques were used to prove it and in [5] it was obtained as an application of a tripleableness theorem for adjoint triples (in the more general setting of closed categories and categories and functors relative to them). F. E. J. Linton [15] gave a short proof of the above theorem as an application of a variant of the tripleableness theorem for triples on $\underline{S}$. In [15], it is mentioned that the theorem is known also to P. Gabriel [7], but we ignore his method of proof.

When F.W. Lawvere [11] gave an elementary (i.e., first-order) characterization of the category $\underline{S}$ of sets and functions, a natural question to pose was whether one could also provide elementary characterizations of certain categories nearly as basic as $\underline{S}$, such as categories of presheaves and Grothendieck topoi.

The theorem above is not an elementary characterization because of the presence of arbitrary coproducts and their preservation in conditions (2c) and (2d). Neither is the proof given by J. Giraud [1] of his characterization of Grothendieck topoi.

An important breakthrough came about around 1970 when F.W. Lawvere and M. Tierney [10] gave an elementary set of axioms for categories called «elementary topoi», so powerful in nature that many constructions for sheaves, done with the availability of limits, became possible within topoi and in the absence of limits (external ones, that is). Still, the question remained of proving a generalization of Giraud's theorem in elementary topoi, i.e. to characterize elementary topoi of the form $Sh_j(\underline{E})$ of j-sheaves for a topology j on an elementary topos $\underline{E}$. On account of the theorem on factorizations of geometric morphisms, a topos $\underline{F}\xrightarrow{f}\underline{E}$ is equivalent to a topos $Sh_j(\underline{E})\to\underline{E}$ iff f is a bounded geometric morphism, i.e., there exists a factorization

$$\underline{F}\xrightarrow{f}\underline{E} = \underline{F}\xrightarrow{i}\underline{E}^{\underline{C}^o}\longrightarrow\underline{E}$$

in $\underline{Top}$ (i.e., of geometric morphisms) with i_* fully faithful. A conjecture of W. Mitchell [18], proven by R. Diaconescu [6], says that an $\underline{E}$-topos $\underline{F}\xrightarrow{f}\underline{E}$ is bounded iff there exists an object G in $\underline{F}$ such that for every object X of $\underline{F}$, the natural map

$$f^* f_*(\tilde{X}^G)\times G \to \tilde{X}$$

obtained using the counit of $f^*\dashv f_*$ and evaluation, is an epimorphism. Here, $\tilde{X}$ is the partial-maps-into-X classifier and the condition says that, internally, G generates all the injectives. Equivalently, the family of all subobjects of G generates everything.

The above result suggested the possibility of obtaining an elementary version of the theorem on presheaves in a similar fashion. F.W. Lawvere conjectured that a certain set of first-order conditions would be adequate for this purpose. In the process of trying to establish his claim, we came up with a definite formulation of Lawvere's conjecture and it is the one we present in this paper. Observe that if a topos $\underline{F}$ is to be compared with toposes of the form $\underline{E}^{\underline{C}^o}$, of internal presheaves on an internal category $\underline{C}$ in $\underline{E}$, we must let $\underline{F}$ be a topos «over $\underline{E}$», i.e., endowed with a structural map, a geometric morphism $\underline{F}\xrightarrow{f}\underline{E}$. The reason why this is not made explicit in the $\underline{Set}$-case is simply that a locally small category $\underline{X}$ admits a unique (up to natural isomorphism) functor $\underline{X} \to \underline{S}$ with a left exact left adjoint (assuming small coproducts exist in $\underline{X}$ as well); namely,

$$hom_{\underline{X}}(1,-): \underline{X} \to \underline{S}.$$

This is not the case for arbitrary topoi, although given a geometric morphism $F \xrightarrow{f} E$ then

$$f_* \approx \underline{F}(1,-): \underline{F} \to \underline{E},$$

where $\underline{F}(-,-)$ is the $\underline{E}$-valued hom-functor lifted according to f. Secondly, we should consider $\underline{F} \xrightarrow{f} \underline{E}$ to be $\underline{E}$-bounded, as the class of presheaves topoi is a subclass of that of bounded topoi. We know that in $\underline{S}^{\underline{C}^o}$ the family of representable functors is a generating family; bounded topoi have an object the family of whose subobjects is generating: certainly the coproduct of all representables is such an object. Yet, we shall have to work with the smaller family of the representables. For this reason we give a condition on a family (internal family) $A \xrightarrow{a} I$ of $\underline{F}$ to be generating, which is weaker than the Mitchell-Diaconescu definition but is more suitable for our purposes. Third question: how to get rid of coproducts in defining atoms, as the family in question must be, in some sense, a family of atoms. We found that this could be done by working only with $\underline{E}$-coproducts, which always exist. Preserving such is a condition which may be expressed in an elementary way. Projectivity presents no problems but it must also be made relative to $\underline{E}$ via $\underline{F} \xrightarrow{f} \underline{E}$. In both cases it is the family $A \xrightarrow{a} I$ as a whole that is considered, not its fibers. In a sense, we accomplish this by requiring that $A \xrightarrow{a} I$ be an (relative to $\underline{E}/_{f_*I}$) atom in $\underline{F}/_I$ a global condition.

The theorem we prove says that a topos $\underline{F} \xrightarrow{f} \underline{E}$ is equivalent to some $\underline{E}^{\underline{C}^o} \to \underline{E}$ iff $\underline{F}$ has an $\underline{E}$-generating and $\underline{E}$-atomic internal family.

We have profited much from the work of R. Diaconescu [6], although it has been necessary to introduce several modifications. J. Bénabou's construction (mentioned in [3]) of the full internal subcategory generated by a family in a topos, which we learned from P.T. Johnstone [8], plays an essential rôle here. Finally, as a guideline for our work we have used our own previous non-elementary version of the theorem, particularly the proof which uses descent techniques [4].

Various constructions in toposes are employed here without much of an explanation for them; these may be found in the book by A. Kock and G.C. Wraith [9] or in the Bangor lectures of G.C. Wraith [20] as well as in the various writings of F.W. Lawvere [12, 13, 14]. Category theoretical background may be acquired in B. Mitchell's [17] or in S. Mac Lane's [16] textbooks. As it seemed unreasonable to refer the reader constantly to the sources, even for the material immediately relevant to what we do here, we have been more explicit about it. Yet, only those proofs not appearing elsewhere are given here in full.

1. Π and exponentiation in a topos.

In a topos $\underline{F}$, pulling back along a morphism $a: A \to I$ is a functor $a^*: \underline{F}/_I \to \underline{F}/_A$, having a right adjoint Π_α. Denote by ϵ^a the counit of the adjointness. Then, for a given $B \xrightarrow{b} A \in \underline{F}/_A$, the morphism

$$\epsilon^\alpha_B: \Pi_\alpha B \underset{I}{\times} A \to B$$

is over A, i.e. $b.\epsilon^\alpha_B = \pi_2$ with

$$\begin{tikzcd}
\Pi_\alpha B \underset{I}{\times} A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \Pi_\alpha B \arrow[d, "\Pi_\alpha(b)"] \\
A \arrow[r, "a"] & I
\end{tikzcd}$$

a pullback. In particular, if $b = X \times A \xrightarrow{\pi_2} A$, one has

$$\epsilon^\alpha_{(X\times A)} = < ev^\alpha_x, \pi_2 > ,$$

where $ev^\alpha_X: \Pi_\alpha(X\times A) \underset{I}{\times} A \to X$ is defined to be $\pi_1 \cdot \epsilon^\alpha_{(X\times A)}$.

A topos is also a cartesian closed category ; in a topos, exponentiation is related to Π as follows. Let $X \xrightarrow{X} 1$ be the terminal map. Then, for objects X, Y of a topos $\underline{F}$, $Y^X \approx \Pi_X(Y \times X)$. Since

$$\begin{tikzcd}
Y \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "Y"] \\
X \arrow[r, "X"'] & 1
\end{tikzcd}$$

is a pullback, $Y^X \approx \Pi_X(X^*(Y))$. More generally, if $\alpha : A \to I$ and $\beta : B \to I$ are morphisms in $\underline{F}$, i.e., objects in $\underline{F}/_I$, $\beta^\alpha = \Pi_\alpha(\alpha^*(\beta))$.

The evaluation map $ev^\alpha_\beta : \beta^\alpha \times \alpha \to \beta$ is given then by

$$\pi_2 \epsilon^{\alpha}_{\alpha^*(\beta)} : \Pi_\alpha(\alpha^*(B)) \underset{I}{\times} A \to B \quad (\text{over } I\ ).$$

The map ev^α_X above is then a special case of this, hence the name «evaluation» for it. Note for this that

$$X \times A \xrightarrow{\pi_2} A \quad \text{is} \quad \alpha^*(X \times I \xrightarrow{\pi_2} I\ ).$$

2. Internal categories and internal presheaves.

The notions of an internal category and its modules make sense in in any category with pullbacks $\underline{E}$. Yet assume $\underline{E}$ to be a topos, for reasons immediately apparent.

A category $\underline{C}$ in $\underline{E}$ is given by :

(i) a pair of objects O, M of $\underline{E}$;

(ii) a pair of morphisms $\partial_0, \partial_1 : M \rightrightarrows O$ of $\underline{E}$;

(iii) a morphism $u : O \to M$;

(iv) a morphism $c : M \underset{O}{\times} M \to M$, where

$$\begin{tikzcd}
M \underset{O}{\times} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{PB}"] & M \arrow[d, "\partial_0"] \\
M \arrow[r, "\partial_1"'] & O
\end{tikzcd}$$

is a pullback, such that the following diagrams commute (indicated by the symbol $\circlearrowright$):

$$\begin{tikzcd}
O \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright" near start] \arrow[dr, phantom, "\circlearrowright" near end] & M \arrow[d, "\partial_0"] \arrow[dl, equal] \\
M \arrow[r, "\partial_1"'] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[d, "\partial_0"] \\
M \arrow[r, "\partial_0"'] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[d, "\partial_1"] \\
M \arrow[r, "\partial_1"'] & O
\end{tikzcd}$$

$$\begin{tikzcd}
O \times_O M \arrow[r, "u \times_O M"] \arrow[dr, "\pi_2"'] \arrow[r, phantom, "\circlearrowright", shift right=4ex] & M \times_O M \arrow[d, "c"] \arrow[r, phantom, "\circlearrowright", shift right=4ex] & M \times_O O \arrow[l, "M \times_O u"'] \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
M \times_O M \times_O M \arrow[r, "M \times_O c"] \arrow[d, "c \times_O M"'] \arrow[dr, phantom, "\circlearrowright"] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

An internal (contravariant) presheaf on $\underline{C}$ is given by:

(i) a morphism $Y \xrightarrow{p} O$ of E;

(ii) a morphism $M \times_O Y \xrightarrow{\theta} Y$ of $\underline{E}$, such that

$$\begin{tikzcd}
Y \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & M \times_O Y \arrow[l, "\theta"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\text{PB}"] & Y \arrow[d, "p"] \\
O & M \arrow[l, "\partial_0"] \arrow[r, "\partial_1"'] & O
\end{tikzcd}$$

$$\begin{tikzcd}
O \times_O Y \arrow[r, "u \times_O Y"] \arrow[dr, "\pi_2"'] \arrow[dr, phantom, "\circlearrowright", shift left=3ex] & M \times_O Y \arrow[d, "\theta"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \times_O Y \arrow[r, "M \times_O \theta"] \arrow[d, "C \times_O Y"'] \arrow[dr, phantom, "\circlearrowright"] & M \times_O Y \arrow[d, "\theta"] \\
M \times_O Y \arrow[r, "\theta"'] & Y
\end{tikzcd}$$

The covariant case is obtained by switching ∂_0 and ∂_1 in this definition.

For an interpretation of internal presheaves as discrete fibrations see [20] ; viewed as discrete fibrations we may regard the category of all such on $\underline{C}$, $\underline{E}^{\underline{C}^o}$, to be a full subcategory of $Cat(\underline{E})/_{\underline{C}}$, obtaining a ready-made definition of internal natural transformations. This can also be given directly, see [6].

It is well-known that, if $\underline{C}$ is a category in a topos $\underline{E}$, the composite

$$\underline{E}/_O \xrightarrow{\partial_1^*} \underline{E}/_M \xrightarrow{\Sigma_{\partial_0}} \underline{E}/_O$$

has the structure of a triple $\mathbf{T}$ and that $\underline{E}^{\underline{C}^o} \approx (\underline{E}/_O)^{\mathbf{T}}$. This triple has a right adjoint $\mathbf{G}$ by means of

$$\underline{E}/_O \xrightarrow{\partial_0^*} \underline{E}/_M \xrightarrow{\Pi_{\partial_1}} \underline{E}/_O$$

and hence $\mathbf{G}$ is a left exact cotriple with $\underline{E}^{\underline{C}^o} \approx (\underline{E}/_O)_{\mathbf{G}}$. By a theorem in [9], $\underline{E}^{\underline{C}^o}$ is then a topos. Another application of this approach to categories of internal presheaves $\underline{E}^{\underline{C}^o}$ is that a theorem about triples (cf. [6]) yields Kan extensions. For a morphism $v: O \to O'$ between the objects of two categories $\underline{C}$ and $\underline{C}'$ in $\underline{E}$, the following diagram commutes, where the top maps are liftings of the bottom ones and v^* is pulling back along the map v :

$$\begin{tikzcd}
\underline{E}^{\underline{C}^o} \arrow[r, shift left=2, "v!"] \arrow[r, shift right=2, "v_*"'] \arrow[d, "U"'] & \underline{E}^{\underline{C}'^o} \arrow[l, "v^*"'] \arrow[d, "U'"] \\
\underline{E}/_O \arrow[r, shift left=2, "\Sigma_v"] \arrow[r, shift right=2, "\Pi_v"'] & \underline{E}/_{O'} \arrow[l, "v^*"']
\end{tikzcd}$$

One has that $v_! \dashv v^* \dashv v_*$. A particular instance with

$$\underline{C}' = \mathbf{1} \quad \text{and} \quad v = O \xrightarrow{O} 1$$

yields functors labelled as in the diagram :

$$\begin{tikzcd}
\underline{E}^{\underline{C}^{\circ}} \arrow[rr, shift left=2, "\varinjlim_{\underline{C}}"] \arrow[d, "U"'] & & \underline{E} \arrow[ll, "\Delta"'] \arrow[d, "id"] \\
\underline{E}/O \arrow[rr, shift left=2, "\Sigma_O"] & & \underline{E} \arrow[ll, "\Delta"'] \arrow[ll, phantom]
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{E}^{\underline{C}^{\circ}} \arrow[rr, shift right=2, "\varprojlim_{\underline{C}} = \Gamma"'] & & \underline{E} \\
\underline{E}/O \arrow[rr, shift right=2, "\Pi_O"'] & & \underline{E}
\end{tikzcd}$$

.

It follows that the pair $\Delta \dashv \Gamma$ is a geometric morphism $\underline{E}^{\underline{C}^{\circ}} \to \underline{E}$; in fact Δ even has a left adjoint. We shall omit giving this canonical geometric morphism its label.

For future applications, let us record here the following remarks about the definition of the functors in the relationship $\varinjlim_{\underline{C}} \dashv \Delta \dashv \Gamma$. Given an internal presheave (Y,θ), $\Gamma(Y,\theta)$ is the object $\Pi_O(Y)$, i.e., Π along $O \xrightarrow{O} 1$ of $Y \xrightarrow{p} O$ and

$$\varinjlim_{\underline{C}}(Y,\theta) = \text{coequalizer of the pair } M \underset{O}{\times} Y \underset{\pi_1}{\overset{\theta}{\rightrightarrows}} Y \,.$$

For an object X of $\underline{E}$, $\Delta X = (X \times O,\ X \times \partial_0)$, a presheaf partly by virtue of the diagrams

$$\begin{tikzcd}
X \times O \arrow[d, "\pi_2"'] & X \times M \arrow[l, "X \times \partial_0"'] \arrow[r, "X \times \partial_1"] \arrow[d, "\pi_2"] \arrow[dl, phantom, "\text{PB}"] \arrow[dr, phantom, "\text{PB}"] & X \times O \arrow[d, "\pi_2"] \\
O & M \arrow[l, "\partial_0"] \arrow[r, "\partial_1"'] & O
\end{tikzcd}$$

which being both pullbacks define also

$$\Delta' : \underline{E} \to \underline{E}^{\underline{C}} \qquad \text{with} \qquad \Delta'(X) = (X \times O\ ,\ X \times \partial_1).$$

3. Toposes over $\underline{E}$ as $\underline{E}$-closed categories with $\underline{E}$-coproducts.

A topos $\underline{F}$ is said to be a topos over another topos $\underline{E}$ if there is a geometric morphism $\underline{F} \xrightarrow{f} \underline{E}$, i.e., a pair

$$\begin{tikzcd}
\underline{F} \arrow[r, shift right, "f_*"'] & \underline{E} \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

with $f^* \dashv f_*$ and f^* left exact. We let (η, ϵ) be the (unit, counit) of the adjointness. It has been shown by W. Mitchell [19] that such a set-up (in fact he shows that one needs only that f^* preserve finite products and 1) endows $\underline{E}$ with the structure of an $\underline{E}$-category, i.e., there is an $\underline{E}$-valued hom-functor $\underline{F}(-,-)$ such that

$$\begin{tikzcd}
\underline{F}^o \times \underline{F} \arrow[r, "\underline{F}(-,-)"] \arrow[dr, "hom_{\underline{F}}(-,-)"'] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & \underline{E} \arrow[d, "hom_{\underline{E}}(1,-)"] \\
 & \underline{S}
\end{tikzcd}$$

up to natural isomorphism. Even if f^* is not left exact, we may think of $\underline{F}(-,-)$ as a sort of $\underline{E}$-valued hom-functor, although it may not lift the ordinary $hom_{\underline{F}}(-,-)$. The definition on objects is

$$\underline{F}(X,Y) = f_*(Y^X).$$

Note that one has a composition induced from the internal composition: $Y^X \times Z^Y \to Z^X$ by applying f_*, which is left exact. Also, $1 \xrightarrow{\ulcorner 1_X \urcorner} X^X$ gives the unit by the same process. The usual laws then follow.

For each X object of $\underline{F}$, the functor $\underline{F}(X,-) : \underline{F} \to \underline{E}$ has a left adjoint, denoted $-\otimes X : \underline{E} \to \underline{F}$, also in the case of a simple adjoint pair $f^* \dashv f_*$; let $E \otimes X = f^*E \times X$. We have, moreover, that

$$f_* \approx \underline{F}(1,-) \quad \text{and} \quad f^* \approx -\otimes 1 .$$

One should think of $E \otimes X$ as the «coproduct of X with itself over the $\underline{E}$-object E» and this is reasonable on account of the natural isomorphism

$$hom_{\underline{F}}(E \otimes X, Y) \approx hom_{\underline{E}}(E, \underline{F}(X,Y)).$$

We shall have the opportunity to apply the above point of view to the adjoint pair

$$\begin{tikzcd} \underline{F}/_I \arrow[r, shift right, "\bar{f}_*"'] & \underline{E}/_{f_*I} \arrow[l, shift right, "\bar{f}^*"'] \end{tikzcd}$$

described by :

$$\bar{f}_*\left(\begin{tikzcd}[column sep=small] X \arrow[rr, "\gamma"] \arrow[dr, "p"'] & {} \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "q"] \\ & I & \end{tikzcd}\right) = \begin{tikzcd} f_*X \arrow[rr, "f_*\gamma"] \arrow[dr, "f_*p"'] & {} \arrow[d, phantom, "\circlearrowleft"] & f_*Y \arrow[dl, "f_*q"] \\ & f_*I & \end{tikzcd}$$

and by :

$$\bar{f}^*\left(\begin{tikzcd}[column sep=small] E \arrow[rr, "\delta"] \arrow[dr, "e"'] & {} \arrow[d, phantom, "\circlearrowleft"] & E' \arrow[dl, "e'"] \\ & f_*I & \end{tikzcd}\right) = \begin{tikzcd}[column sep=small] f^*E \arrow[rrrr, "f^*\delta"] \arrow[dr, "f^*e"'] & & & & f^*E' \arrow[dl, "f^*e'"] \\ & f^*f_*I \arrow[dr, "\epsilon"'] & \circlearrowleft & f^*f_*I \arrow[dl, "\epsilon"] & \\ & & I & & \end{tikzcd}$$

We leave to the reader to show adjointness. Note also that the above pair is obtainable as

$$\begin{tikzcd} \underline{F}/_I \arrow[r, shift right, "\epsilon^*"'] & \underline{F}/_{f^*f_*I} \arrow[l, shift right, "\Sigma_\epsilon"'] \arrow[r, shift right, "\bar{\bar{f}}_*"'] & \underline{E}/_{f_*I} \arrow[l, shift right, "\bar{\bar{f}}^*"'] \end{tikzcd}$$

where $(\bar{\bar{f}}^*, \bar{\bar{f}}_*)$ is the geometric morphism which is obtained by pulling back $(\bar{f}^*, \bar{f}_*)$ along (Δ, Γ) in $\underline{Top}$ (Theorem 5.1, [6]). Of course $(\Sigma_\epsilon, \epsilon^*)$ need not be geometric and neither is $(\bar{f}^*, \bar{f}_*)$. Yet we shall think of $\underline{F}/_I$ as an $\underline{E}/_{f_*I}$-category which has $\underline{E}/_{f_*I}$-coproducts, by virtue of the above remarks.

With this interpretation, given an I-indexed family in $\underline{F}$, i. e., a morphism $A \xrightarrow{a} I$, i. e., an object of $\underline{F}/_I$, then $\underline{F}/_I(a, -)$ has a left adjoint $- \otimes a$. If $\underline{E} \xrightarrow{e} f_*I$ is an object of $\underline{E}/_{f_*I}$, $e \otimes a$ is then (since products in $\underline{F}/_I$ are pullbacks in $\underline{F}$) the appropriate diagonal in the pullback drawn here after. The interpretation of $f^*E \times A \to I$ is that of «coproduct of members of the family $A \xrightarrow{a} I$ over the $\underline{E}$-object E and according to the assignment

$$\begin{tikzcd}
f^*E \times_I A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{PB}"] & A \arrow[d, "a"] \\
f^*E \arrow[r, "\epsilon_I . f^* e"'] & I
\end{tikzcd}$$

$E \xrightarrow{e} f_* I$». It is this sort of coproducts which will replace the external coproducts of the non-elementary treatment of the subject.

4. The full internal subcategory generated by an internal family in a topos.

Let $A \xrightarrow{a} I$ be an I-indexed internal family in $\underline{F}$, i.e., a morphism of $\underline{F}$, where $\underline{F}$ is a topos. Define a category $\underline{A}$ in $\underline{F}$ and an internal inclusion $u_{\underline{A}}$ of $\underline{A}$ into $\underline{F}$ as in what follows.

Let $\underline{A}$ have I for object of objects and A_1 for object of morphisms with

$$A_1 = (I \times a)^{(a \times I)} \text{ in } \underline{F}/_{I \times I}.$$

Let us recall how exponentiation is described by means of Π in a topos (cf. Section 1):

$$\begin{tikzcd}
 & & \prod_{a \times I}(A \times A) = A_1 \arrow[dl, "<\partial_0, \partial_1>"] \\
A \times A \arrow[r, "a \times A"] \arrow[d, "A \times a"'] \arrow[dr, phantom, "\mathrm{PB}"] & I \times A \arrow[d, "I \times a"] & \\
A \times I \arrow[r, "a \times I"'] & I \times I &
\end{tikzcd}$$

this gives already the two maps

$$\partial_0, \partial_1 : A_1 \rightrightarrows I.$$

Composition may be defined by descending to $\underline{F}/_{I \times I \times I}$ (as indicated in [8]), i.e., let $A_1 \times A_1 \xrightarrow{c} A_1$ be given (modulo pullbacks which commute with exponentials) by the exponential adjoint of

$$(I \times I \times a)^{(I \times a \times I)} \times (I \times a \times I)^{(a \times I \times I)} \times (a \times I \times I) \xrightarrow{ev.(1 \times ev)} (I \times I \times a).$$

The identity then comes from

$$1 \xrightarrow{\ulcorner 1_a \urcorner} a^a \quad \text{in} \quad \underline{F}/_I .$$

These data define a category $\underline{A}$ in $\underline{F}$. We now define $u_{\underline{A}} \in |\underline{F}^{\underline{A}}|$ as

$$(A, \xi_A) \text{ with } \xi_A : A_1 \underset{I}{\times} A \to A \text{ as } \pi_2 . \epsilon^{a \times I}_{A \times A} .$$

In order to prove that

$$\begin{tikzcd}
A \arrow[d, "a"'] & A_1 \underset{I}{\times} A \arrow[l, "\xi_A"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\text{PB}"] & A \arrow[d, "a"] \\
I & A_1 \arrow[l, "\partial_1"] \arrow[r, "\partial_0"'] & I
\end{tikzcd}$$

decompose the left square as in

$$\begin{tikzcd}
\underset{a \times I}{\Pi} (A \times A) \underset{I}{\times} A \arrow[rr, "\epsilon^{a \times I}"] \arrow[dr, "\pi'_2"'] \arrow[dd, "\pi_1"'] \arrow[ddr, phantom, "\circlearrowleft"] & & A \times A \arrow[dl, "a \times I"] \arrow[dd, "\pi_2"] \arrow[dl, phantom, bend right, "\circlearrowleft"] \\
& A \times I \arrow[d, "\pi_2"] \arrow[dr, phantom, "\text{PB}"] & \\
A_1 \arrow[r, "\partial_1"] & I & A \arrow[l, "a"']
\end{tikzcd}$$

where the meaning and commutativity of the left bottom square is given by the following way of decomposing the pullback

$$\begin{tikzcd}
A_1 \underset{I}{\times} A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{PB}"] & A \arrow[d, "a"] \\
A_1 \arrow[r, "\partial_0"'] & I
\end{tikzcd}$$

as in the diagram

$$\begin{tikzcd}
A_1 \underset{I}{\times} A \arrow[r, "\pi'_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{PB}"] & A \times I \arrow[r, "\pi_1"] \arrow[d, "a \times I"] \arrow[dr, phantom, "\text{PB}"] & A \arrow[d, "a"] \\
A_1 \arrow[r, "<\partial_0 , \partial_1>"'] & I \times I \arrow[r, "\pi_1"'] & I
\end{tikzcd}$$

Proofs of the remaining facts pertaining the presheaf nature of $u_{\underline{A}}$ are left to the reader. They involve some manipulation of the ϵ, recalling they are natural.

REMARK. The construction of the category $\underline{D}$ in a topos $\underline{F}$ given in [6], internally the full subcategory of $\underline{F}$ whose objects are the subobjects of an object G, is a particular case of the above construction of Bénabou for the case the family is given by $\epsilon \xrightarrow{n} \Omega^G$ with $\epsilon \rightarrowtail^{<n,e>} \Omega^G \times G$ the subobject classified by $ev: \Omega^G \times G \to \Omega$. Similarly the functor ϵ from $\underline{D}$ to $\underline{F}$ is $u_{\underline{D}}$. This remark allows us to make use of the general framework of [6].

5. The functor $Hom_{\underline{A}}(u_{\underline{A}}, -): \underline{F} \to \underline{F}^{\underline{A}^{\circ}}$ and its left adjoint $- \otimes_{\underline{A}} u_{\underline{A}}$.

These are defined in general in [6] for any presheaf $(Y, \theta) \in |F^A|$. For the case of $u_{\underline{A}} = (A, \xi_A) \in |\underline{F}^{\underline{A}}|$ they are described below and given an alternative construction.

For X an object of $\underline{F}$,

$$Hom_{\underline{A}}(u_{\underline{A}}, X) = (\Pi_a(X \times A), \xi^A_X),$$

where

$$\begin{tikzcd}
X \times A \arrow[r, "X \times a"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\text{PB}"] & X \times I \arrow[d, "\pi_2"] & \Pi_a(X \times A) = (\Delta X)^a \arrow[dl, "j"] \\
A \arrow[r, "a"'] & I &
\end{tikzcd}$$

hence a map $\xi^A_X: \Pi_a(X \times A) \times_I A_1 \to \Pi_a(X \times A)$ is a map

$$\xi^A_X: \Delta X^a \times (I \times a)^{(a \times I)} \to \Delta X^a \text{ in } \underline{F}/_I,$$

where

$$\begin{tikzcd}
\Pi_a(X \times A) \times_I A_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\text{PB}"] & \Pi_a(X \times A) \arrow[d, "j"] \\
A_1 \arrow[r, "\partial_1"'] & I
\end{tikzcd}$$

It is explicitly described via its transpose in the diagram below :

$$\begin{tikzcd}
 & & X\times A & & & & \Pi_\alpha(X\times A) \arrow[ddd, "j"] \\
\Pi_\alpha(X\times A)\underset{I}{\times}A_1\times A \arrow[urr, "{<ev^\alpha_X(1\times\xi_A,\pi_3>}"] \arrow[rrrr] \arrow[dr, "\pi_2"'] \arrow[drrrrr, phantom, "\text{PB}"] & & & & \Pi_\alpha(X\times A)\underset{I}{\times}A_1 \arrow[urr, "\xi^A_X"] \arrow[dr, "\pi_2"'] & & \\
 & A_1\underset{I}{\times}A \arrow[rrrr] \arrow[dr, "\pi_2"'] \arrow[drrrrr, phantom, "\text{PB}"] & & & & A_1 \arrow[dr, "\partial_0"] & \\
 & & A \arrow[from=1-3, "\pi_2"', crossing over] \arrow[rrrr, "\alpha"'] & & & & I
\end{tikzcd}$$

The left adjoint is defined, for $(Y,\theta)\in\underline{F}^{\underline{A}^{\circ}}$, as in the coequalizer diagram

$$\begin{tikzcd}
Y\underset{I}{\times}A_1\underset{I}{\times}A \arrow[r, shift left, "Y\underset{I}{\times}\xi_A"] \arrow[r, shift right, "\theta\underset{I}{\times}A"'] & Y\underset{I}{\times}A \arrow[r] & (Y,\theta)\otimes_{\underline{A}}u_{\underline{A}}
\end{tikzcd}$$

REMARK. It may be of interest to note an alternative construction of the functor $Hom_{\underline{A}}(u_{\underline{A}},X)$. The object part is describable as the left vertical arrow in the pullback

$$\begin{tikzcd}
I\underset{\Omega^A}{\times}\tilde{X}^A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{PB}"] & \tilde{X}^A \arrow[d, "dom^A"] \\
I \arrow[r, "\bar{\alpha}"'] & \Omega^A
\end{tikzcd},$$

where $\bar{\alpha}$ is the exponential adjoint of the characteristic function of the graph of α , i.e. of $A\xrightarrowtail{<1,\alpha>}A\times I$. To see this, observe that the fiber above i of $(\Delta X)^\alpha$ is X^{A_i} where A_i is the fiber above i of $A\xrightarrow{\alpha}I$, i.e., $A_i=\alpha^*(i)$. On the other hand, the fiber above $I\underset{\Omega^A}{\times}\tilde{X}^A\xrightarrow{\pi_1}I$ is given by «all pairs $\left(i, \; A \leftarrowtail A' \xrightarrow{f} X\right)$ with $A' = A_i$ »,

internal version, i.e., X^{A_i}.

6. The internal family $M \xrightarrow{<\partial_1, \partial_0>} O \times O$ in $\underline{E}^{\underline{C}^o}$.

Let $\underline{E}$ be a topos, $\underline{C}$ a category in $\underline{E}$ with the data as in Section 2. Part of this data, especifically the diagrams

$$\begin{tikzcd}
M \arrow[d, "\partial_0"'] & M \underset{O}{\times} M \arrow[l, "c"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\mathrm{PB}"] & M \arrow[d, "\partial_0"] \\
O & M \arrow[l, "\partial_0"] \arrow[r, "\partial_1"'] & O
\end{tikzcd}$$

plus the equations

$$c.(u \underset{O}{\times} M) = \pi_2 \quad \text{and} \quad c.(M \underset{O}{\times} c) = c.(c \underset{O}{\times} M),$$

say, exactly, that (M, c) is an object of $\underline{E}^{\underline{C}^o}$.

We wish to consider the «family of representables» in $\underline{E}^{\underline{C}^o}$. This is given, at the level of $\underline{E}$ and internally, by the family $M \xrightarrow{\partial_1} O$. The corresponding family in $\underline{E}^{\underline{C}^o}$ is

$$(M, c) \xrightarrow{<\partial_1, \partial_0>} \Delta O = O \times O.$$

Let us examine this more carefully. We now claim that the morphism

$$<\partial_1, \partial_0> : M \longrightarrow O \times O \quad \text{of } \underline{E}$$

is a natural transformation

$$<\partial_1, \partial_0> : (M, c) \to \Delta O,$$

i.e., a morphism of $\underline{E}^{\underline{C}^o}$, thought of as an ΔO-indexed family in $\underline{E}^{\underline{C}^o}$.

Indeed, note the commutativity of the diagram here after. Noting that in it the appropriate diagrams are pullbacks, it can be shown that

$$(M \xrightarrow{<\partial_1, \partial_0>} O \times O, \; c)$$

$$\begin{tikzcd}
& O\times O & & O\times M \arrow[ll, "O\times\partial_0"'] \arrow[rr, "O\times\partial_1"] \arrow[ddl, "\pi_2"] & & O\times O \arrow[ddl, "\pi_2"] \\
M \arrow[ur, "<\partial_1{,}\partial_0>"] \arrow[d, "\partial_0"'] & & M\underset{O}{\times}M \arrow[ll, "c"'] \arrow[rr, "\pi_2"] \arrow[ur, "<\partial_1.\pi_2{,}\pi_1>"] \arrow[d, "\pi_1"'] & & M \arrow[ur, "<\partial_1{,}\partial_0>"] \arrow[d, "\partial_0"] & \\
O & & M \arrow[ll, "\partial_0"] \arrow[rr, "\partial_1"'] & & O &
\arrow[from=1-2, to=3-1, "\pi_2"]
\end{tikzcd}$$

is, in fact, a «functor from $\Delta\underline{C}$ to $\underline{E}^{\underline{C}^o}$» (where $\Delta\underline{C}$ is (by left exactness of Δ) a category in $\underline{E}^{\underline{C}^o}$), i.e., a presheaf on $\Delta\underline{C}$ in $\underline{E}^{\underline{C}^o}$. The obvious interpretation of the family is then the natural inclusion of $\underline{C}$ in $\underline{E}^{\underline{C}^o}$ as «the full subcategory determined by the representable functors».

Note, for later use, that the family

$$(M, c)\xrightarrow{<\partial_1,\,\partial_0>}\Delta O$$

acts as $M\xrightarrow{\partial_1}O$. By this we mean: suppose we are given $Z\xrightarrow{p}O\in\underline{E}/_O$, then the object part of the diagram defining $\Delta Z\underset{\Delta O}{\times}(M, c)$ has, as object part, the top diagram in the following decomposition of the pullback:

$$\begin{tikzcd}
Z\underset{O}{\times}M \arrow[r, "\pi_2"] \arrow[d, "<\pi_1{,}\partial_0.\pi_2>"'] \arrow[dd, bend right=60, "\pi_1"'] & M \arrow[d, "<\partial_1{,}\partial_0>"] \arrow[dd, bend left=60, "\partial_1"] \\
Z\times O \arrow[r, "p\times O"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{PB}"] & O\times O \arrow[d, "\pi_1"] \\
Z \arrow[r, "p"'] & O
\end{tikzcd}$$

This describes in part the functor $\underline{E}/_O\to\underline{E}^{\underline{C}^o}$, left adjoint to the forgetful functor. Given $Z\xrightarrow{p}O$, the morphism $Z\underset{O}{\times}M\to O$ (as above) can be made into an object of $\underline{E}^{\underline{C}^o}$ via the action

$$Z \underset{0}{\times} M \underset{0}{\times} M \xrightarrow{<\pi_1, c.\pi_{23}>} Z \underset{0}{\times} M .$$

The Classification Theorem of [6] establishes a natural and bijective correspondence between geometric morphisms $\Phi : \underline{F} \to \underline{E}^{\underline{C}^o}$ over $\underline{E}$ and flat functors from $f^*\underline{C}$ to $\underline{F}$. The flat functor corresponding to a given Φ is $\chi_\Phi = \Phi^*(M, c)$ and the geometric morphism associated to a flat functor χ is Φ_χ given by the pair :

$$\begin{tikzcd}
\underline{F} \arrow[r, shift right, "Hom_{f^*\underline{C}}(X, -)"'] & \underline{F}^{(f^*\underline{C})^o} \arrow[l, shift right, "-\otimes_{f^*\underline{C}} X"'] \arrow[r, shift right, "\overline{\overline{f}}_*"'] & \underline{E}^{\underline{C}^o} \arrow[l, shift right, "\overline{\overline{f}}^*"']
\end{tikzcd}$$

These two processes are inverses to one another.

If $\Phi = 1_{\underline{E}^{\underline{C}^o}}$ then $\chi_\Phi = (M, c)$, and conversely.. Corresponding then to (M, c) is the identity adjoint pair, given as the pair

$$\begin{tikzcd}
\underline{E}^{\underline{C}^o} \arrow[r, shift right, "Hom_{\Delta\underline{C}}((M, c), -)"'] & (\underline{E}^{\underline{C}^o})^{(\Delta\underline{C})^o} \arrow[l, shift right, "-\otimes_{\Delta\underline{C}}(M, c)"'] \arrow[r, shift right, "\overline{\overline{\Gamma}}"'] & \underline{E}^{\underline{C}^o} \arrow[l, shift right, "\overline{\overline{\Delta}}"']
\end{tikzcd}$$

This gives two decompositions of the identity $1_{\underline{E}^{\underline{C}^o}}$. That given by the left adjoint implies that, for an object (Z, ζ) of $\underline{E}^{\underline{C}^o}$,

$$[\,*] \qquad (Z, \zeta) = (\Delta Z, \Delta\zeta) \otimes_{\Delta\underline{C}} (M, c).$$

By previous remarks, $[\,*]$ translates into a coequalizer diagram in $\underline{E}^{\underline{C}^o}$:

$$\begin{tikzcd}
(Z \underset{0}{\times} M \underset{0}{\times} M, <\pi_{12}, c.\pi_{23}>) \arrow[r, shift left, "\zeta \underset{0}{\times} M"] \arrow[r, shift right, "Z \underset{0}{\times} c"'] & (Z \underset{0}{\times} M, <\pi_1, c.\pi_{23}>) \arrow[r, "\zeta"] & (Z, \zeta).
\end{tikzcd}$$

The decomposition of $1_{\underline{E}^{\underline{C}^o}}$ given by the right adjoint says that

$$[**] \qquad (Z, \zeta) = \overline{\overline{\Delta}}\,\overline{\overline{\Gamma}}(Hom_{\Delta\underline{C}}((M, c), (Z, \zeta))).$$

At the object level, this says that the following is a pullback in $\underline{E}$:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{PB}"] & \Gamma(Hom_{\Delta\underline{C}}((M,c),(Z,\zeta))) \arrow[d] \\
O \arrow[r, "\eta_O"'] & \Gamma\Delta O
\end{tikzcd}$$

a version of the Yoneda Lemma.

Combining [*] with [**] we get

$$(Z,\zeta) = \bar{\bar{\Delta}}\bar{\bar{\Gamma}}(Hom_{\Delta\underline{C}}((M,c),(Z,\zeta))) \otimes_{\Delta\underline{C}} (M,c)$$

and consequently an epimorphism whose object part is an epimorphism

$$[***] \qquad \Delta\Gamma(\Pi_{\partial_1}(Z \underset{O}{\times} M)) \underset{O}{\times} M \xrightarrow{q} Z,$$

reminiscent of the canonical epimorphism

$$\Sigma_{E_X} H^C \xrightarrow{p_X} X \text{ in } \underline{S}^{\underline{C}^o}$$

[4], where

$$E_X = \bigcup_C Hom_{\underline{S}^{\underline{C}^o}}(H^C, X) \text{ and } p_X . inj_y = y .$$

In fact, it will play the same rôle here.

7. The canonical adjoint pair $\underline{F} \rightleftarrows \underline{E}^{\underline{C}^o}$ induced by an internal family in $\underline{F}$.

For $\underline{F} \xrightarrow{f} \underline{E}$ (geometric) and $A \xrightarrow{a} I$ in $\underline{F}$, letting $\underline{A} \in Cat(\underline{F})$, $u_{\underline{A}} \in |\underline{F}^{\underline{A}}|$ be as in Section 4 and $\underline{C} = f_*\underline{A}$ (a category in $\underline{E}$ since f_* is left exact) there exists (see [6]) an adjoint pair $\Phi^* \dashv \Phi_*$ fitting into a commutative diagram

$$\begin{tikzcd}
\underline{F} \arrow[rr, shift right, "\Phi_*"'] \arrow[dr, shift left, "f_*"] & & \underline{E}^{\underline{C}^o} \arrow[ll, shift right, "\Phi^*"'] \arrow[dl, shift right, "\Delta"'] \\
& \underline{E} \arrow[ul, shift left, "f^*"] \arrow[ur, shift right, "\Gamma"'] &
\end{tikzcd}$$

The pair is given by

$$\begin{tikzcd}[column sep=huge] \underline{F} \arrow[r, shift right, "Hom_{\underline{A}}(u_{\underline{A}},-)"'] & \underline{F}^{\underline{A}^\circ} \arrow[l, shift right, "-\otimes_{\underline{A}} u_{\underline{A}}"'] \arrow[r, shift right, "\epsilon^*"'] & \underline{F}^{f^*\underline{C}^0} \arrow[l, shift right, "\epsilon_!"'] \arrow[r, shift right, "\overline{\overline{f}}_*"'] & \underline{E}^{\underline{C}^0} \arrow[l, shift right, "\overline{\overline{f}}^*"'] \end{tikzcd} .$$

The functor Φ_* is then given on objects by:

$$\Phi_*(X) = (f_*\Pi_\alpha(X\times A),\ f_*\xi^A_X)$$

for an object X of $\underline{F}$. (Verify for this that

$$\overline{\overline{f}}_*(\epsilon^*(Y,\theta)) = (f_*Y,\ f_*\theta)$$

and resort to the definition of $Hom_{\underline{A}}(u_{\underline{A}},-)$ in Section 5.) If (Z,ζ) is an object of $\underline{E}^{\underline{C}^0}$, then

$$\Phi^*(Z,\zeta) = coeq\,(\begin{tikzcd}[column sep=huge] f^*Z \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \arrow[r, shift left, "f^*\zeta \underset{I}{\times} A"] \arrow[r, shift right, "f^*Y \underset{f^*f_*I}{\times} \xi_A(\epsilon\times A)"'] & f^*Z\underset{I}{\times}A \end{tikzcd}$$

Hence, for an X of $\underline{F}$, the object $\Phi^*\Phi_*X$ of $\underline{F}$ is defined by the coequalizer

$$\begin{tikzcd}[column sep=huge] f^*f_*\Pi_\alpha(X\times A) \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \arrow[r, shift left, "f^*f_*\xi^A_X \underset{I}{\times} A"] \arrow[r, shift right, "1 \underset{f^*f_*I}{\times} (\xi_A\cdot(\epsilon \underset{I}{\times} A))"'] & f^*f_*\Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[d, "q_{\Phi_*X}"'] \\ & \Phi^*\Phi_*X \end{tikzcd} .$$

Let us define, for $X \in |\underline{F}|$, a morphism

$$p^\alpha_X = f^*f_*\Pi_\alpha(X\times A)\underset{I}{\times}A \xrightarrow{\epsilon \underset{I}{\times} A} \Pi_\alpha(X\times A)\underset{I}{\times}A \xrightarrow{ev^\alpha_X} X .$$

We claim that the diagram

$$\begin{tikzcd}[column sep=huge] f^*f_*\Pi_\alpha(X\times A) \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \arrow[r, "f^*f_*\xi^A_X \underset{I}{\times} A"] \arrow[d, "1\times(\xi_A\cdot(\epsilon \underset{I}{\times} A))"'] & f^*f_*\Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[d, "p^\alpha_X"] \\ f^*f_*\Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[r, "p^\alpha_X"'] & X \end{tikzcd}$$

commutes. Observe for this that, by the definition of ξ^A_X, i.e., as the exponential adjoint of $\epsilon^\alpha.(1\times\xi_A)$ for the adjoint pair $\alpha^*\dashv\Pi_\alpha$ whose counit is ϵ^α, the diagram

$$\begin{tikzcd}
\Pi_\alpha(X\times A)\underset{I}{\times}A_1\underset{I}{\times}A \arrow[r, "\xi^A_X\underset{I}{\times}A"] \arrow[d, "1\times\xi_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[d, "\epsilon^\alpha"] \\
\Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[r, "\epsilon^\alpha"'] & X\times A
\end{tikzcd}$$

Then use the fact that $ev^\alpha_X=\pi_1\cdot\epsilon^\alpha_{X\times A}$.

It then follows the existence of a unique morphism $\bar\epsilon_X$ as in:

$$|*| \qquad \begin{tikzcd}
f^*f_*\Pi_\alpha(X\times A)\underset{I}{\times}A \arrow[r, "q_{\Phi_*X}"] \arrow[dr, "p^\alpha_X"'] & \Phi^*\Phi_*X \arrow[d, "\bar\epsilon_X"] \arrow[d, phantom, shift right=25, "\circlearrowleft"] \\
 & X
\end{tikzcd}$$

For $(Z,\zeta)\in|\underline{E}^{\underline{C}^o}|$, there is given a coequalizer diagram from the equation [*] of Section 6; this is

$$(1) \qquad \begin{tikzcd}
Z\underset{f_*I}{\times}f_*A_1\underset{f_*I}{\times}f_*A_1 \arrow[r, shift left, "\zeta\underset{f_*I}{\times}f_*A_1"] \arrow[r, shift right, "Z\underset{f_*I}{\times}f_*c"'] & Z\underset{f_*I}{\times}f_*A_1 \arrow[r, "\zeta"] & (Z,\zeta),
\end{tikzcd}$$

where we have omitted some structure maps (see Section 6). On the other hand, $\Phi^*(Z,\zeta)\in|\underline{F}|$ fits also in a coequalizer diagram:

$$(2) \quad \begin{tikzcd}
f^*Z\underset{I}{\times}f^*f_*A_1\underset{I}{\times}A \arrow[r, shift left, "f^*\zeta\underset{I}{\times}A"] \arrow[r, shift right, "1\underset{I}{\times}\xi_A(\epsilon\underset{I}{\times}A)"'] & f^*Z\underset{I}{\times}A \arrow[r, "q_{(Z,\zeta)}"] & \Phi^*(Z,\zeta),
\end{tikzcd}$$

hence Φ_* carries it into a pair of equal maps, namely: (3)

$$\begin{tikzcd}
f_*\Pi_\alpha(f^*Z\times f^*f_*A_1\underset{I}{\times}A\times A) \arrow[r, shift left, "f_*\Pi_\alpha(f^*\zeta\underset{I}{\times}A\times A)"] \arrow[r, shift right, "f_*\Pi_\alpha(1\times\xi_A(\epsilon\times1)\times1)"'] & f_*\Pi_\alpha(f^*Z\underset{I}{\times}A\times A) \arrow[d, "\Phi_*(q_{(Z,\zeta)})"] \\
 & \Phi_*\Phi^*(Z,\zeta)
\end{tikzcd}$$

We shall next give, for each $E \xrightarrow{e} f_*I$ of $\underline{E}$, a canonical morphism

$$\chi_e^{\alpha} : E \underset{f_*I}{\times} f_*A_1 \to f_*\Pi_{\alpha}((f^*E \underset{I}{\times} A) \times A)$$

with

$$\begin{tikzcd}
E \underset{f_*I}{\times} f_*A_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{PB}"] & f_*A_1 \arrow[d, "f_*\partial_0"] \\
E \arrow[r, "e"'] & f_*I
\end{tikzcd}$$

defined as the twice exponential adjoint (relative to $f^* \dashv f_*$ first and $\alpha^* \dashv \Pi_{\alpha}$ next) of

$$\hat{\chi}_e^{\alpha} = f^*E \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \xrightarrow{1 \times \epsilon^{\alpha \times I} \cdot (\epsilon \times 1)} f^*E \underset{I}{\times} A \times A .$$

Equivalently,

$$\hat{\chi}_e^{\alpha} = 1 \times < \xi_A \cdot (\epsilon \times 1), \pi_2 > .$$

By naturality of χ_e^{α} with respect to morphisms $E \xrightarrow{e} f_*I$ in $\underline{E}/_{f_*I}$, the diagram (4)

$$\begin{tikzcd}
Z \underset{f_*I}{\times} f_*A_1 \underset{f_*I}{\times} f_*A_1 \arrow[r, "\zeta \underset{f_*I}{\times} f_*A_1"] \arrow[d, "\chi_Z^{\alpha} \underset{f_*I}{\times} f_*A_1"'] & Z \underset{f_*I}{\times} f_*A_1 \arrow[d, "\chi_Z^{\alpha}"] \\
f_*\Pi_{\alpha}(f^*Z \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \times A) \arrow[r, "f_*\Pi_{\alpha}(f^*\zeta \underset{I}{\times} A \times A)"'] & f_*\Pi_{\alpha}(f^*Z \underset{I}{\times} A_1 \times A)
\end{tikzcd}$$

commutes. We now claim that also the diagram (5):

$$\begin{tikzcd}
Z \underset{f_*I}{\times} f_*A_1 \underset{f_*I}{\times} f_*A_1 \arrow[r, "Z \underset{f_*I}{\times} f_*c"] \arrow[d, "\chi_Z^{\alpha} \underset{f_*I}{\times} f_*A_1"'] & Z \underset{f_*I}{\times} f_*A_1 \arrow[d, "\chi_Z^{\alpha}"] \\
f_*\Pi_{\alpha}(f^*Z \underset{f^*f_*I}{\times} f^*f_*A_1 \underset{I}{\times} A \times A) \arrow[r, "f_*\Pi_{\alpha}(1 \times \xi_A \cdot (\epsilon \times 1) \times 1)"'] & f_*\Pi_{\alpha}(f^*Z \underset{I}{\times} A_1 \times A)
\end{tikzcd}$$

commutes. This is proved by passing to the twice transpose of this diagram, recalling that

$$|**| \qquad \begin{tikzcd} A_1 \underset{I}{\times} A_1 \underset{I}{\times} A \arrow[r, "c \underset{I}{\times} A"] \arrow[d, "1 \times ev"'] \arrow[dr, phantom, "\circlearrowleft"] & A_1 \underset{I}{\times} A \arrow[d, "\epsilon^{\alpha \times I}"] \\ A_1 \underset{I}{\times} A \arrow[r, "ev"'] & A \times A \end{tikzcd}$$

by the definition of c .

As a consequence of (4) and (5) when comparing (1) and (3), one has that

$$\Phi_*(q_{(Z,\zeta)}) \cdot \chi_Z^\alpha \cdot (\zeta \times 1) = \Phi_*(q_{(Z,\zeta)}) \cdot \Phi_*(f^*\zeta \times 1) \cdot \chi_Z^\alpha \underset{f^*I}{\times} f_*A_1 =$$

$$= \Phi_*(q_{(Z,\zeta)}) \cdot \Phi_*(1 \times \xi_A \cdot (\epsilon \times 1)) \cdot \chi_Z^\alpha \underset{f_*I}{\times} f_*A_1 =$$

$$= \Phi_*(q_{(Z,\zeta)}) \cdot \chi_Z^\alpha \cdot (1 \times f_*c),$$

hence that there exists a unique morphism $\bar{\eta}_{(Z,\zeta)}$ as in the commutative diagram

$$\begin{tikzcd} Z \underset{f_*I}{\times} f_*A_1 \arrow[r, "\zeta"] \arrow[d, "\chi_Z^\alpha"'] & (Z,\zeta) \arrow[d, "\bar{\eta}_{(Z,\zeta)}"] \\ \Phi_*(f^*Z \underset{I}{\times} A) \arrow[r, "\Phi_*(q_{(Z,\zeta)})"'] & \Phi_*\Phi^*(Z,\zeta) \end{tikzcd} .$$

The maps $\bar{\eta}_{(Z,\zeta)}$ and $\bar{\epsilon}_X$ define natural transformations

$$\bar{\eta} : 1 \to \Phi_*\Phi^* \quad \text{and} \quad \bar{\epsilon} : \Phi^*\Phi_* \to 1$$

and they are, respectively, the unit and counit of $\Phi^* \dashv \Phi_*$.

8. $\underline{E}$-generating internal families in a topos over $\underline{E}$.

Let $\underline{F} \xrightarrow{f} \underline{E}$ be a geometric morphism of topos. An internal I-indexed family $A \xrightarrow{\alpha} I$ of $\underline{F}$ is said to be *$\underline{E}$-generating for $\underline{F}$* if and only if for every object X of $\underline{F}$ the canonical morphism

$$p_X^\alpha : f^* f_* \Pi_\alpha (X \times A) \underset{I}{\times} A \to X$$

(defined in Section 7) is an epimorphism. By the remark made in Section 5, the diagram

$$\begin{tikzcd}
\Pi_\alpha(X \times A) \arrow[r] \arrow[d] & \tilde{X}^A \arrow[d, "dom^A"'] \\
I \arrow[r, "\bar{a}"] & \Omega^A
\end{tikzcd}$$

is a pullback. This serves the purpose of linking our definition with that of Mitchell-Diaconescu [18 and 6] of «an object of generators» or «a generator over $\underline{E}$» of a topos $\underline{F}$ over $\underline{E}$ by means of $\underline{F} \xrightarrow{f} \underline{E}$. The internal family of all subobjects of A is the morphism $\epsilon \xrightarrow{n} \Omega^A$ where

$$\epsilon \rightarrowtail^{<n,\, e>} \Omega^A \times A$$

is classified by the evaluation map. It is then easy to see that the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & \epsilon \arrow[d, "n"] \\
I \arrow[r, "\bar{a}"] & \Omega^A
\end{tikzcd}$$

is a pullback; by definition, $\bar{a}$ is given by the correspondence

$$\frac{I \times A \xrightarrow{\bar{a} \times A} \Omega^A \times A \xrightarrow{ev} \Omega}{A \xrightarrowtail{<a,\, A>} I \times A\,.}$$

Hence, there exists a canonical morphism

$$f^* f_* \Pi_\alpha (X \times A) \underset{I}{\times} A \xrightarrow{\Phi} f^* f_* \tilde{X}^A \underset{\Omega^A}{\times} \epsilon$$

such that $p_X^n . \Phi = p_X^\alpha$, where

$$\begin{tikzcd}
f^* f_* \tilde{X}^A \underset{\Omega^A}{\times} \epsilon \arrow[r, "p_X^n"] \arrow[d, tail] & X \arrow[d, tail, "\eta"] \\
f^* f_* \tilde{X}^A \times A \arrow[r, "p_X^A"'] & \tilde{X}
\end{tikzcd}$$

is a pullback. Since the map p_X^A is required to be epi for A to be an «object of generators», we have

(i) if an internal family $A \xrightarrow{a} I$ is generating, so is the (larger) internal family $\epsilon \xrightarrow{n} \Omega^A$ of «all subobjects of the union of the members of the original family»;

(ii) if an object A is generating in the Mitchell-Diaconescu sense, then the internal family of all subobjects of A is generating in our sense.

It is not true, however, that the converse of (ii) holds, or that in (i), $A \xrightarrow{a} I$ generating implies A an object of generators. The latter condition is definitely stronger as is evidenced by the fact that the Mitchell-Diaconescu theorem asserts, precisely, that if $\underline{F} \xrightarrow{f} \underline{E}$ has an object of generators, the canonical Φ_* in the adjoint pair

$$\begin{tikzcd} \underline{F} \arrow[r, shift left, "\Phi_*"] & \underline{E}^{\underline{C}^o} \arrow[l, shift left, "\Phi^*"] \end{tikzcd}$$

(with $\underline{C}$ induced by the internal family $\epsilon \xrightarrow{n} \Omega^A$) is fully faithful. On the other hand, consider what it means to assume that the family $1 \to 1$ in $\underline{F}$ is $\underline{E}$-generating: it amounts to the condition

$$f^* f_* X \xrightarrow{\epsilon X} X \quad \text{epi for each } X,$$

or f_* faithful. Since, in this case, $\Phi_* = f_*$, we see that the condition cannot imply f_* (i.e., Φ_*) fully faithful.

It is our aim to give sufficient conditions for Φ_* - where $A \xrightarrow{a} I$ is $\underline{E}$-generating - to be fully faithful. An object B of a locally small category $\underline{X}$ has been called (cf. [4]) *abstractly unary* if the functor $hom_{\underline{X}}(B, -)$ commutes with set-indexed coproducts - indeed not an elementary condition. There is a canonical map

$$\sum_{k \in K} hom_{\underline{X}}(B, C_k) \xrightarrow{\chi} hom_{\underline{X}}(B, \sum_{k \in K} C_k)$$

with

$$\chi \cdot inj_k = hom_{\underline{X}}(B, inj_k), \quad k \in K,$$

so that requiring that B be abstractly unary amounts to asking that each such χ (with K a set, C_k an object of $\underline{X}$ for each $k \in K$) be an isomorphism.

For an elementary version of this property, also restricting ourselves to «coproducts of members of the given family $A \xrightarrow{a} I$», to say that $A \xrightarrow{a} I$ is *a family of $\underline{E}$-abstractly unary objects* amounts to the following:

let $E \xrightarrow{e} f_*I$ be a morphism of $\underline{E}$, i.e., $e \in |\underline{E}/_{f_*I}|$; define χ_e^a by means of the following adjointness correspondences:

$$\begin{array}{c} e \times \bar{f}_*(a^a) \xrightarrow{\chi_e^a} \bar{f}_*((e \otimes a)^a) \\ \hline \bar{f}^*(e \times \bar{f}_*(a^a)) \longrightarrow (e \otimes a)^a \\ \hline \end{array}$$

$$\begin{tikzcd}
\bar{f}^*(e \times \bar{f}_*(a^a)) \times a \arrow[r] \arrow[d, "canon \times 1"'] & e \otimes a = \bar{f}^* e \times a \\
\bar{f}^* e \times \bar{f}^* \bar{f}_*(a^a) \times a \arrow[r, "1 \times \epsilon \times 1"] & \bar{f}^* e \times a^a \times a \arrow[u, "1 \times ev"']
\end{tikzcd} ;$$

require that for each e as above, χ_e^a be an isomorphism. (The notation $e \otimes a$ is explained in Section 3, as well as the adjoint pair $\bar{f}^* \dashv \bar{f}_*$.)

Note that χ_e^a is precisely the morphism defined in Section 7 as

$$\chi_e^a : E \underset{f_*I}{\times} f_*A \longrightarrow f_*\Pi_a((f^*E \underset{I}{\times} A) \times A).$$

A family $A \xrightarrow{a} I$ of $\underline{F}$ is said to be *a family of f_*-abstractly unary objects* if χ_e^a is iso for every e of the form $e = f_*\beta$ with $\beta \in \underline{F}/_I$. In the case of the family $1 \to 1$, it is a family of $\underline{E}$-abstractly unary objects iff the unit η of $f^* \dashv f_*$ is iso and a family of f_*-abstractly unary objects iff ηf_*, equivalently $f_*\epsilon$, is iso. Under the condition of being $\underline{E}$-generating, which says that f_* is faithful, the latter yields ϵ iso or f_* fully faithful. This latter condition plays a similar rôle in the general situation, as we shall see immediately.

We need, for immediate and later purposes, to record the following information.

LEMMA. *In $\underline{E}^{\underline{C}^{o}}$, the internal family $(M, c) \xrightarrow{<\partial_1, \partial_0>} \Delta O$ is $\underline{E}$-generating.*

PROOF. The morphism q of [***] of Section 6 is $q_{\Phi_*(Z,\zeta)}$ of section 7. Since the adjoint pair $\Phi^* \dashv \Phi_*$ is the identity in this case and since the family in question is the internal family which induces this adjoint pair in the manner of Section 7, it follows that it must be $\underline{E}$-generating. To see this, note that, in the diagram

$$\begin{tikzcd}
\bar{\Delta}\bar{\Gamma} Hom_{\Delta\underline{C}}((M,c),(Z,\zeta))\times(M,c) \arrow[r, "q_{\Phi_*(Z,\zeta)}"] \arrow[dr, "p^{\partial_1}_{(Z,\zeta)}"'] & \Phi^*\Phi_*(Z,\zeta) \arrow[d, "\bar{\epsilon}_{(Z,\zeta)}"] \\
 & (Z,\zeta)
\end{tikzcd}$$

which commutes by Section 7, [*], $\bar{\epsilon}_{(Z,\zeta)}$ is the identity, Hence, as the morphism $q_{\Phi_*(Z,\zeta)}$ is a coequalizer, so is $p^{\partial_1}_{(Z,\zeta)}$, hence epi. ■

THEOREM. *Let* $\underline{F} \xrightarrow{f} \underline{E}$ *be an* $\underline{E}$*-topos. If* $A \xrightarrow{\alpha} I \in \underline{F}$ *is an* $\underline{E}$*-generating internal family of* f_**-abstractly unary objects, then the induced (defined in Section 7)* $\Phi_*\colon \underline{F} \to \underline{E}^{\underline{C}^0}$ *is fully faithful.*

PROOF. Since $A \xrightarrow{\alpha} I$ is $\underline{E}$-generating, p^α_X is epi for each X, hence, by the commutative diagram of Section 7 defining $\bar{\epsilon}_X$, it follows that $\bar{\epsilon}_X$ is epi. This says that Φ_* is faithful. Hence, to show Φ_* fully faithful, we must prove that $\bar{\epsilon}_X$ is iso for each X.

Denote $f_*\Pi_\alpha(X\times A)$ by E^α_X and let the following be an exact diagram involving p^α_X :

$$(1) \qquad \begin{tikzcd}
K \arrow[r, shift left, "<a_1, b_1>"] \arrow[r, shift right, "<a_2, b_2>"'] & f^*E^\alpha_X \times_I A \arrow[r, "p^\alpha_X"] & X.
\end{tikzcd}$$

Consider the topos

$$\underline{F}/_J \quad \text{with} \quad J = f^*E^\alpha_X \times f^*E^\alpha_X .$$

Since $\underline{F}/_J \approx \underline{F}^{\underline{J}}$ for an appropriate category $\underline{J}$ in $\underline{F}$ (cf. [20]) the lemma applies to give $J \xrightarrow{\Delta} J\times J$ (over J) as an $\underline{F}$-generating family. Let

$$K \xrightarrow{<a_1, a_2>} J$$

be regarded as an object of $\underline{F}/_J$; it follows (after some simplifications) that there exist an object $E^{\Delta}_K \in |\underline{F}|$ and an epimorphism

$$E^{\Delta}_K \times f^* E^{\alpha}_X \xrightarrow{r} K$$

over J . Letting $Y = E^{\Delta}_K$ and considering the canonical epi

$$f^* E_Y \underset{I}{\times} A \xrightarrow{p^{\alpha}_Y} Y ,$$

one may conveniently compose with the above to give an epi

$$(2) \qquad f^* E \underset{I}{\times} A \xrightarrow{p'_K} K$$

satisfying $a_1 p'_K = a_2 p'_K = f^* \pi_2$ (let $E = E^{\alpha}_Y \times E^{\alpha}_X$).

Next, we shall look at the diagram :

$$(3) \qquad \begin{tikzcd}
 & & & \Phi^* \Phi_* X \arrow[d, "\bar{\epsilon}_X"] \\
f^* E \underset{I}{\times} A \arrow[r, "p'_K"] \arrow[dr, dashed, shift left] \arrow[dr, dashed, shift right] & K \arrow[r, shift left, "<a_1, b_1>"] \arrow[r, shift right, "<a_2, b_2>"'] & f^* E^{\alpha}_X \underset{I}{\times} A \arrow[r, "p^{\alpha}_X"] \arrow[ur, "q"] & X \\
 & f^* E^{\alpha}_X \underset{I}{\times} f^* f_* A_1 \underset{I}{\times} A \arrow[ur, shift left, "f^* f_* \xi^A_X \times 1"] \arrow[ur, shift right, "1 \times \xi_A . (\epsilon \times 1)"'] & &
\end{tikzcd} .$$

Ignoring the dotted arrows (to be defined below), it is our aim to establish the following :

$$q . <a_1, b_1> = q . <a_2, b_2> .$$

This is, in fact, all we need to ensure $\bar{\epsilon}_X$ iso.

Let the dotted arrows be given by two morphisms denoted

$$(4) \qquad f^* E \underset{I}{\times} A \overset{<f^* \bar{a}_i , f^* \bar{b}_i \times 1>}{\rightrightarrows} f^* E^{\alpha}_X \underset{I}{\times} f^* f_* A_1 \underset{I}{\times} A ,$$

where $\bar{a}_i, \bar{b}_i$ $(i = 1, 2)$ arise by double exponential adjointness as follows, using the fact that $A \xrightarrow{\alpha} I$ is a family of f_*-abstractly unary objects, so that the appropriate χ^{-1} exists with

$$e = f_*\beta : E^\alpha_X \to f_* I$$

and $\beta : \Pi_\alpha (X \times A) \to I$ the canonical map:

$$\begin{array}{c} f^*E \underset{I}{\times} A \xrightarrow{<a_i p'_K,\, b_i p'_K,\, b_i p'_K>} f^*E^\alpha_X \underset{I}{\times} A \times A \\ \hline \begin{tikzcd} E \arrow[r] \arrow[dr, "<\bar{a}_i, \bar{b}_i>"'] & f^*\Pi_\alpha (f^*E^\alpha_X \underset{I}{\times} A \times A) \arrow[d, "\chi^{-1}"] \\ & E^\alpha_X \underset{f_*I}{\times} f_*A_1 \end{tikzcd} \end{array}$$

The two maps defined in (4) have the properties

$$f^*\bar{a}_i = a_i.p'_K = f^*\pi_2$$

and

$$\xi_A.(\epsilon \times 1).(f^*\bar{b}_i \times 1) = b_i.p'_K$$

for $i = 1, 2$. It follows that

$$q.<a_i, b_i>.p'_K = q.(1 \times \xi_A(\epsilon \times 1)).<f^*\bar{a}_i, f^*\bar{b}_i \times 1>$$

and so, that

$$q.<a_i, b_i>.p'_K = q.(f^*f_*\xi^A_X \times 1).<f^*\bar{a}_i, f^*\bar{b}_i \times 1>$$

for $i = 1, 2$, by the definition of $q = q_{\Phi^* X}$.

Hence it will be enough to prove (5)

$$(f^*f_*\xi^A_X \underset{I}{\times} A).<f^*\bar{a}_1, f^*\bar{b}_1 \times 1> = (f^*f_*\xi^A_X \underset{I}{\times} A).<f^*\bar{a}_2, f^*\bar{b}_2 \times 1>.$$

From the equation

$$p^\alpha_X.<a_1, b_1>.p'_K = p^\alpha_X.<a_2, b_2>.p'_K$$

follows that

$$ev_X.<\epsilon.f^*\pi_2, b_1.p'_K> = ev_X.<\epsilon.f^*\pi_2, b_2.p'_K>$$

and hence the result, by virtue of p'_K epi and the commutative diagrams below $(i = 1,2)$:

$$\begin{tikzcd}
f^*E \underset{I}{\times} A \arrow[r, "{<f^*\pi_2, f^*\bar{b}_i \times 1>}"] \arrow[rddd, "{<\epsilon . f^*\pi_2, \bar{b}_i . p'_K>}"'] & f^*E^{\alpha}_{X} \underset{I}{\times} f^*f_*A_1 \underset{I}{\times} A \arrow[r, "f^*f_*\xi^A_X \underset{I}{\times} A"] \arrow[d] & f^*E^{\alpha}_{X} \underset{I}{\times} A \arrow[dd, "\epsilon \underset{I}{\times} A"] \\
 & f^*f_*(\Pi_\alpha(X\times A) \underset{I}{\times} A_1) \underset{I}{\times} A \arrow[d, "\epsilon \underset{I}{\times} A"'] & \\
 & \Pi_\alpha(X\times A) \underset{I}{\times} A_1 \underset{I}{\times} A \arrow[r, "\xi^A_X \underset{I}{\times} A"] \arrow[d, "\xi^A_X \underset{I}{\times} A"] & \Pi_\alpha(X\times A) \underset{I}{\times} A \arrow[d, "ev_X"] \\
 & \Pi_\alpha(X\times A) \underset{I}{\times} A \arrow[r, "ev_X"'] & X
\end{tikzcd}$$

■

9. $\underline{E}$-atomic families in a topos over $\underline{E}$.

Recall that our aim is to give necessary and sufficient conditions on a geometric morphism $\underline{F} \xrightarrow{f} \underline{E}$ for it to be «of the form» $\underline{E}^{\underline{C}^{o}} \to \underline{E}$ for some category $\underline{C}$ in $\underline{E}$. We have already that, given a family $A \xrightarrow{a} I$, with $\underline{C}$ and

$$\begin{tikzcd}
\underline{F} \arrow[r, shift left, "\Phi_*"] & \underline{E}^{\underline{C}^{o}} \arrow[l, shift left, "\Phi^*"]
\end{tikzcd}$$

induced by it, if a is an $\underline{E}$-generating family, Φ_* is faithful and if, further, a is a family of f_*-abstractly unary objects then Φ_* is fully faithful. We seek necessary and sufficient conditions for such a pair Φ^*, Φ_* to be an adjoint equivalence of categories.

Let $A \xrightarrow{a} I$ be an internal family in $\underline{F}$ (over $\underline{E}$). We shall say that $A \xrightarrow{a} I$ *is $\underline{E}$-atomic* (or «a *family of $\underline{E}$-atoms*») iff:

(i) $A \xrightarrow{a} I$ is a family of $\underline{E}$-abstractly unary objects;

(ii) $A \xrightarrow{a} I$ *is a family of $\underline{E}$-projective objects*, i.e. (definition): for every morphism $X \xrightarrow{g} Y$ of $\underline{F}$, if g is an epimorphism, so is

$$f_*\Pi_\alpha(g \times A) : f_*\Pi_\alpha(X \times A) \to f_*\Pi_\alpha(Y \times A).$$

LEMMA. *In* $\underline{E}^{\underline{C}^{o}}$ *, the family* $(M, c) \xrightarrow{<\partial_1, \partial_0>} \Delta O$ *is* $\underline{E}$*-atomic.*

PROOF. (i) Given a morphism $E \xrightarrow{e} \Gamma \Delta O$ in $\underline{E}$, note that

$$\chi_e^{\partial_1} : E \underset{O}{\times} \Gamma \Pi_{\partial_1} (M \underset{O}{\times} M) \to \Gamma \Pi_{\partial_1} (E \underset{O}{\times} M \underset{O}{\times} M)$$

is the object part of

$$\chi_e^{\partial_1} : \Delta E \underset{\Delta O}{\times} \Phi_*(M, c) \to \Phi_*(\Delta E \underset{\Delta O}{\times} (M, c))$$

where $\Phi_* : \underline{E}^{\underline{C}^{o}} \to \underline{E}^{\underline{C}^{o}}$ is the right adjoint of the adjoint pair induced by (M, c) by the Classification Theorem of Diaconescu [6] (see Section 6). As Φ_* must be the identity in this case, so is $\chi_e^{\partial_1}$.

(ii) Given $g : (Z, \zeta) \to (Y, \theta)$ in $\underline{E}^{\underline{C}^{o}}$, then also

$$\Gamma \Pi_{\partial_1} (g \underset{O}{\times} (M, c)) : \Gamma \Pi_{\partial_1} (Z \underset{O}{\times} M) \to \Gamma \Pi_{\partial_1} (Y \underset{O}{\times} M)$$

is epi, since

$$\Gamma \Pi_{\partial_1} (g \underset{O}{\times} (M, c)) = \Phi_*(g) = g . \quad \blacksquare$$

We now prove :

THEOREM. *Let* $\underline{F} \xrightarrow{f} \underline{E}$ *be a geometric morphism of topos. Let* $A \xrightarrow{a} I$ *be an internal family of* $\underline{F}$ *which is* $\underline{E}$*-generating and* $\underline{E}$*-atomic. Then, the unit* $\bar{\eta} : 1 \to \Phi_* \Phi^*$ *of the adjointness of the functors*

$$\begin{tikzcd} \underline{F} \arrow[r, shift right, "\Phi_*"'] & \underline{E}^{\underline{C}^{o}} \arrow[l, shift right, "\Phi^*"'] \end{tikzcd} \qquad (\text{over } \underline{E})$$

(defined in Section 7) is an isomorphism.

PROOF. We start out by the consideration of the diagram defining $\bar{\eta}$ (discussed in detail in Section 7) where, by assumption of $\underline{E}$-atomicity of a , each χ is an isomorphism (we have omitted some structure maps in this diagram, drawn here after). In order to proceed from here, we shall find it convenient to obtain an alternative description of $q_{(Z,\zeta)}$, defined before as the coequalizer of the pair

$$\begin{tikzcd}
f^*Z \times_I f^*f_*A_1 \times_I A \arrow[r, shift left, "f^*\zeta \times_I A"] \arrow[r, shift right, "1\times\xi_A\cdot(\epsilon\times 1)"'] & f^*Z \times_I A .
\end{tikzcd}$$

$$\begin{tikzcd}
Z \times_{f_*I} f_*A_1 \times_{f_*I} f_*A_1 \arrow[r, shift left, "\zeta \times_{f_*I} f_*A_1"] \arrow[r, shift right, "Z \times_{f_*I} f_*c"'] \arrow[d, "\chi"'] & Z \times_{f_*I} f_*A_1 \arrow[r, "\zeta"] \arrow[d, "\chi"'] & (Z,\zeta) \arrow[d, "\bar{\eta}(Z,\zeta)"'] \\
\Phi_*(f^*Z \times_I f^*f_*A_1 \times_I A) \arrow[r, shift left, "\Phi_*(f^*\zeta \times_I A)"] \arrow[r, shift right, "\Phi_*(1\times\xi_A\cdot(\epsilon\times 1))"'] & \Phi_*(f^*Z \times_I A) \arrow[r, "\Phi_*(q_{(Z,\zeta)})"] & \Phi_*\Phi^*(Z,\zeta)
\end{tikzcd}$$

Using now that Φ_* is fully faithful on account of a being $\underline{E}$-generating (Section 8, Theorem), we proceed as follows : Let

$$\begin{tikzcd}
K \arrow[r, shift left, "K_0"] \arrow[r, shift right, "K_1"'] & Z \times_{f_*I} f_*A_1 \arrow[r, "\zeta"] & (Z,\zeta)
\end{tikzcd}$$

be an exact diagram in $\underline{E}^{\underline{C}^{\circ}}$ (again omitting structure maps : we should write (K,k) and $(Z \times_{f_*I} f_*A_1, <\pi_1, f_*c.\pi_{23}>))$, i. e.,

$$(K_0, K_1) = \text{kernel pair } (\zeta) \quad \text{and} \quad \zeta = coeq(K_0, K_1).$$

Consider the canonical epi in $\underline{E}^{\underline{C}^{\circ}}$, $p_K^{\partial_1} : K \times_{f_*I} f_*A_1 \to K$ and recall that

$$\chi : K \times_{f_*I} f_*A_1 \approx \Phi_*(f^*K \times_I A)$$

is iso. This gives the following commutative diagram :

$$\begin{tikzcd}
K \times_{f_*I} f_*A_1 \arrow[r, "p_K^{\partial_1}"] \arrow[d, shift left, "\chi"] & K \arrow[r, shift left, "K_0"] \arrow[r, shift right, "K_1"'] \arrow[dr, shift left, "a_0"] \arrow[dr, shift right, "a_1"'] & Z \times_{f_*I} f_*A_1 \arrow[r, "\zeta"] \arrow[d, shift right, "\chi"'] & (Z,\zeta) \\
\Phi_*(f^*K \times_I A) \arrow[u, shift left, "\chi^{-1}"] \arrow[ur, "r"'] & & \Phi_*(f^*Z \times_I A) \arrow[u, shift right, "\chi^{-1}"'] \arrow[ur, "p"']
\end{tikzcd}$$

with :

$$r = p_K^{\partial_1} \cdot \chi^{-1}, \quad a_i = \chi . K_i \quad (i = 0, 1) \quad \text{and} \quad p = \zeta . \chi^{-1} .$$

Note that

$$(a_0, a_1) = \text{kernel pair } (p) \quad \text{and} \quad p = coeq\,(a_0, a_1) .$$

Since $\Phi *$ is fully faithful, there exist uniquely determined morphisms

$$\begin{tikzcd} f^*K \underset{I}{\times} A \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & f^*Z \underset{I}{\times} A \end{tikzcd} \quad \text{in } \underline{F}$$

such that

$$\Phi *(a_i) = a_i . r \quad (i = 0, 1).$$

The desired property which says that the coequalizer of a_0 , a_1 is precisely

$$\begin{tikzcd} f^*Z \underset{I}{\times} A \arrow[r, "q(Z{,}\zeta)"] & \Phi^*(Z, \zeta) \end{tikzcd}$$

is a consequence now of the following :

LEMMA. *Let $\Phi : \underline{X} \to \underline{Y}$ be a fully faithful functor between toposes, preserving epimorphisms. Consider the diagrams (top one in $\underline{Y}$, bottom one in $\underline{X}$):*

$$\begin{tikzcd} \Phi(T) \arrow[r, "t"] & S \arrow[r, "s"] \arrow[d, "v"'] & \Phi(X) \arrow[r, shift left, "\Phi(\gamma_0)"] \arrow[r, shift right, "\Phi(\gamma_1)"'] \arrow[d, "u"'] & \Phi(Y) \arrow[r, "p"] & Z \\ & \Phi(R) \arrow[r, "r"] & K \arrow[ur, shift left, "a_0"] \arrow[ur, shift right, "a_1"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & & X \arrow[r, shift left, "\gamma_0"] \arrow[r, shift right, "\gamma_1"'] & Y \arrow[r, "q"] & Q \\ T \arrow[urr, bend left, "\ulcorner st \urcorner"] \arrow[r, "\ulcorner vt \urcorner"] & R \arrow[urr, shift left, "a_0"] \arrow[urr, shift right, "a_1"'] & & & \end{tikzcd}$$

where :

$$p = coeq(\Phi(\gamma_0), \Phi(\gamma_1)), \quad (a_0, a_1) = K.p.(p), \quad q = coeq(\gamma_0, \gamma_1)$$

(hence $\Phi(q)$ epi though not necessarily the coequalizer of $\Phi(\gamma_0)$, $\Phi(\gamma_1)$),

u such that $\Phi(\gamma_i) = a_i . u$ *(induced), r epi,*

$\Phi(a_i) = a_i . r\ (i = 0, 1),\ \Phi(\ulcorner st \urcorner) = s . t,\ \Phi(\ulcorner vt \urcorner) = v . t.$

It follows that $q = coeq(a_0, a_1)$.

PROOF OF THE LEMMA. Note that, since $s . t$ is epi, $\ulcorner st \urcorner$ is also epi, since Φ faithful, hence reflects epimorphisms.

Also, since $q = coeq(\gamma_0, \gamma_1)$ and $\ulcorner st \urcorner$ epi, we have

$$q = coeq(\gamma_0 . \ulcorner st \urcorner, \gamma_1 . \ulcorner st \urcorner).$$

Hence, $q = coeq(a_0 . \ulcorner vt \urcorner, a_1 . \ulcorner vt \urcorner)$. So, $q = coeq(a_0, a_1)$. ■

We now look at the diagrams :

$$\begin{tikzcd}
\Phi_*(f^*K \times_I A) \arrow[r, "r"] \arrow[dr, "\Phi_*(r')"'] & K \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] \arrow[d, "\xi"] & \Phi_*(f^*Z \times_I A) \arrow[r, "p"] \arrow[dr, "\Phi_*(q_{(Z,\zeta)})"'] & (Z,\zeta) \arrow[d, "\bar{\eta}_{(Z,\zeta)}"] \\
 & \Phi_*(K') \arrow[ur, shift left, "\Phi_* b_0"] \arrow[ur, shift right, "\Phi_* b_1"'] & & \Phi_*\Phi^*(Z,\zeta)
\end{tikzcd}$$

$$\begin{tikzcd}
f^*K \times_I A \arrow[rr, shift left, "a_0"] \arrow[rr, shift right, "a_1"'] \arrow[dr, "r'"'] & & f^*Z \times_I A \arrow[r, "q_{(Z,\zeta)}"] & \Phi^*(Z,\zeta) \\
 & K' \arrow[ur, shift left, "b_0"] \arrow[ur, shift right, "b_1"'] & &
\end{tikzcd}$$

where (b_0, b_1) = kernel pair$(q_{(Z,\zeta)})$, hence

$$(\Phi_* b_0, \Phi_* b_1) = \text{kernel pair } \Phi_*(q_{(Z,\zeta)})$$

since Φ_* preserves kernel pairs (having a left adjoint). Let ξ be the unique morphism such that $\Phi_* b_i . \xi = a_i\ (i = 0, 1)$. Let r' be the unique morphism such that $b_i . r' = a_i\ (i = 0, 1)$. It follows that

$$\Phi_* b_i . \Phi_* r' = a_i . r\ \ (i = 0, 1).$$

To show $\xi . r = \Phi_*(r')$, observe first that

$$<\Phi_* b_0, \Phi_* b_1> . \xi . r = <a_0, a_1> . r = <a_0 . r, a_1 . r> =$$

$$= <\Phi_* a_0, \Phi_* a_1> = \Phi_* <a_0, a_1> = \Phi_* <b_0 . r', b_1 . r'> =$$

$$= < \Phi_* b_0 , \Phi_* b_1 > . \Phi_* r' .$$

Since $<\Phi_* b_0 , \Phi_* b_1>$ is a kernel pair, hence a (congruence) relation, the induced

$$<\Phi_* b_0 , \Phi_* b_1> : \Phi_* K' \rightarrowtail \Phi_*(f^*Z \underset{I}{\times} A) \times \Phi_*(f^*Z \underset{I}{\times} A)$$

is monic. Hence, $\Phi_* r' = \xi . r$ as required.

The aim now is to show that ξ is an isomorphism since then it will follow from this that both p and $\Phi_*(q_{(Z,\zeta)})$ have the same kernel pair, hence that, as coequalizers, they be the same ; more detailed : since

$$\Phi_*(q_{(Z,\zeta)}) = coeq(a_0 , a_1)$$

there exists a unique

$$\delta : \Phi_*\Phi^*(Z,\zeta) \to (Z,\zeta) \quad \text{such that} \quad p = \delta . \Phi_*(q_{(Z,\zeta)}) .$$

But then, by the uniqueness, $\delta = (\bar{\eta}_{(Z,\zeta)})^{-1}$.

To do so, we shall continue the descent argument further down, as in the following diagram :

$$\begin{tikzcd}
\Phi_*(f^*K'' \underset{I}{\times} A) \arrow[r, "s"] & K'' \arrow[r, shift left, "\gamma_0"] \arrow[r, shift right, "\gamma_1"'] & \Phi_*(f^*K \underset{I}{\times} A) \arrow[r, "r"] \arrow[d, "\Phi_* r''"'] \arrow[dr, "\Phi_* r'"] & K \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] \arrow[d, "\xi"] \arrow[dl, dashed, "\rho"'] & \Phi_*(f^*Z \underset{I}{\times} A) \arrow[r, "p"] \arrow[dr, "\Phi_* q"'] & (Z,\zeta) \arrow[d, "\bar{\eta}_{(Z,\zeta)}"] \\
 & & \Phi_* Q \arrow[r, "\Phi_* \omega"'] & \Phi_* K' \arrow[ur, shift left] \arrow[ur, shift right, "\Phi_* b_i"'] & & \Phi_*\Phi^*(Z,\zeta)
\end{tikzcd}$$

$$\begin{tikzcd}
f^*K'' \underset{I}{\times} A \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & f^*K \underset{I}{\times} A \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] \arrow[d, "r''"'] \arrow[dr, dashed, "r'"] & f^*Z \underset{I}{\times} A \arrow[r, "q"] & \Phi^*(Z,\zeta) \\
 & Q \arrow[r, "\omega"'] & K \arrow[u, shift left, "b_0"] \arrow[u, shift right, "b_1"'] &
\end{tikzcd}$$

where : (c_0 , c_1) = unique maps with $\Phi^*(c_i) = \gamma_i . s$, $i = 0, 1$,

(γ_0 , γ_1) = kernel pair(r), s = canonical epi, $r'' = coeq(c_0 , c_1)$

Since $r' . c_0 = r' . c_1$ (this can be seen by passing to the Φ_*-images:

$$\Phi*(r'.c_0) = \Phi*r'.\gamma_0.s = \xi.r.\gamma_0.s = \xi.r.\gamma_1.s =$$
$$= \Phi*r'.\gamma_1.s = \Phi*(r'.c_1))$$

and $r'' = coeq(c_0, c_1)$, there exists a unique ω such that $\omega.r'' = r'$. The same situation repeats itself above, namely, $\Phi*\omega.\Phi*r'' = \Phi*r'$. Since

$$\Phi*r''.\gamma_0.s = \Phi*r''.\gamma_1.s$$

(this is so since $\gamma_i.s = \Phi*c_i$ and since $r''.c_0 = r''.c_1$), it follows (as $r = coeq(\gamma_0.s, \gamma_1.s)$) that there exists a unique ρ (dotted arrow) with $\rho.r = \Phi*r''$.

We also have that $\Phi*\omega.\rho = \xi$ (note that

$$\Phi*\omega.\rho.r = \Phi*\omega.\Phi*r'' = \Phi*(\omega.r'') = \Phi*(r') = \xi.r).$$

Hence, if we show that ρ is iso, if we then prove $\Phi*\omega$ iso, it would follow that ξ is iso (ξ is already monic, though). Since ξ is monic, ρ is monic and since $\rho.r = \Phi*r''$ and r'' epi, $\Phi*r''$ epi, hence ρ epi. So, ρ is iso. Now

$$a_i = \Phi*b_i.\xi \quad \text{and} \quad \xi = \Phi*\omega.\rho$$

hence

$$<a_0,a_1> = \Phi*<b_0.\omega, b_1.\omega>.\rho ,$$

so the latter is a congruence relation, as $<a_0,a_1>$ is one. Since $\Phi*$ is faithful and ρ iso, it follows that $<b_0.\omega, b_1.\omega>$ is a congruence relation, hence the kernel pair of its own coequalizer. We know that $q = coeq(a_0, a_1)$ hence

$$q = coeq(b_0.\omega.r'', b_1.\omega.r'') \quad \text{and} \quad q = coeq(\omega.b_0, \omega.b_1).$$

But (b_0, b_1) = kernel pair (q). Thus, ω is iso, hence $\Phi*\omega$ is iso as required. This completes the proof. ■

10. The main Theorem.

We have already given a proof of the following

THEOREM. *Let* $\underline{F} \xrightarrow{f} \underline{E}$ *be a geometric morphism of elementary toposes. Then, the following are equivalent:*

(a) there exists a category $\underline{C}$ *in* $\underline{E}$ *and a factorization*

$$\begin{tikzcd} \underline{F} \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \underline{E}^{\underline{C}^{\circ}} \arrow[dl] \\ & \underline{E} & \end{tikzcd}$$

in $\underline{Top}$*, with* $\Phi = (\Phi^*, \Phi_*)$ *an adjoint equivalence of categories;*

(b) there exists an internal family $A \xrightarrow{a} I$ *in* $\underline{F}$*,* $\underline{E}$*-generating and* $\underline{E}$*-atomic.*

PROOF. In order to prove that the conditions (b) are necessary for (a), we need only invoke the lemmas proven in Sections 8 and 9, noting for this that conditions (b) are stable under equivalences of categories. As for the implication (b) $\Longrightarrow$ (a), this is the contents of the theorems of Sections 8 and 9 combined. ■

We may now ask if for the case $\underline{E} = \underline{S}$, the category of sets, the above theorem reduces to the theorem proved by us in [4]. The answer is that it does not, and that it gives a better result, in the sense that it is an elementary characterization. In a sense, the coproducts which we needed in [4] have not entirely disappeared: The assumption that $\underline{F}$ be a topos over $\underline{S}$, i.e., $\underline{F} \xrightarrow{f} \underline{S}$ geometric, already says that $\underline{S}$-indexed copowers exist. However, this is now strictly first-order.

A final observation is that a by-product of our theorem is a form of the metatheorem of Lawvere's elementary theory of the category of sets [11]: discarding those among the eight axioms which are part of the topos structure on the category, we are left with axioms which say that 1 is a projective generator and that coproducts are disjoint (1 is abstractly unary), i.e., with the special case of our theorem for

$$E = S \quad \text{and} \quad A \xrightarrow{a} I = 1 \to 1,$$

namely: Given a topos $\underline{F}$ over $\underline{S}$ (which can be done in essentially one way), $\underline{F}$ is equivalent to $\underline{S}$ iff 1 is an $\underline{S}$-generator and an $\underline{S}$-atom in $\underline{F}$. As in Lawvere's metatheorem, there is here a cocompleteness assumption: The existence of a geometric $\underline{F} \xrightarrow{f} \underline{S}$ says that $\underline{F}$ has all copowers of 1 indexed by a set.

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