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TRANSFORMATION GROUPOIDS AND BUNDLES OF BANACH SPACES

by Anthony Karel SEDA

1. INTRODUCTION.

In this paper we study objects $(G, S, p, .)$ in the category $\mathcal{T}$ of topological transformation groupoids whose fibres are endowed with Banach space structures over some fixed field K. Thus, $p: S \to X$ is a fibre space, $p^{-1}(x)$ is a K-Banach space for each $x \in X = Ob(G)$ and $(.)$ is an action of a groupoid G on S via p. We will impose several other conditions on S (see Section 2 for precise definitions) and, in fact, for the most part S will be a proto-bundle in the sense of Dauns, Hofmann and Fell, see [6] and [7]. Proto-bundles are currently receiving a lot of attention in connection with non-commutative generalizations of the Gelfand-Naimark representation theory of C*-algebras, and the representations of rings and algebras by sections. However, our present interest in them stems from the fact that they are precisely the objects (in the case of Hilbert space fibres) on which one can attempt to represent topological groupoids and to develop a theory of of unitary representations of locally compact topological groupoids. Thus, one way to approach representations (strongly continuous representations) of G is to consider continuous (strongly continuous) linear actions of G on a proto-bundle $p: S \to X$ with Hilbert space fibres. Indeed, it is possible to establish certain analogues, for compact topological groupoids, of the well known Peter-Weyl theory for compact groups. We do not pursue this line of thought here, however, other than to observe that the results of this paper and of [9] are foundational in this direction. In fact, we will make several applications of the results of [9] as we proceed, further emphasising the point of view that what can be done with Haar measure in the category of transformation groups can often be successfully carried out, with Haar systems of measures, in our present category.

Actually, proto-bundles are not the ultimate object of study either here or in the Dauns-Hofmann theory, simply because they fail to have sufficiently many sections (this phrase is made precise in Section 3). Roughly speaking, this latter deficiency is remedied as follows. The first imposition that one makes is to assign arbitrarily small neighborhoods, determined by the norm function on S, to the zero element of each fibre; thus one obtains pre-bundles. Finally, one specifies the existence of local sections passing through each point of S to obtain Banach bundles. One fact that emerges here is that, over paracompact spaces X, these three definitions coincide in the presence of suitably restricted actions of G.

The contents of this paper are organised in the following way. In Section 2 we consider generalities and two results worth noting in the case G has compact components are:

(i) the orbit space S/G is metrizable;

(ii) the norms on the fibres can be chosen in such a way that the norm function $\| \ \|$ on S is continuous, rather than just upper semi-continuous.

In Section 3 we consider sections of the projection p. Finally, in Section 4 we obtain a generalisation of the well known Tietze-Gleason theorem concerning equivariant extensions in the category of transformation groups.

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2. GENERALITIES.

Let G denote a Hausdorff topological groupoid with object space X and let π and π' denote the initial and final maps of G respectively. We refer to [9, 10, 11] for notation and terminology. There are two assumptions about G that it will often be convenient to make. One is that for each $x \in X$ the restriction π'_x of the final map π' to the subspace $st_G x$ is open, where $st_G x = \pi^{-1}(x)$. It follows then (and conversely) that, for each $x \in X$, the restriction π_x of the initial map to the subspace $cost_G x$ is also open, where $cost_G x = \pi'^{-1}(x)$. If G is understood, we will sometimes write

$$st(x) \text{ for } st_G\, x \quad\text{and}\quad ct(x) \text{ for } cost_G\, x.$$

The other useful assumption is that G is locally transitive, that is, each point x of X has a neighborhood U such that the full subgroupoid $G(U)$ of G over U is transitive. From which it follows that the transitive components (see [9]) are both open and closed.

We will say that G *has compact components* if each transitive component of G is compact. If G is locally transitive and has compact components, then X is paracompact and, hence, normal. If π_x is open for each x and G is locally transitive, then π and π' are open, but not conversely.

Finally, we note that if G has compact components or is locally trivial, then each π_x is open. It has been observed by M.K. Dakin and the author that one advantage of working in the category of transitive topological groupoids for which π_x is open for each x is that this category admits arbitrary products. That is not the case for the category of transitive locally trivial topological groupoids.

Next let $p: S \to X$ be a fibre space; thus p is surjective and continuous, and suppose $(.): G \times_X S \to S$ is an action of G on S via p, see [10, 11]. We will always suppose that $(.)$ is continuous and we let

$$\phi_\alpha : p^{-1}(\pi(\alpha)) \to p^{-1}(\pi'(\alpha))$$

denote the homeomorphism $\phi_\alpha(s) = \alpha . s$ induced by $\alpha \in G$. Given an element $s \in S$, we define the *G-orbit* $\hat{s} = G.s$ of s by

$$G.s = \{\, \alpha . s \mid \pi(\alpha) = p(s) \,\},$$

see [11]. Let S/G denote the set of G-orbits and let $\rho: S \to S/G$ denote the canonical orbit map giving S/G the quotient topology of S. All these basic definitions are, of course, due to Ehresmann.

In [8], McClendon discusses fibre spaces in which each fibre $S_x = p^{-1}(x)$ has a metric δ_x compatible with the subspace topology, such that the induced function $\delta: S \times_X S \to \mathrm{R}$ is continuous, where R denotes the real line. He calls such spaces *metric families* and we have considered them independently in [10].

Following [6, 7], we say that $p: S \to X$ *is a proto-bundle* if each fibre S_x has the structure of a Banach space over a fixed field K (usually the field R or the complex field C) whose norm, $\| \ \|_x$, is compatible with the subspace topology. Moreover, *the addition function*

$$+: S \times_X S \to S$$

and the *scalar product function* $K \times S \to S$ are continuous and also the zero section $0: X \to S$ is continuous, where $0(x)$ is the zero element of S_x . However, it is usual here only to require upper semi-continuity of the function called *norm*, $\| \ \| : S \to$ R , induced by the norms on the fibres, rather than continuity (but see Theorem 2 and the proof of Theorem 1 d below).

Finally, we recall from [9] that a *Haar system of measures* for a locally compact Hausdorff topological groupoid G is a family $\{m, \mu, \mu_x\}$ of Baire measures m on G, μ on X and μ_x on $ct(x)$, for each $x \in X$, such that for each Baire subset E of G we have :

(i) $m(E) = \int_X \mu_x(E_x)\,d\mu(x)$,

(ii) $\mu_y(\alpha E_x) = \mu_x(E_x)$ whenever $x, y \in X$ and $\alpha \in G(x,y)$, where $E_x = E \cap ct(x)$.

The existence and classification of such systems was dealt with in [9] ; they were originally conceived in [12] with exactly the sort of applications in mind which we will shortly give.

THEOREM 1. *Let G be a locally transitive topological groupoid acting on a fibre space $p: S \to X$ and suppose that each π_x is open. Then :*

(a) p is an open map.

(b) Suppose $s \in S$ and $p(s) = x$. Let U be a neighborhood of the identity 1_x in $st(x)$ and let V be a neighborhood of s in $p^{-1}(x)$. Then

$$U.V = (.)(U \times V)$$

is a neighborhood of s in S.

(c) $\rho: S \to S/G$ is an open map.

(d) If, further, G has compact components and S is either a metric family or a proto-bundle, then S/G is metrizable.

PROOF. (a) Let O be an open set in S , let $s \in O$ and let $p(s) = x$. By continuity of the action $(.)$ restricted to $st(x)\times p^{-1}(x)$ and noting that $I_x . s = s$, there are neighborhoods U of I_x in $st(x)$ and V of s in $p^{-1}(x)$ such that

$$\alpha . t \in O \quad \text{whenever} \quad \alpha \in U \quad \text{and} \quad t \in V.$$

Whence

$$p(\alpha . t) = \pi_x'(\alpha) \in p(O) \quad \text{and so} \quad x \in \pi_x'(U) \subset p(O).$$

Thus, p is an open map.

(b) We can suppose G is transitive, and clearly $s = I_x . s \in U . V$. The action of G restricts to a continuous function

$$(.)\colon ct(x)\times_X S \to p^{-1}(x).$$

Hence, there are neighborhoods A of I_x in $ct(x)$ and B of s in S such that $(.)(A \times_X B) \subset V$ and we can further suppose that $p(B) = \pi_x(A)$. The inverse map in G gives a homeomorphism

$$Inv\colon ct(x) \to st(x)$$

and so $Inv(A)$ is a neighborhood of I_x in $st(x)$ and again we can make the further supposition that $Inv(A) \subset U$. But then, if $b \in B$, there exists

$$\alpha \in A \quad \text{such that} \quad \pi(\alpha) = p(b) \quad \text{and} \quad \alpha . b \in V.$$

Moreover,

$$\alpha^{-1} \in U \quad \text{and so} \quad \alpha^{-1}.(\alpha . b) = b \in U . V.$$

Hence, $s \in B \subset U . V$ and $U . V$ is a neighborhood of s in S.

(c) Let Z denote the object set of the transitive component of G determined by some object x. Then $p^{-1}(Z)$ is open in S and we have

$$\rho^{-1}(A) = p^{-1}(Z), \quad \text{where} \quad A = \rho(p^{-1}(Z)).$$

Hence, A is open in S/G and it suffices to show that $\rho(O)$ is open in A for each open subset O of $p^{-1}(Z)$. For each $y \in Z$, $\rho : p^{-1}(y) \to A$ is surjective by transitivity on Z and is open since the orbit map of a group action is open. Thus,

$$\rho(O) = \bigcup_{y \in Z} \rho(O \cap p^{-1}(y))$$

is open in A.

(d) Let us first suppose that G is transitive and that S is a proto-bundle. For each $x \in X$ define δ_x by $\delta_x(s,t) = \|s-t\|_x$. Then δ_x is a metric on S_x and $\delta : S \times_X S \to R$ is upper semi-continuous. Now let $\{m, \mu, \mu_x\}$ be a Haar system of measures for G and define δ'_x on S_x, for each x, by

$$\delta'_x(s,t) = \int_{ct(x)} \delta(a^{-1}.s, a^{-1}.t)\, d\mu_x(a)$$

as in [10]. Since $ct(x)$ is compact, upper semi-continuity of δ is enough to ensure that this integral exists, and it follows from [10] that δ'_x is an invariant metric for each x, and that $\delta' : S \times_X S \to R$ is continuous. Moreover, the argument given in the main theorem of [10] remains unchanged here and we can conclude that δ_x and δ'_x are equivalent metrics for each $x \in X$. Thus, both cases here can be reduced to the case of a metric family in which the metrics are invariant, that is

$$\delta_y(a.s, a.t) = \delta_x(s,t)$$

whenever $a \in G(x,y)$ and $s, t \in S_x$.

Next define $d(\hat{s}, \hat{t})$ for $\hat{s}, \hat{t} \in S/G$ by

$$d(\hat{s},\hat{t}) = inf\{ \delta_x(u,v) \mid u \in \hat{s} \cap S_x,\ v \in \hat{t} \cap S_x,\ x \in X \}.$$

One readily checks that d is a metric on S/G, and we claim that the topology induced by d coincides with the quotient topology of ρ. To show this, it suffices to show that ρ is both open and continuous relative to d. Let $s \in S$ and let $B(\hat{s},\eta)$ denote the η-ball about $\hat{s}$ in S/G relative to d. By continuity of δ at (s,s) there is a neighborhood V of s in S such that

$$\delta(u,v) < \eta \quad \text{for all } u, v \in V \text{ with } p(u) = p(v),$$

and hence $d(\hat{u},\hat{v}) < \eta$. Thus, $\rho(V) \subset B(\hat{s},\eta)$, and so ρ is continuous.

Next let O be open in S, let $s \in O$ and let $p(s) = x$. As in part (a), find a neighborhood U of 1_x in $st(x)$ and an

$$\eta > 0 \quad \text{such that } U.B(s,\eta) \subset O,$$

where $B(s,\eta)$ denotes the η-ball around s in S_x relative to δ_x. Let $\hat{t}$ be in $B(\hat{s},\eta)$; then $d(\hat{s},\hat{t}) < \eta$ and hence

$$\delta_y(u,v) < \eta \quad \text{for some } u \in \hat{s} \cap S_y \text{ and } v \in \hat{t} \cap S_y.$$

There exists $a \in G$ such that $a \,.\, u = s$; then $\delta_x(s, a\,.\,v) < \eta$ and so

$$\delta_x(s, r) < \eta \quad \text{with} \quad r \in \hat{t}.$$

Hence $r \in B(s, \eta)$ and it follows that

$$\rho(U\,.\,B(s,\eta)) = B(\hat{s}, \eta) \subset \rho(O)\ ;$$

so ρ is open.

To complete the proof, we apply the previous argument to each transitive component and note then that S/G is the union of a locally finite family of closed metrizable subspaces and is, hence, metrizable by a theorem of Nagata, see [5]. //

In view of this theorem it is now apparent that, if $(G, S, p, .)$ is an object of $\mathcal{T}$, then S is fibred over both $Ob(G)$ and S/G in a canonical way, and if G has compact components then the space S/G has nice properties. Thus, S resembles a product space fibred over both projections. This point of view has been exploited elsewhere in obtaining general constructions of measures in S and, in particular, a generalization of the classical theorems of Fubini and Tonelli concerning product measures. In this theory, Fubini's theorem occurs as a special case of a theorem concerning invariant measures for the action of G.

If G is locally transitive and has compact components, and acts *linearly* on a proto-bundle $p : S \to X$ (thus ϕ_a is linear for each $a \in G$), then by integrating $\| \ \|$ as in the proof of Theorem 1, (d), we obtain G-invariant norms $\| \ \|'_x$, for each x, such that the induced function $\| \ \|' : S \to \mathrm{R}$ is continuous. Moreover, $\| \ \|'_x$ is topologically equivalent to $\| \ \|_x$ for each x. We can, in fact, obtain continuity of $\| \ \|'$ without the compactness restriction on G if we assume invariance. This is an immediate conclusion from the following more general result which can be proved in the same way as we proved Lemma 2 of [10], noting that compactness was only used there to ensure openness of each π_x.

THEOREM 2. *Suppose G is locally transitive and acts on a fibre space $p : S \to X$. Suppose further that each π_x is open and that, for each $x \in X$, we have continuous functions $f_x : S_x \to \mathrm{R}$ satisfying*

$$f_y(\alpha . s) = f_x(s) \quad \textit{whenever} \quad s \in S_x \ \textit{and} \ \alpha \in G(x,y).$$

Then the induced function $f: S \to \mathrm{R}$ *is continuous.*

A proto-bundle $p: S \to X$ is called a *pre-bundle*, see [6,7], if p is an open map satisfying: For any $x \in X$, the family

$$\{ U(0|_V, \eta) \mid V \text{ is a neighborhood of } x \text{ and } \eta > 0 \}$$

is a neighborhood basis of $0(x)$, where $U(0|_V, \eta)$ denotes the set

$$\{ s \in S \mid p(s) \in V \text{ and } \|s\| < \eta \}.$$

THEOREM 3. *Suppose* G *and* S, *where* S *is a proto-bundle, satisfy the hypothesis of Theorem 2 with* $f_x = \|\ \|_x$. *Then* S *is a pre-bundle.*

PROOF. Given any neighborhood O of $0(x)$ in S, find a neighborhood U of I_x in $st(x)$ and an open ball $B(0(x), \eta)$ in S_x such that

$$\alpha . s \in O \quad \text{whenever} \quad \alpha \in U \ \text{and} \ s \in B(0(x), \eta).$$

By the invariance of the norms we have

$$\|\alpha . s\|_{\pi'(\alpha)} = \|s\|_x < \eta .$$

But each ϕ_α is a homeomorphism and so, if $V = \pi'_x(U)$, we have

$$U(0|_V, \eta) \subset O.$$

It now follows from Theorem 1 that $p: S \to X$ is a pre-bundle. //

3. SECTIONS.

We next consider the important results of Douady and Dal Soglio-Herault, see [4,6], and the way they apply here. If $p: S \to X$ is a pre-bundle over a paracompact base space X, then given any point $s \in S$ there is a bounded section $\sigma: X \to S$ (which means that σ is continuous and $p\sigma$ is the identity function on X) such that $\sigma(p(s)) = s$, that is, σ *passes through* s. Thus, there is a bounded section passing through each point s of a pre-bundle S and this is paraphrased by saying that S *has enough sections*, or S *has sufficiently many sections*. The term *bounded* used here means that $\|\sigma(x)\|_x$ is bounded on X and, in fact, if we define $\|\sigma\|$ by

$$\|\sigma\| = sup\,\{\ \|\sigma(x)\|_x \mid x \in X\ \},$$

then we obtain a norm on the space $\Gamma(X)$ of all bounded sections, which turns $\Gamma(X)$ into a K-Banach space, and similar statements can be made for any subset $A \subset X$.

Given an arbitrary proto-bundle $p: S \to X$, the additive group of all sections σ acts on S as a group of homeomorphisms under the operation

$$(\sigma, s) \mapsto \sigma \oplus s = \sigma(p(s)) + s\,.$$

This has the consequence that, in the case of pre-bundles over paracompact spaces, we can specify a neighborhood base at each point of S as follows. First we introduce the following notation : if σ is a section, V is an open set in X and $\eta > 0$, we define the set $U(\sigma|_V, \eta)$ by

$$U(\sigma|_V, \eta) = \{\ s \in S \mid p(s) \in V \text{ and } \|\ s - \sigma(p(s))\ \| < \eta\ \}.$$

Clearly we have the relation

$$U(\sigma|_V, \eta) = \sigma \oplus U(0|_V, \eta).$$

From which it follows that, if S is a pre-bundle and $s \in S$, then s has a neighborhood base of the form

$$\{\ U(\sigma|_V, \eta) \mid V \text{ is a neighborhood of } p(s) \text{ and } \eta > 0\ \},$$

where σ is some (fixed) section passing through s.

A *Banach bundle*, see [6], is a proto-bundle $p: S \to X$ satisfying:

1° For each $x \in X$, the family

$$\{\ U(0|_V, \eta) \mid V \text{ is a neighborhood of } x \text{ and } \eta > 0\ \}$$

is a neighborhood basis of $0(x)$,

2° for each $s \in S$, there is a neighborhood V of $p(s)$ and a local section $\sigma: V \to S$ such that $\sigma(p(s)) = s$.

It easily follows from 2 that p is an open map and, hence, every Banach bundle is a pre-bundle. Conversely, if X is paracompact, then every pre-bundle over X is a Banach bundle. If, further, there is an action of G on S as in Theorem 3 and X is paracompact, then the proto-bundle S is a Banach bundle. In particular, if G has compact components and acts linearly we can always re-norm S in such a way that S becomes a Banach bundle.

Assuming we have an action of G on S, another question which arises naturally is that of the existence of sections $\sigma : X \to S$ which are G-*invariant* in the sense that

$$\alpha . \sigma(\pi(\alpha)) = \sigma(\pi'(\alpha)) \quad \text{for each } \alpha \in G.$$

We show next how such sections can be constructed using Haar systems of measures. So suppose as usual that G is locally transitive, has compact components and acts linearly on a proto-bundle S. Given a section $\sigma : X \to S$ define $\bar{\sigma} : X \to S$ by

$$\bar{\sigma}(x) = \int_{ct(x)} \alpha . \sigma(\pi(\alpha))\, d\mu_x(\alpha).$$

It is clear that this (vector-valued) integral exists in S_x and so $\bar{\sigma}(x) \in S_x$ for each x. The invariance property of a Haar system of measures and the fact that integration commutes with a linear mapping imply that $\bar{\sigma}$ satisfies

$$\alpha . \bar{\sigma}(\pi(\alpha)) = \bar{\sigma}(\pi'(\alpha)) \quad \text{for each } \alpha \in G.$$

It now easily follows that $\bar{\sigma}$ is continuous and is, therefore, an invariant section. On the other hand, if σ is an invariant section to start with, then $\bar{\sigma}$ as defined above coincides with σ (since G has compact components we take

$$\mu_x(cost_G\, x) = 1 \quad \text{for each } x \in X\ )$$

and thus we obtain all invariant sections by this process of integration. Let $\Gamma_G(X)$ denote the subvector space of $\Gamma(X)$ consisting of G-invariant sections. If

$$\sigma_n \in \Gamma_G(X) \quad \text{for } n = 1, 2, 3, \ldots \quad \text{and } \|\sigma_n - \sigma\| \to 0$$

then

$$\sigma_n(x) \to \sigma(x) \text{ in } S_x \quad \text{for each } x \in X.$$

Hence, for any $\alpha \in G(x,y)$,

$$\alpha . \sigma_n(x) \to \alpha . \sigma(x), \quad \text{that is, } \sigma_n(y) \to \alpha . \sigma(x).$$

So $\alpha . \sigma(x) = \sigma(y)$ and so $\sigma \in \Gamma_G(X)$ which is, therefore, a Banach space.

Thus, there is a homomorphism $\Gamma(X) \to \Gamma_G(X)$ which is the identity on $\Gamma_G(X)$. Indeed, if $C(X)$ denotes the Banach algebra of all bounded continuous K-valued functions, then $\Gamma_G(X)$ is a $C(X)$-submodule of the $C(X)$-module $\Gamma(X)$.

Let us call a $C(X)$-submodule M of $\Gamma(X)$ *G-invariant* if, given $\sigma \in M$ and $\alpha \in G$, there is an element

$$\omega \in M \quad \text{such that} \quad \omega(\pi'(\alpha)) = \alpha . \sigma(\pi(\alpha)) .$$

(Strictly speaking this is an abuse of terminology since G does not act on $\Gamma(X)$.)

Clearly, $M = \Gamma_G(X)$ satisfies this property, but so does $M = \Gamma(X)$ if S is a pre-bundle over a paracompact space X, and this idea therefore generalises $\Gamma_G(X)$.

Using this idea we can modify the hypothesis of the bundle version of the Stone-Weierstrass theorem given by Hofmann in [6]. And we will take the trouble to record this change, but refer to [6] for the details of the proof.

We need the following terminology: a submodule M of $\Gamma(X)$ is called *fully additive* if for any locally finite family $\{\sigma_j, j \in J\}$ of elements of M, the sum $\sum_j \sigma_j$ is again in M.

THEOREM 4. *Let G be a transitive topological groupoid acting on a proto-bundle $p: S \to X$, where X is paracompact. Suppose M is a G-invariant fully additive submodule of $\Gamma(X)$ such that the set $\{\sigma(x) \mid \sigma \in M\}$ is dense in S_x for some one $x \in X$. Then M is norm-dense in $\Gamma(X)$.*

PROOF. Since M is G-invariant and G is transitive, the set $\{\sigma(y) \mid \sigma \in M\}$ is dense in S_y for all $y \in X$. The conclusion now follows from [6]. //

If we consider non-locally transitive groupoids G, then we can take a disjoint union of groups, each acting trivially on the fibres of S, to conclude that $\Gamma(X) = \Gamma_G(X)$. At the other extreme, it can happen that $\Gamma_G(X)$ consists only of the zero section for any linear action of any compact transitive groupoid. An example of this situation is provided by a compact differentiable manifold X whose tangent bundle S admits no never-zero sections. However, one interesting intermediate situation is that of a G-vector bundle, where G is a compact group - as defined in [1] - though we do not suppose here that G is finite. So let G be a compact Hausdorff topological group and let X be a compact Hausdorff G-space. Denote by $\hat{G}$ the groupoid $G \times X$ whose object space is X and whose morphism set $\hat{G}(x, y)$ consists of the set

$$\{ (g, x) \mid g.x = y \},$$

the operation in $\hat{G}$ being

$$(h, y)(g, x) = (hg, x) \text{ iff } g.x = y.$$

Then $\hat{G}$ is compact Hausdorff but not necessarily locally transitive since we make no restriction on the action of G on X. If $p: S \to X$ is a proto-bundle and we have a continuous linear action $(.)$ of $\hat{G}$ on S, then there is an action of G on S defined by

$$g.s = (g, p(s)).s,$$

which is linear on the fibres of S and which, moreover, is such that p is equivariant.

Conversely, such an action of G on S determines an action of $\hat{G}$ on S in the obvious way. If it is the case that S is locally trivial and has finite dimensional fibres (that is, S *is a vector bundle*), then S is a G-vector bundle.

Now a Haar system of measures can be constructed for $\hat{G}$ in which each measure μ_x is, essentially, Haar measure ν on G, see [9]. A simple calculation shows that

$$\int_{cost_{\hat{G}} x} a.\sigma(\pi(a))\, d\mu_x(a) = \int_G g.\sigma(g^{-1}.x)\, d\nu(g)$$

for any element σ of $\Gamma(X)$. From which it follows that $\Gamma_{\hat{G}}(X)$ is precisely the set $\Gamma(X)^G$ of Atiyah, see [1].

It is worth noting in this context that we have defined elsewhere the notion of representation ring for a groupoid G, and from this point of view $K_G(X)$ (equivariant K-theory) appears as the representation ring of $\hat{G}$.

The idea of considering invariant sections for actions of a groupoid A via a morphism $\omega: A \to Ob(G)$ of groupoids occurs in [3], and certain results are conditional on the existence of invariant sections. It would seem likely that our results can be brought to bear on the results of [3] (assuming suitable topologies) if the sets $\omega^{-1}(x)$ have Banach space structures. We will not pursue this here, but I have to thank Professor Ronald Brown for many suggestions concerning this circle of ideas.

4. THE TIETZE-GLEASON THEOREM.

In this section we will consider the problem of extending equivariant mappings defined on subbundles, and we must, therefore, frame the necessary definitions and terminology. Indeed, we have yet to define the morphisms in $\mathcal{T}$, and we do this next.

Let $(G, S, p, .)$ and $(H, S', p', .)$ be two objects of $\mathcal{T}$. Then, a *morphism* or *equivariant map* $(G, S, p, .) \to (H, S', p', .)$ is a pair (ω, f), where ω is a morphism $G \to H$ of topological groupoids and f is a fibre-preserving, continuous function mapping S into S' which commutes with ω and the actions of G and H. More precisely, ω and f satisfy:

a) ω is a functor and both ω and $Ob(\omega)$ are continuous, where $Ob(\omega)$ denotes the induced map on objects.

b) The diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & S' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Ob(G) & \xrightarrow{\quad Ob(\omega) \quad} & Ob(H) \end{array}$$

commutes.

c) $f(a . s) = \omega(a) . f(s)$ whenever $\pi(a) = p(s)$.

Given some fixed topological groupoid G, we can form the subcategory $\mathcal{T}(G)$ of $\mathcal{T}$ with objects $(G, S, p, .)$ and morphisms $(1, f)$, where 1 denotes the identity homorphism $G \to G$. From now on, we will remain in $\mathcal{T}(G)$ and hence the term *equivariant map* means a morphism $(1, f)$ in $\mathcal{T}(G)$ which we will write simply as f.

Finally, we say that a subset A of S *is G-invariant* if

$$\alpha . s \in A \quad \text{whenever} \quad s \in A \quad \text{and} \quad \alpha . s \quad \text{is defined.}$$

The action of G then restricts to an action on A.

THEOREM 5. *Let G be a locally transitive topological groupoid with compact components acting on fibre spaces $p: S \to X$ and $p': S' \to X$, where S' is a proto-bundle with linear G-action. Suppose further that A is a G-invariant subset of S and that $f: A \to S'$ is an equivariant map. Then, if f has a fibre-preserving continuous extension $g: S \to S'$, the expression*

$$\hat{g}(s) = \int_{ct(p(s))} \alpha . g(\alpha^{-1} . s)\, d\mu_{p(s)}(\alpha), \quad s \in S,$$

determines an equivariant extension $\hat{g}$ of f.

PROOF. We may perform an initial integration of the norm of S' and so we may suppose that S' is a pre-bundle with invariant norms for which $\| \;\|$ is a continuous function. Clearly the integral defining $\hat{g}(s)$ exists for each $s \in S$, and $\hat{g}(s) \in p'^{-1}(p(s))$, or in other words $\hat{g}$ is fibre-preserving. Also if $s \in A$, then

$$\alpha . g(\alpha^{-1} . s) = \alpha . f(\alpha^{-1} . s) = f(s) \quad \text{whenever} \quad \pi'(\alpha) = p(s),$$

from which it follows that $\hat{g}(s) = f(s)$ and so $\hat{g}$ is an extension of f. Next using the invariance of the measures μ_x and the linearity of the action of G, we have

$$\begin{aligned} \beta . \hat{g}(s) &= \beta . \int \alpha . g(\alpha^{-1} . s)\, d\mu_x(\alpha) \qquad (\text{where } p(s) = x) \\ &= \int \beta\alpha . g(\alpha^{-1}\beta^{-1} . (\beta . s))\, d\mu_y(\beta\alpha) \quad (\text{where } \pi'(\beta) = y) \\ &= \int \gamma . g(\gamma^{-1} . (\beta . s))\, d\mu_y(\gamma) = \hat{g}(\beta . s) \end{aligned}$$

and so $\hat{g}$ is equivariant (we have omitted the domains of integration here to ease notation).

It remains only to show that $\hat{g}$ is continuous on S. To do this, let $s \in S$, and $p(s) = x$ and let O be an arbitrary neighborhood of $\hat{g}(s)$ in S'. Since X is paracompact and S' is a pre-bundle, there is a section σ passing through $\hat{g}(s)$ and hence there is a basic neighborhood $U(\sigma|_W, \eta) \subset O$. The restriction of $\hat{g}$ to $p^{-1}(x)$ is continuous, by uniform continuity, and hence $\| \alpha . \hat{g}(t) - \sigma(\pi'(\alpha)) \|$ is continuous on $st(x) \times p^{-1}(x)$. Since this latter function vanishes when $\alpha = I_x$ and $t = s$, there are neighborhoods U of I_x in $st(x)$ and V of s in $p^{-1}(x)$ such that

$$\| \alpha . \hat{g}(t) - \sigma(\pi'(\alpha)) \| < \eta \quad \text{whenever} \quad (\alpha, t) \in U \times V,$$

and we can suppose $\pi'_x(U) \subset W$. But then

$$\| \hat{g}(\alpha . t) - \sigma(\pi'(\alpha)) \| < \eta \quad \text{and so} \quad \hat{g}(U . V) \subset U(\sigma|_W, \eta).$$

By Theorem 1 b, $U . V$ is a neighborhood of s and, hence, $\hat{g}$ is continuous at s. This completes the proof of the theorem. //

As a consequence of this theorem we now establish as our final result a direct generalisation of the well-known Tietze-Gleason equivariant extension theorem for actions of compact groups, see [2].

THEOREM 6. *Suppose G, S and S' are as in Theorem 5 and that in addition S is a normal space, A is closed in S and S' is a vector bundle. Then f has an equivariant extension $\hat{g}: S \to S'$.*

PROOF. Let $\{U_j\}$ be a locally finite open cover of X such that $p'^{-1}(U_j)$ is trivial for each j and, using normality of X, let $\{V_j\}$ be an open cover of X with the property that $\bar{V}_j \subset U_j$ for each j. Then there are positive integers n_j and vector bundle isomorphisms

$$\phi_j : p'^{-1}(U_j) \to U_j \times K^{n_j}.$$

Thus, for those j such that $A \cap p^{-1}(\bar{V}_j) \neq \emptyset$ we have a continuous function

$$p_2 \phi_j f : A \cap p^{-1}(\bar{V}_j) \to K^{n_j},$$

where p_2 denotes the projection on the second factor. Since closed subspaces of a normal space are normal, the Tietze extension theorem provides an extension

$$h_j : p^{-1}(\bar{V}_j) \to K^{n_j} \quad \text{of} \quad p_2 \phi_j f.$$

Define g_j on $p^{-1}(V_j)$ by $g_j(s) = (p(s), h_j(s))$; then

$$\phi_j^{-1} g_j : p^{-1}(V_j) \to p'^{-1}(V_j)$$

is continuous and fibre-preserving. Now let $\{f_j\}$ be a partition of unity subordinate to $\{V_j\}$ and define $g : p^{-1}(p(A)) \to S'$ by