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An abstract setting for homotopy pushouts and pullbacks

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AN ABSTRACT SETTING FOR HOMOTOPY PUSHOUTS AND PULLBACKS

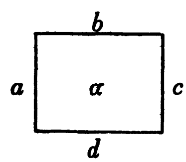
by Christopher B. SPENCER

INTRODUCTION.

Starting with a 2-category, a double category of homotopy commutative squares having additional structure in the form of a connection, generalising the connections of double categories defined in [2, 3], can be constructed. I shall show that the category $\mathfrak{D}$ of such objects is equivalent to the category of 2-categories. My main aim is to present the objects of $\mathfrak{D}$ as a general setting for various results in homotopy theory dealing with homotopy pushouts and pullbacks. See for example [7, 8, 9, 10, 11, 13, 14, 16].

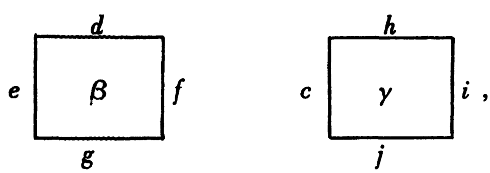
NOTATION.

I continue the notation and conventions of [3]. A double category D is thus viewed as a collection of squares D_2 with two operations, $\circ$ and $+$, giving rise to vertical and horizontal category structures, together with vertical and horizontal edge categories V, H over the same class of objects C_0. A square α together with its edges is represented in the diagram



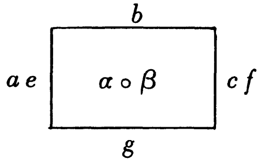
$$\epsilon_0\alpha = b, \quad \epsilon_1\alpha = d, \quad \delta_0\alpha = a, \quad \delta_1\alpha = c,$$

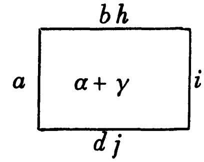
and given squares



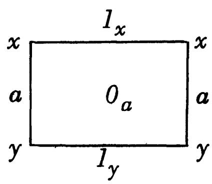
$$\alpha \circ \beta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \text{and} \quad \alpha + \gamma = [\alpha \ \gamma]$$

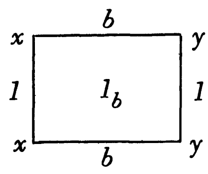
are defined and have edges as follows





The identities on V and H are both denoted by 1_x, or simply 1. On D_2 the identities with respect to $+$ and $\circ$ have edges

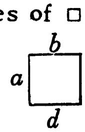




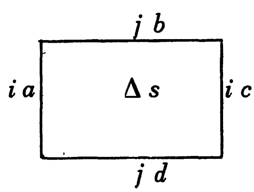
respectively, and $0_{1_x} = 1_{1_x}$ is written $\odot_x$, or simply $\odot$.

A 2-category may be regarded as a double category into which H is the trivial one point category.

1. CONNECTIONS.

Let D be a double category and A a category. An *A-connection on* D (Brown) is a morphism of double categories $\Delta : \square A \to D$ where $\square A$ denotes the double category of commutative squares in A. Given an edge $a : x \to y$ of A, Δ assigns vertical and horizontal edges $i\,a$, $j\,a$ of D to the corresponding vertical and horizontal edges of $\square A$ represented by a. Thus Δ assigns to each commuting square $s =$ 

in A (thus, $bc = ad$) a square $\Delta(s)$ with edges



Functions $\Gamma, \Gamma' : A \to D_2$ for which Γa, $\Gamma' a$ have edges given by

$$\begin{array}{ccc} & ja & \\ ia & \boxed{\quad \Gamma a \quad} & 1 \\ & 1 & \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ 1 & \boxed{\quad \Gamma' a \quad} & ia \\ & ja & \end{array}$$

are determined by restricting Δ to squares of $\square A$ of the form

$$\begin{array}{ccc} & a & \\ a & \boxed{\qquad} & 1 \\ & 1 & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & 1 & \\ 1 & \boxed{\qquad} & a \\ & a & \end{array}$$

respectively.

The morphism properties of Δ ensure the following properties of the functions Γ, Γ' :

$$\begin{array}{lll} & \text{(i)} & (\Gamma a + 1_{jb}) \circ \Gamma b = \Gamma ab, \\ & \text{(ii)} & \Gamma' a \circ (1_{ja} + \Gamma' b) = \Gamma' ab, \\ (1.1) & \text{(iii)} & \Gamma 1_x = \Gamma' 1_x = \odot_x, \\ & \text{(iv)} & \Gamma' a + \Gamma a = 1_{ja}, \\ & \text{(v)} & \Gamma' a \circ \Gamma a = 0_{ia}, \end{array}$$

where $a : x \to y$ and $b : y \to z$ are edges in A. By defining

$$(1.2) \qquad \Delta(s) = (0_{ia} \circ \Gamma' d) + (\Gamma b \circ 0_{ic})$$

for s in $\square A$ with edges

$$\begin{array}{ccc} & b & \\ a & \boxed{\quad s \quad} & c \\ & d & \end{array}$$

the connection Δ can be recovered from the functions Γ, Γ' satisfying the above conditions.

REMARKS. 1. Conditions (i) and (ii) may be compared with the transport condition for a connection on a special double groupoid as defined in [2,3]. In this situation a function Γ' satisfying the above properties is obtained from Γ by taking $\Gamma' = -(\Gamma a^{-1})^{-1}$.

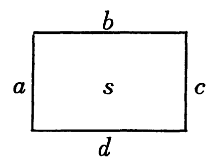
2. In a previous version of this note I had worked entirely with the func-

tions Γ, Γ' in slightly less general setting and I am grateful to R. Brown for his more elegant notion of A-connection.

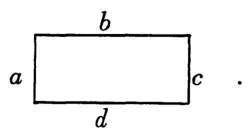
For the remainder of this note I shall consider only double categories D of the special type in which $H = V = D_1$ and all connections on D will be D_1-connections for which $i = j =$ identity.

As for double groupoids with connections we have the notion of degenerate square. Here a square is called *degenerate* if it has a decomposition $\alpha = [\alpha_{ij}]$ in which α_{ij} is either 0_a, 1_a, Γa or $\Gamma' a$ for some edge a in D_1. The following result generalises Proposition 2 of [2].

PROPOSITION 1.1. *Given the square*



in $\square D_1$, $\Delta(s)$ *is the unique degenerate square of* D *having the edges*



PROOF. Since Δ is a morphism of double categories,

$$0_a = \Delta(0_a), \quad 1_a = \Delta(1_a).$$

Thus by the construction of Γ and Γ' all degenerate squares α have a decomposition $\alpha = [\Delta(s_{ij})]$ where s_{ij} and $s = [s_{ij}]$ are squares of $\square D_1$. Again by the morphism properties of Δ, $\alpha = \Delta(s)$.

2. 2-CATEGORIES AND DOUBLE CATEGORIES.

Firstly I describe the category $\mathfrak{D}$ of those double categories relevant to our discussion. An object of $\mathfrak{D}$ is a pair (D, Δ) where D is a double category and $\Delta: D_1 \to D_2$ is a (special) connection on D. Morphisms of $\mathfrak{D}$ are morphisms of double categories preserving the connections. Note that morphisms preserve the connection Δ if and only if they preserve the associated functions Γ, Γ'.

Let 2-$\mathcal{C}$ denote the category of 2-categories.

THEOREM 2.1. *There exists an equivalence of categories* $\rho : 2\text{-}\mathcal{C} \rightleftarrows \mathfrak{D} : \omega$ *such that* ρ *is a right adjoint of* ω.

PROOF. Given a 2-category C I define below a double category with connection $\rho(C) = (D, \Delta)$:

Take D to be the double category $Q(C)$ of up-squares of C $([1], C)$. That is $D_0 = C_0$, $D_1 = C_1$ and the squares with edges $a\,\underset{d}{\overset{b}{\boxed{\quad}}}\,c$ are quintuples

$$(\alpha;\, a \,{}^{b}_{d}\, c) \quad \text{such that} \quad \alpha \in C_2 \text{ has edges } ad\,\underset{1}{\overset{1}{\boxed{\;\alpha\;}}}\,bc.$$

Vertical and horizontal composition are defined respectively by:

$$(\alpha;\, a \,{}^{b}_{d}\, c) \circ (\beta;\, e \,{}^{d}_{g}\, f) = ((0_a \circ \beta) + (\alpha \circ 0_f);\, ae \,{}^{b}_{g}\, cf)$$

and

$$(\alpha;\, a \,{}^{b}_{d}\, c) + (\gamma;\, c \,{}^{h}_{j}\, i) = ((\alpha \circ 0_j) + (0_b \circ \gamma);\, a \,{}^{bh}_{dj}\, i).$$

It is straightforward to check this gives the structure of a double category in which the identities and zeros are

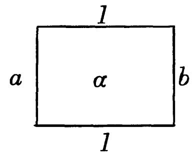
$$(0_b;\, 1 \,{}^{b}_{b}\, 1) \quad \text{and} \quad (0_a;\, a \,{}^{1}_{1}\, a), \quad \text{respectively.}$$

The connection Δ is obtained from

$$\Gamma a = (0_a;\, a \,{}^{a}_{1}\, 1), \quad \Gamma' a = (0_a;\, 1 \,{}^{1}_{a}\, a)$$

and equation (1.2). Properties (i)-(v) are immediate and clearly ρ extends to a functor $\rho : 2\text{-}\mathcal{C} \to \mathfrak{D}$.

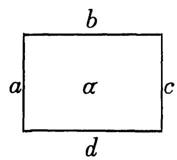
Conversely, given a double category D take $\omega(D)$ to be the 2-category obtained by taking the sub-double category of D consisting of squares of the form



(D' in [12]). Again ω extends to a functor $\omega : \mathfrak{D} \to 2\text{-}\mathcal{C}$ in an obvious way.

Corresponding to an observation in Proposition 2.4 of [12] there is a natural isomorphism $\psi : \omega\rho \to 1_{2\text{-}\mathcal{C}}$ determined by the identity maps on the squares, edges and vertices.

Next I obtain a natural transformation $\phi : 1_{\mathfrak{D}} \to \rho\omega$. Let $\bar{D}$ be an object (D, Γ, Γ') of $\mathfrak{D}$. Define $\phi(\bar{D}) : \bar{D} \to \rho\omega(\bar{D})$ to be the identity on the vertices and edges; and given a square



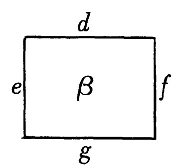
set

$$\phi(\bar{D})(\alpha) = (\Gamma' b \circ \alpha \circ \Gamma d \,;\, a \;\begin{smallmatrix} b \\ d \end{smallmatrix}\; c).$$

Then

$$\phi(\bar{D})(\alpha \circ \beta) = (\Gamma' b \circ \alpha \circ \beta \circ \Gamma g \,;\, ae \;\begin{smallmatrix} b \\ g \end{smallmatrix}\; cf)$$

where



while

$$\phi(\bar{D})(\alpha) \circ \phi(\bar{D})(\beta) = (\delta \,;\, ae \;\begin{smallmatrix} b \\ g \end{smallmatrix}\; cf)$$

where

$$\delta = (0_a \circ \Gamma' d \circ \beta \circ \Gamma g) + (\Gamma' b \circ \alpha \circ \Gamma d \circ 0_f)$$
$$= \Gamma' d \circ \alpha \circ (\Gamma' d + \Gamma d) \circ \beta \circ \Gamma g = \Gamma' d \circ \alpha \circ \beta \circ \Gamma g$$

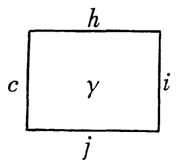
by (1.1) (iv). Thus

$$\phi(\bar{D})(\alpha \circ \beta) = \phi(\bar{D})(\alpha) \circ \phi(\bar{D})(\beta).$$

Also

$$\phi(\bar{D})(\alpha + \gamma) = (\Gamma' bh \circ (\alpha + \beta) \circ \Gamma dj \ ; \ a \ \begin{matrix} bh \\ dj \end{matrix} \ i)$$

where



while

$$\phi(\bar{D})(\alpha) + \phi(\bar{D})(\gamma) = (\epsilon \ ; \ a \ \begin{matrix} bh \\ dj \end{matrix} \ i),$$

where

$$\begin{aligned} \epsilon &= (\Gamma' b \circ \alpha \circ \Gamma d \circ 0_j) + (0_b \circ \Gamma' h \circ \gamma \circ \Gamma j) \\ &= \Gamma' bh \circ (\alpha + \gamma) \circ \Gamma dj, \end{aligned}$$

by the interchange law in D and the transport conditions (1.1) (i) and (ii). I have now proved $\phi(\bar{D})$ is a morphism of double categories. Also, applying condition (1.1) (v), it is readily shown that

$$\phi(\bar{D})\Gamma a = (0_a \ ; \ a \ \begin{matrix} a \\ 1 \end{matrix} \ 1) \quad \text{and} \quad \phi(\bar{D})\Gamma' a = (0_a \ ; \ 1 \ \begin{matrix} 1 \\ a \end{matrix} \ a)$$

and hence $\phi(\bar{D})$ preserves the connections.

Since $\phi(\bar{D})$ is bijective on faces with inverse $\eta : \rho\omega(\bar{D}) \to \bar{D}$ defined on faces by

$$\eta(\alpha \ ; \ a \ \begin{matrix} b \\ d \end{matrix} \ c) = (0_a \circ \Gamma' d) + \alpha + (\Gamma b \circ 0_c),$$

$\phi(\bar{D})$ is an isomorphism of double categories and the first part of the Theorem is proved.

Finally the identities

(a) $(\rho\psi)(\phi\rho) = 1_\rho$ and (b) $(\psi\omega)(\omega\phi) = 1_\omega$

are easily verified (the proof of (b) requires (1.1) (iii)) showing that ρ is a right adjoint of ω. This completes the proof.

Now let $2\text{-}\mathcal{C}^!$ be the full sub-category of $2\text{-}\mathcal{C}$ consisting of those 2-categories in which, for each pair of vertices x, y, the squares

$$\begin{array}{c} 1_x \\ \boxed{\quad \alpha \quad} \\ 1_y \end{array}$$

form a groupoid under + ([4], page 81) (inverses will accordingly be denoted by -), and let $\mathfrak{D}^!$ be the full sub-category of $\mathfrak{D}$ whose objects are double categories D (with connections) for which the 2-category $\omega(D)$ is an object of $2\text{-}\mathcal{C}^!$.

COROLLARY 2.2. *The functors* ρ, ω *restrict to an equivalence of categories* $2\text{-}\mathcal{C}^! \underset{\omega^!}{\overset{\rho^!}{\rightleftarrows}} \mathfrak{D}^!$ *and* $\rho^!$ *is a right adjoint of* $\omega^!$.

Objects of either categories $2\text{-}\mathcal{C}^!$ or $\mathfrak{D}^!$ may be taken as a framework for abstract homotopy theory. For example R. M. Vogt's result on strong homotopy equivalences [15] in an object C of $2\text{-}\mathcal{C}^!$ translates as follows.

An edge $a: x \to y$ in C_1 is a *homotopy equivalence* if there is a homotopy inverse $\bar{a}: y \to x$ and squares

$$\begin{array}{c} 1_x \\ a\bar{a}\ \boxed{\quad \delta \quad}\ 1_x \\ 1_x \end{array} \qquad\qquad \begin{array}{c} 1_y \\ \bar{a}a\ \boxed{\quad \epsilon \quad}\ 1_y \\ 1_y \end{array}.$$

(That is, in the language of [4], a represents an equivalence in $\overline{\omega(D)}$, the category $\omega(D)$ modulo homotopy.) I call $(a, \bar{a}, \delta, \epsilon)$ *a strong homotopy equivalence* if

$$0_{\bar{a}} \circ \delta = \epsilon \circ 0_{\bar{a}} \quad \text{and} \quad 0_a \circ \epsilon = \delta \circ 0_a .$$

PROPOSITION 2.3. *Given any homotopy equivalence* a *with homotopy inverse* $\bar{a}$ *and a homotopy* $\begin{array}{c} 1 \\ a\bar{a}\ \boxed{\ \delta\ }\ 1 \\ 1 \end{array}$, *then* $(a, \bar{a}, \delta, \epsilon)$ *is a strong homotopy* *equivalence, where*

$$\epsilon = (-0_{\bar{a}a} \circ \bar{\epsilon}) + (0_{\bar{a}} \circ \delta \circ 0_a) + \bar{\epsilon}$$

and $\begin{array}{ccc} & 1 & \\ \bar{a}a & \boxed{\bar{\epsilon}} & 1 \\ & 1 & \end{array}$ *is arbitrary.*

PROOF. Follow Vogt's argument verbatim.

However to handle pushout and pullback squares and homotopy commutative squares in general I believe it is more convenient to work with squares in objects of $\mathfrak{D}^!$ (the connections allow one to turn everything into a square). We consider below some general properties of these objects.

For each object (D, Δ) of $\mathfrak{D}^!$ there is a *reflection* $r: D_2 \to D_2$ such that on edges r behaves as follows:

$$\begin{array}{ccc} & b & \\ a & \boxed{\quad \alpha \quad} & c \\ & d & \end{array} \qquad \qquad \begin{array}{ccc} & a & \\ b & \boxed{\quad r(\alpha) \quad} & d \\ & c & \end{array}$$

and $r(\alpha)$ is defined by

$$r(\alpha) = (0_b \circ \Gamma' c) - (\Gamma' b \circ \alpha \circ \Gamma d) + (\Gamma a \circ 0_d).$$

In the case of double groupoids with connection, $r(\alpha) = -\tau(\alpha)$ where τ is the rotation of Theorem C in [3]. Corresponding to that theorem we have the

THEOREM 2.4. *The reflection r satisfies:*

(i) $r(\alpha + \beta) = r(\alpha) \circ r(\beta)$ *whenever* $\alpha + \beta$ *is defined,*

(ii) $r(\alpha \circ \gamma) = r(\alpha) + r(\gamma)$ *whenever* $\alpha \circ \gamma$ *is defined,*

(iii) $r^2 = id$,

(iv) r *determines an isomorphism of 2-categories* $r: \omega(D) \to \omega^v(D)$, *where* $\omega^v(D)$ *denotes the 2-category of squares*

$$\begin{array}{ccc} & a & \\ 1 & \boxed{\qquad\quad} & 1 \\ & b & \end{array}$$

with the operations + *and* ∘ *on* $\omega(D)$ *interchanged,*

(v) $r\Gamma = \Gamma$, $r\Gamma' = \Gamma'$,

(vi) $(\Gamma' a + \alpha) \circ (r(\alpha) + \Gamma d) = \Gamma b + \Gamma' c$,

(vii) $(\Gamma' a \circ r(\alpha)) + (\alpha \circ \Gamma d) = \Gamma b \circ \Gamma' c$.

PROOF. By Corollary 2.2 it suffices to consider double categories $D = \rho(C)$ arising from a 2-category C in $2\text{-}\mathcal{C}^!$. It is readily checked that under the isomorphism $\phi(D): D \to \rho\omega(D)$ the rotation on $\rho\omega(D)$ becomes

$$r(\alpha;\ a \overset{b}{\underset{d}{}}\ c) = (-\alpha;\ b \overset{a}{\underset{c}{}}\ d).$$

The condition (2.4)(iii) is immediate and, for (i),

$$r((\alpha;\ a \overset{b}{\underset{d}{}}\ c) + (\beta;\ c \overset{e}{\underset{g}{}}\ f)) = ((-0_b \circ \beta) - (\alpha \circ 0_g);\ be \overset{a}{\underset{f}{}}\ dg) =$$

$$= (-\alpha;\ b \overset{a}{\underset{c}{}}\ d) \circ (-\beta;\ e \overset{c}{\underset{f}{}}\ g) =$$

$$= r(\alpha;\ a \overset{b}{\underset{d}{}}\ c) \circ r(\beta;\ c \overset{e}{\underset{g}{}}\ f).$$

(ii) follows from (i) and (iii); and (iv) follows from (i), (ii) and (iii). The remaining properties are easily verified directly.

3. PUSHOUT AND PULLBACK SQUARES.

Throughout this Section I will work in a double category D with connection Δ (and associated functions Γ, Γ') such that (D, Δ) is an object of $\mathfrak{D}^!$.

DEFINITION 3.1. A *pullback square in* D is an element $\alpha \in D_2$ such that for any element $\beta \in D_2$ with

$$\epsilon_1 \beta = \epsilon_1 \alpha, \quad \partial_1 \beta = \partial_1 \alpha,$$

there exists $\gamma_1, \gamma_2 \in D_2$ with

$$\epsilon_0 \gamma_1 = \epsilon_0 \gamma_2 = c \text{ (say)}, \quad \epsilon_1 \gamma_1 = 1, \quad \epsilon_1 \gamma_2 = 1, \quad \partial_1 \gamma_1 = \partial_0 \alpha, \partial_1 \gamma_2 = \epsilon_0 \alpha$$

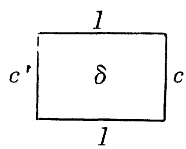
such that

$$(3.1) \qquad \beta = \begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & \alpha \end{bmatrix}$$

and, in addition, if

$$\beta = \begin{bmatrix} \Gamma' c' & r(\gamma_2') \\ \gamma_1' & \alpha \end{bmatrix}$$

is another such representation, then there exists



such that

$$\delta + r(\gamma_i) = r(\gamma_i') \quad (i = 1, 2).$$

Dually, I call α a *pushout square* if any $\bar{\beta} \in D_2$ with $\epsilon_0 \bar{\beta} = \epsilon_0 \alpha$, $\partial_0 \bar{\beta} = \partial_0 \alpha$ may be written

$$\bar{\beta} = \begin{bmatrix} \alpha & r(\bar{\gamma}_2) \\ \bar{\gamma}_1 & \Gamma c \end{bmatrix}$$

where

$$\epsilon_0 \bar{\gamma}_1 = \epsilon_1 \alpha, \ \epsilon_0 \bar{\gamma}_2 = \delta_0 \alpha, \ \partial_0 \bar{\gamma}_1 = 1, \ \partial_0 \bar{\gamma}_2 = 1, \ \partial_1 \bar{\gamma}_1 = \partial_1 \bar{\gamma}_2 = c,$$

and for any other such representation there exists $\bar{\delta} \in \omega(D)_2$ such that

$$\bar{\gamma}_i + \bar{\delta} = \bar{\gamma}_i' \quad (i = 1, 2).$$

The usual uniqueness up to homotopy pushout and pullback squares holds.

PROPOSITION 3.2. *Let α, α' be pullback squares with*

$$\epsilon_1 \alpha = \epsilon_1 \alpha', \ \partial_1 \alpha = \partial_1 \alpha'.$$

Then

$$\alpha' = \begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & \alpha \end{bmatrix}$$

in which $c: \partial_0 \partial_0 \alpha' \to \partial_0 \partial_0 \alpha$ *is a homotopy equivalence.*

PROPOSITION 3.3. *Let* α, α' *be pushout squares with*

$$\epsilon_0 \alpha = \epsilon_0 \alpha', \quad \partial_0 \alpha = \partial_0 \alpha'.$$

Then

$$\alpha' = \begin{bmatrix} \alpha & r(\bar{\gamma}_2) \\ \bar{\gamma}_1 & \Gamma c \end{bmatrix}$$

in which $c: \partial_1 \epsilon_1 \alpha \to \partial_1 \epsilon_1 \alpha'$ *is a homotopy equivalence.*

PROPOSITION 3.4. *If* α *be a pullback (pushout) square then so is* $r(\alpha)$ *a pullback (pushout) square.*

PROOF. I consider only the pullback case. Let α be a pullback square and σ an element of D_2 such that

$$\epsilon_1 \sigma = \epsilon_1 r(\alpha) = \partial_1 \alpha, \quad \partial_1 \sigma = \partial_1 r(\alpha) = \epsilon_1 \alpha .$$

Then I may write

$$r(\sigma) = \begin{bmatrix} \Gamma' c & r(\bar{\gamma}_2) \\ \bar{\gamma}_1 & \alpha \end{bmatrix}$$

and applying r to this equation obtain

$$\sigma = r(r(\sigma)) = \begin{bmatrix} \Gamma' c & r(\bar{\gamma}_1) \\ \bar{\gamma}_2 & r(\alpha) \end{bmatrix} = \begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & r(\alpha) \end{bmatrix}$$

where I have put $\gamma_1 = \bar{\gamma}_2$, $\gamma_2 = \bar{\gamma}_1$. Thus equation (3.1) in Definition 3.1 is satisfied. Now suppose

$$\sigma = \begin{bmatrix} \Gamma' c' & r(\gamma_2') \\ \gamma_1' & r(\alpha) \end{bmatrix} .$$

Then

$$r(\sigma) = \begin{bmatrix} \Gamma' c' & r(\gamma_1') \\ \gamma_2' & \alpha \end{bmatrix}$$

implying the existence of $\delta \in \omega(D)_2$ such that

$$\delta + r(\gamma_i) = r(\gamma_i') \quad (i = 1, 2)$$

and completing the proof.

The «uniqueness up to homotopy» part of Definition 3.1 may be extended to allow the γ's to have more general edges. More precisely, we have the

LEMMA 3.5. *Let α be a pullback square and suppose*

$$\begin{bmatrix} \Gamma' c & r(\alpha_2) \\ \alpha_1 & \alpha \end{bmatrix} = \begin{bmatrix} \Gamma' c' & r(\alpha_2') \\ \alpha_1' & \alpha \end{bmatrix}$$

where $d_i = \epsilon_1(\alpha_i) = \epsilon_1(\alpha_i')$ $(i = 1, 2)$, *then there exists* $\delta \in \omega(D)_2$ *with*

$$\delta + r(\alpha_i) = r(\alpha_i') \quad (i = 1, 2).$$

The dual result also holds.

PROOF. I consider only the pullback case. Since

$$\begin{bmatrix} \Gamma' c & r(\alpha_2) + \Gamma d_2 \\ \alpha_1 \circ \Gamma d_1 & \alpha \end{bmatrix} = \begin{bmatrix} \Gamma' c' & r(\alpha_2') + \Gamma d_2 \\ \alpha_1' \circ \Gamma d_1 & \alpha \end{bmatrix}$$

and α is a pullback square, there exists $\delta \in \omega(D)_2$ such that

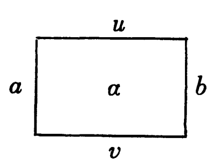
(i) $\delta + r(\alpha_1 \circ \Gamma d_1) = r(\alpha_1' \circ \Gamma d_1)$, and

(ii) $\delta + r(\alpha_2 + \Gamma d_2) = r(\alpha_2' + \Gamma d_2)$.

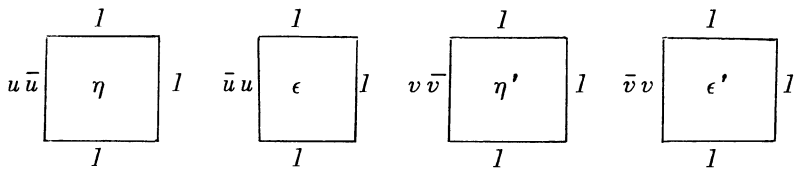
From (i), on composing with $\Gamma' d_1$, we obtain $\delta + r(\alpha_1) = r(\alpha_1')$. Similarly using (ii) we may show $\delta + r(\alpha_2) = r(\alpha_2')$.

PROPOSITION 3.6. *Let α be an element of D_2 such that one pair of opposite edges are homotopy equivalences. Then α is both a pullback and a pushout square.*

PROOF. By Proposition 3.4 and duality it suffices to show that the element α of D_2 with edges



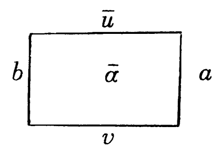
is a pullback square if u, v are homotopy equivalences. By Proposition 1.3 we may assume we have strong homotopy equivalences $(u, \bar{u}, \eta, \epsilon)$ and $(v, \bar{v}, \eta', \epsilon')$. Then η, ϵ, η', ϵ' have edges as follows



and

$$(3.2)\qquad \begin{aligned} 0_{\bar{u}} \circ \eta &= \epsilon \circ 0_{\bar{u}}, \quad \eta \circ 0_u = 0_u \circ \epsilon \\ 0_{\bar{v}} \circ \eta' &= \epsilon' \circ 0_{\bar{v}}, \quad \eta' \circ 0_v = 0_v \circ \epsilon'. \end{aligned}$$

I begin by constructing a square



such that

$$(3.3)\qquad r(\epsilon)^{-1} \circ (\bar{\alpha} + \alpha) \circ r(\epsilon') = 0_b$$

and

$$(3.4)\qquad r(\eta)^{-1} \circ (\alpha + \bar{\alpha}) \circ r(\eta') = 0_a .$$

Let $\gamma = \Gamma' u \circ \alpha \circ \Gamma v$ and set

$$\bar{\alpha} = (-\epsilon \circ 0_b) + (\Gamma \bar{u} \circ (-\gamma) \circ \Gamma' \bar{v}) + (0_a \circ \eta').$$

Now $\alpha = (0_a \circ \Gamma' v) + \gamma + (\Gamma u \circ 0_b)$ and

$$\begin{aligned} \eta' + \Gamma' v &= (0_{v\bar{v}} \circ \Gamma' v) + (\eta' \circ 0_v) \\ &= (0_{v\bar{v}} \circ \Gamma' v) + (0_v \circ \epsilon'), \quad \text{by (3.2)}, \\ &= 0_v \circ ((0_{\bar{v}} \circ \Gamma' v) + \epsilon'). \end{aligned}$$

Thus,

$$\begin{aligned} (\Gamma \bar{u} \circ (-\gamma) \circ \Gamma' \bar{v}) + (0_a \circ \eta') + (0_a \circ \Gamma' v) + \gamma &= \\ = \Gamma \bar{u} \circ 0_{av} \circ (\Gamma' \bar{v} v + \epsilon'), \end{aligned}$$

since it is equal to

$$\begin{bmatrix} \Gamma\bar{u} & \odot & \odot & \odot \\ -\gamma & 0_{av} & 0_{av} & \gamma \\ \Gamma'\bar{v} & 0_{\bar{v}}\circ\Gamma'v & \epsilon' & \odot \end{bmatrix}.$$

From which I obtain

$$\bar{\alpha} + \alpha = (-\epsilon + \Gamma\bar{u}u)\circ 0_b \circ (\Gamma'\bar{v}v + \epsilon').$$

Now

$$r(\epsilon) = -\epsilon + \Gamma\bar{u}u \quad \text{and} \quad r(\epsilon') = -\epsilon' + \Gamma\bar{v}v,$$

and hence, $r(\epsilon)^{-1}\circ(-\epsilon + \Gamma\bar{u}u) = \odot$ and

$$(\Gamma'v\bar{v} + \epsilon')\circ r(\epsilon') = (\Gamma'\bar{v}v + \epsilon')\circ(-\epsilon' + \Gamma\bar{v}v)$$
$$= -\epsilon' + (\Gamma'\bar{v}v + \Gamma\bar{v}v) + \epsilon' = \odot.$$

Thus I have at last arrived at equation (3.3). (3.4) follows by symmetry.

After the above preliminaries I now proceed to prove α is a pull-back square. Let

$$\begin{array}{ccc} & e & \\ d & \boxed{\beta} & b \\ & v & \end{array},$$

then if $\gamma_1 = (\beta + \bar{\alpha})\circ r(\eta')$ and $r(\gamma_2) = \Gamma e \circ ((0_{\bar{u}}\circ\Gamma'u) + \epsilon)$ we have

$$\begin{bmatrix} \Gamma'e\bar{u} & r(\gamma_2) \\ \gamma_1 & \alpha \end{bmatrix} = \begin{bmatrix} 1_e & \Gamma'\bar{u} + (0_{\bar{u}}\circ\Gamma'u) & \epsilon \\ \beta & \bar{\alpha} + \alpha & 0_b \\ 1_v & r(\eta') & \odot \end{bmatrix}$$

employing (3.1),

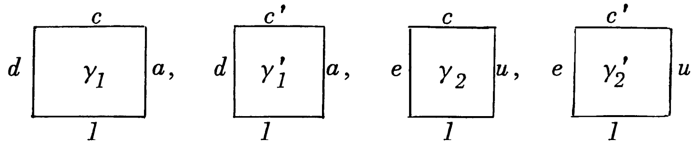
$$= \begin{bmatrix} 1_e & \Gamma'\bar{u}u & \epsilon \\ 1_e & -\epsilon + \Gamma\bar{u}u & \odot \\ \beta & 0_b & 0_b \end{bmatrix}, \text{ by (3.3)},$$

$$= \beta - \epsilon + \Gamma'\bar{u}u + \Gamma\bar{u}u + \epsilon = \beta.$$

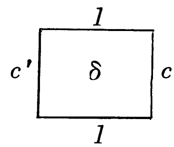
Finally, suppose

$$\begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & \alpha \end{bmatrix} = \begin{bmatrix} \Gamma' c' & r(\gamma_2') \\ \gamma_1' & \alpha \end{bmatrix}$$

where the γ 's have edges as follows



Then define



by

$$\delta = (0_{c'} \circ -\eta) + (\bar{\delta} \circ 0_{\bar{u}}) + (0_c \circ \eta),$$

where $\bar{\delta} = -(\Gamma' c' \circ \gamma_2') + (\Gamma' c \circ \gamma_2)$. Then

$$\begin{aligned}(\delta + r(\gamma_2)) \circ \Gamma u &= (0_{c'} \circ -\eta \circ 0_u) + (\bar{\delta} \circ 0_{\bar{u}u}) + (0_c \circ \eta \circ 0_u) + \\ &\quad + (\Gamma' u \circ \Gamma u) - (\Gamma' c \circ \gamma_2) + \Gamma e \\ &= (0_{c'u} \circ -\epsilon) - (\Gamma' c' \circ \gamma_2' \circ 0_{\bar{u}u}) + (\Gamma' c \circ \gamma_2 \circ 0_{\bar{u}u}) \\ &\quad + (0_{cu} \circ \epsilon) - (\Gamma' c \circ \gamma_2) + \Gamma e, \quad \text{by } (3.1), \\ &= -(\Gamma' c' \circ \gamma_2') + \Gamma e.\end{aligned}$$

Therefore,

$$\begin{aligned}\delta + r(\gamma_2) &= (0_{c'} \circ \Gamma' u) + ((\delta + r(\gamma_2)) \circ \Gamma u) \\ &= (0_{c'} \circ \Gamma' u) - (\Gamma' c' \circ \gamma_2') + \Gamma e = r(\gamma_2').\end{aligned}$$

Furthermore, $r(\bar{\delta}) \circ (\gamma_1 + a) = (\gamma_1' + a)$. Thus

$$(r(\bar{\delta}) \circ (\gamma_1 + a)) + \bar{a} = \gamma_1' + a + \bar{a}.$$

So by (3.4),

$$(r(\bar{\delta}) + 1_{\bar{u}}) \circ (1_c + r(\eta)) \circ \gamma_1 = \gamma_1' + (r(\eta) \circ 0_a).$$

Applying the reflection r this becomes

$$(\bar{\delta} \circ 0_{\bar{u}}) + (0_c \circ \eta) + r(\gamma_1) = r(\gamma_1') \circ (\eta + 1_a).$$

Therefore,

$$\delta + r(\gamma_1) = -(0_{c'} \circ \eta) + (\bar{\delta} \circ 0_{\bar{u}}) + (0_c \circ \eta) + r(\gamma_1)$$
$$= -(0_{c'} \circ \eta) + (r(\gamma_1') \circ (\eta + 1_a)) = r(\gamma_1').$$

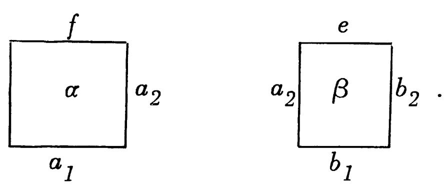
This completes the proof.

The next result puts Lemma 4 of [7] into our present setting.

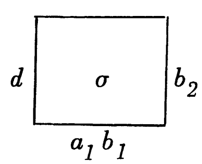
PROPOSITION 3.7. *Let $\gamma = \alpha + \beta$ where α, β are pullback (pushout) squares. Then γ is a pullback (pushout) square.*

Similarly $\gamma' = \alpha' \circ \beta'$ is a pullback (pushout) square if α', β' are pullback (pushout) squares.

PROOF. By Proposition 3.4 and duality it suffices to consider the following case. Let $\gamma = \alpha + \beta$ where α, β are pullback squares and let α, β have edges



Then given a square σ with edges



we require γ_1, γ_2 in D_2, c in D_1 such that

$$\sigma = \begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & \alpha + \beta \end{bmatrix}.$$

Since β is a pullback square I may write

$$\sigma \circ (\Gamma a_1 + 1_{b_1}) = \begin{bmatrix} \Gamma' \bar{c} & r(\bar{\gamma}_2) \\ \bar{\gamma}_1 & \beta \end{bmatrix}$$

and then since α is a pullback square I may also write

$$(0_d \circ \Gamma' a_1) + \bar{\gamma}_1 = \begin{bmatrix} \Gamma'\bar{c} & r(\bar{\bar{\gamma}}_2) \\ \bar{\bar{\gamma}}_1 & \alpha \end{bmatrix}.$$

Thus

$$\sigma = (0_d \circ \Gamma' a_1) + (\sigma \circ (\Gamma a_1 + 1_{b_1})) = \begin{bmatrix} \Gamma'\bar{\bar{c}} & r(\gamma_2) \\ \gamma_1 & \alpha+\beta \end{bmatrix}.$$

where $\gamma_1 = \bar{\bar{\gamma}}_1$ and

$$r(\gamma_2) = \begin{bmatrix} \Gamma'\bar{c} & r(\bar{\gamma}_2) \\ r(\bar{\bar{\gamma}}_2) & 1_e \end{bmatrix}.$$

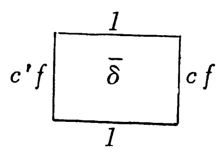
Now suppose

$$\begin{bmatrix} \Gamma' c' & r(\gamma_2') \\ \gamma_1' & \alpha+\beta \end{bmatrix} = \begin{bmatrix} \Gamma' c & r(\gamma_2) \\ \gamma_1 & \alpha+\beta \end{bmatrix}$$

are two representatives of σ. Then

$$\sigma = \begin{bmatrix} \Gamma' c f & r(\tilde{\gamma}_2) \\ \gamma_1 + \alpha & \beta \end{bmatrix}$$

where $r(\tilde{\gamma}_2) = r(\gamma_2) \circ (\Gamma f + 1_e)$. Thus since β is a pullback, by Proposition 3.5, there exists $\bar{\delta}$ in $\omega(D)_2$ with edges



and satisfying

(3.5) $$\bar{\delta} + r(\gamma_1 + \alpha) = r(\gamma_1' + \alpha)$$

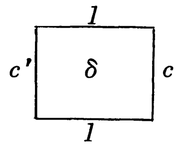
and

(3.6) $$\bar{\delta} + (r(\gamma_2) \circ (\Gamma f + 1_e)) = r(\gamma_2') \circ (\Gamma f + 1_e).$$

From (3.5) we have

$$\gamma_1' + \alpha = \begin{bmatrix} \Gamma' c' & \Gamma c' + 1_f \\ \gamma_1' & \alpha \end{bmatrix} = \begin{bmatrix} \Gamma' c & r(\bar{\delta}) \circ (\Gamma c + 1_f) \\ \gamma_1 & \alpha \end{bmatrix}.$$

Thus since α is a pullback square there exists δ in $\omega(D)_2$ with edges



and satisfying

$$\delta + r(\gamma_1) = r(\gamma_1') \tag{3.7}$$

and

$$\delta + (r(\bar{\delta}) \circ (\Gamma c + 1_f)) = \Gamma c' + 1_f. \tag{3.8}$$

Now from the definition of r, $r(\bar{\delta}) = \Gamma' c f - \bar{\delta} + \Gamma c' f$ and substitution in (3.8) gives

$$\delta + ((\Gamma' c f - \bar{\delta} + \Gamma c' f) \circ (\Gamma c + 1_f)) = \Gamma c' + 1_f,$$

the left hand side of which may be expressed as

$$\delta + (0_c \circ \Gamma' f) - \bar{\delta} + \Gamma c' f \;= (\delta \circ \Gamma' f) - \bar{\delta} + \Gamma c' f.$$

Hence

$$(\delta \circ \Gamma' f) - \bar{\delta} + (\Gamma' c' f \circ \Gamma c' f) = \Gamma' c' f \circ (\Gamma c' + 1_f)$$

and so

$$(\delta \circ \Gamma' f) = \bar{\delta} + 0_{c'} \circ \Gamma' f. \tag{3.9}$$

Now

$$\delta + r(\gamma_2) = \begin{bmatrix} \delta & r(\gamma_2) \\ \Gamma' f & \Gamma f + 1_e \end{bmatrix}$$

$$= (\delta \circ \Gamma' f) - \bar{\delta} + (r(\gamma_2') \circ (\Gamma f + 1_e)), \text{ by (3.6)},$$

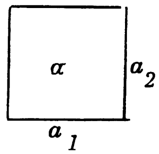
$$= (0_{c'} \circ \Gamma' f) + (r(\gamma_2') \circ (\Gamma f + 1_e)), \quad \text{by (3.9)}.$$

Therefore,

$$\delta + r(\gamma_2) = r(\gamma_2'). \tag{3.10}$$

Finally, (3.7) and (3.10) show that δ has the required properties to establish the «uniqueness up to homotopy» part of Definition 3.1.

My last result puts Lemma 5 of [7] into the present setting. This result requires the existence of pushouts and pullbacks in (D, Δ). That is, I say pullbacks exist if given edges a_1, a_2 with common final points there exists a pullback square



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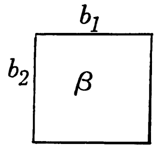
Similarly I say *pushouts exist* if given edges b_1, b_2 with common initial points there exists a pushout square



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PROPOSITION 3.8. *Suppose pullbacks exist in* (D, Γ, Γ') *and let* $\gamma = \alpha + \beta$ *where* γ, β *are pullback squares, then* α *is also a pullback square.*

Dually, if pushouts exist and γ, α *are pushout squares, then* β *is a pushout square.*

PROOF. Again I consider only the pullback case. Let α' be a pullback square such that $\epsilon_1 \alpha' = \epsilon_1 \alpha$, $\partial_1 \alpha' = \partial_1 \alpha$ and let

$$\omega = \partial_0 \epsilon_0 \alpha, \quad \omega' = \partial_0 \epsilon_0 \alpha'.$$

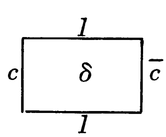
Then since α' is a pullback square there exist γ_1, γ_2 in D_2 and $\bar{c} : \omega \to \omega'$ in D_1 such that

$$\alpha = \begin{bmatrix} \Gamma' \bar{c} & r(\gamma_2) \\ \gamma_1 & \alpha' \end{bmatrix}.$$

Since $\alpha + \beta$ is a pullback square, by Proposition 3.2 there exist squares $\bar{\gamma}_1$, $\bar{\gamma}_2$ and a $c : \omega \to \omega'$ such that

$$\alpha + \beta = \begin{bmatrix} \Gamma' c & r(\bar{\gamma}_2) \\ \bar{\gamma}_1 & \alpha' + \beta \end{bmatrix} = \begin{bmatrix} \Gamma' \bar{c} & r(\gamma_2) + 1_e \\ \gamma_1 & \alpha' + \beta \end{bmatrix},$$

where $e = \epsilon_0 \beta$. Then, since by the previous proposition $\alpha' + \beta$ is a pullback square, there exists



showing that c is also a homotopy equivalence. Thus by Proposition 3.6, γ_1 is a pullback square and so applying Proposition 3.7 to

$$\alpha = (\Gamma'\bar{c} + r(\gamma_2)) \circ (\gamma_1 + \alpha')$$

we see that α is a pullback square.

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