CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

BRIAN J. DAY

Varieties of an enriched category

Cahiers de topologie et géométrie différentielle catégoriques, tome 19, nº 2 (1978), p. 169-177

<http://www.numdam.org/item?id=CTGDC_1978__19_2_169_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE *Vol. XIX - 2 (1978)*

VARIETIES OF AN ENRICHED CATEGORY

by Brian J. DAY

ABSTRACT.

The purpose of this article is to describe the effect of translating several results on varieties of a category (by Y. Diers) into the $\mathcal{V}$-context where $\mathcal{V}$ is a complete symmetric monoidal closed category. Under suitable completeness hypotheses on a $\mathcal{V}$-category $\mathcal{C}$ we show that if $N: \mathfrak{A} \to \mathcal{C}$ is a $\mathcal{V}$-dense functor it is then possible to define the notion of a $\mathcal{V}$-identity (relative to N) such that a full subcategory of $\mathcal{C}$ is a variety of models for a class of $\mathcal{V}$-identities if and only if it is closed under $\mathcal{V}$-limits, $\mathcal{V}$-subobjects and N-absolute $\mathcal{V}$-colimits. Applications to universal algebra are discussed.

INTRODUCTION.

The concept of a variety of objects relative to a dense functor is due to Y. Diers [5]. It turns out that this very general approach to the study of varieties has interesting applications to characterization results in earlier works of G. Birkhoff, A.I. Mal'cev, A. Shafaat and W.S. Hatcher on varieties of algebras defined by classes of identities and by classes of implications (see the references to Diers [5]). The concept is also sufficiently wide to include work by F.E.J. Linton [8] on equational categories.

The present article is aimed at providing a generalisation of several of Diers'results in the case where all the categorical algebra is *relative* to a (fixed) symmetric monoidal closed ground category (see MacLane [9] for the basic categorical algebra, and see Eilenberg and Kelly [7], Day and Kelly [3] and Dubuc [6] for its relative enrichment). The final results derived in this article are formally similar to those of Diers but the technique of proof is different; for example, we have to use mean tensor products in the sense of Borceux and Kelly [2].

The results differ slightly from Diers' in that we require the existence of certain N-absolute regular factorisations. However, this does not seriously hinder applications. Our main application of the enriched results is to finitary universal algebra in a closed category (Borceux and Day [1]).

1. N-IDENTITIES.

For notational convenience we shall henceforth suppose that *all* categorical algebra used is *relative* to a fixed symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}, \otimes, I, [-,-], \dots)$.

Let $N : \mathcal{A} \to \mathcal{C}$ be a dense functor. An *N-identity (of type X)* is a pair of objects $A_0, A_1 \in \mathcal{A}$ and a pair of morphisms

$$(\omega_1, \omega_2) : X \to \mathcal{C}(NA_1, NA_0) \quad \text{in } \mathcal{V}.$$

A *model of an N-identity* is a $C \in \mathcal{C}$ such that

$$\begin{array}{ccccc} & & \mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, NA_0) & & \\ & {\scriptstyle 1\otimes\omega_1} \nearrow & & \searrow {\scriptstyle M} & \\ \mathcal{C}(NA_0, C) \otimes X & & & & \mathcal{C}(NA_1, C) \\ & {\scriptstyle 1\otimes\omega_2} \searrow & & \nearrow {\scriptstyle M} & \\ & & \mathcal{C}(NA_0, C) \otimes C(NA_1, NA_0) & & \end{array}$$

commutes. Similarly we talk about the models of a *class of N-identities*; such a model is simply an object $C \in \mathcal{C}$ which is a model for each N-identity in the class.

The collection of all models of a class of N-identities is called an N-variety. Suppose $\mathcal{C}$ is small-complete.

PROPOSITION 1.1. *An N-variety is, as a full subcategory of $\mathcal{C}$, closed under products, cotensoring, subobjects, N-absolute colimits in $\mathcal{C}$, and N-absolute epimorphic images.*

PROOF. The proof for limits and subobjects is straightforward. Let $Gk * Hk$

be a mean tensor product with each Hk a model. Then

$$\mathcal{C}(NA_0, Hk)\otimes X \underset{1\otimes\omega_2}{\overset{1\otimes\omega_1}{\rightrightarrows}} \mathcal{C}(NA_0, Hk)\otimes\mathcal{C}(NA_1, NA_0) \xrightarrow{M} \mathcal{C}(NA_1, Hk)$$

are equal for all $k \in \mathcal{K}$. Thus both legs are equal on taking $Gk*-k$, and the result follows. Similarly the result for N-absolute epimorphic images follows.

2. N-REFLECTIVE SUBCATEGORIES.

Let $I: \mathfrak{M} \to \mathcal{C}$ be the inclusion of a full subcategory. We say that I is *N-reflective* if it has a left N-adjoint R:

$$\mathfrak{M}(RA, M) \sim \mathcal{C}(NA, IM).$$

An N-reflective subcategory is called *properly N-reflective* if

$$\mathcal{C}(IRA, C) \sim \mathcal{C}(NA, C) \quad \text{for all } A \in \mathfrak{A}$$

implies $C \in \mathfrak{M}$.

PROPOSITION 2.1. *If $\mathcal{C}$ has regular factorisations then any N-reflective subcategory of $\mathcal{C}$ which is closed under subobjects and N-absolute colimits is properly N-reflective.*

PROOF. Call the subcategory $\mathfrak{M}$. Because $\mathcal{C}$ has regular factorisations and $\mathfrak{M}$ is closed in $\mathcal{C}$ under taking subobjects, each N-adjunction unit: $\eta_A: NA \to IRA$ is, by factorisation, a regular epimorphism. Now factor $R: \mathfrak{A} \to \mathfrak{M}$ into a bijection on objects $\hat{R}: \mathfrak{A} \to \mathfrak{A}'$ followed by a fully faithful functor $R': \mathfrak{A}' \to \mathfrak{M}$, and suppose

$$\mathcal{C}(IRA, C) \sim \mathcal{C}(NA, C) \quad \text{for all } A \in \mathfrak{A}\ ;$$

then

$$\mathcal{C}(IR'A', C) \sim \mathcal{C}(NA', C) \quad \text{for all } A' \in \mathfrak{A}'$$

(this *not* natural in $A' \in \mathfrak{A}'$). Also $\hat{R}$ induces

$$C \sim \mathcal{C}(NA, C)*NA \longrightarrow \mathcal{C}(IRA, C)*IRA$$
$$\longrightarrow \mathcal{C}(IR'A', C)*IR'A' \longrightarrow C.$$

Therefore $\mathcal{C}(IR'A', C)*IR'A' \sim C$; it remains to prove that the left-hand side is an N-absolute colimit. But, for each $A' \in \mathcal{A}'$, we have

$$\begin{aligned}
&\mathcal{C}(IR'A, C) * \mathcal{C}(NA', IR'A) \cong \\
&\quad \cong \mathcal{C}(IR'A, C) * \mathcal{M}(RA', R'A) \quad (\text{since } R \dashv_N I) \\
&\quad \cong \mathcal{C}(IR'A, C) * \mathcal{A}'(A', A) \\
&\quad \cong \mathcal{C}(IR'A', C) \quad (\text{by the representation theorem}) \\
&\quad \cong \mathcal{C}(NA', C).
\end{aligned}$$

Thus $C \in \mathcal{M}$, as required. //

PROPOSITION 2.2. *If $\mathcal{C}$ has N-absolute regular factorizations then a properly N-reflective subcategory of $\mathcal{C}$ is an N-variety if and only if it is closed under subobjects.*

PROOF. Necessity follows from Proposition 1.1. Let $\mathcal{M}$ be a properly N-reflective subcategory of $\mathcal{C}$ closed under subobjects. Let $\mathcal{K}$ be the class of identities:

$$\mathcal{C}(NA_1, K) \underset{\mathcal{C}(1,\beta)}{\overset{\mathcal{C}(1,\alpha)}{\rightrightarrows}} \mathcal{C}(NA_1, NA_0)$$

where (α, β) is the kernel pair of $\eta_{A_0} : NA_0 \to IRA_0$. If $C \in \mathcal{M}$, then

$$\mathcal{C}(\eta_{A_0}, 1) : \mathcal{C}(IRA_0, C) \cong \mathcal{C}(NA_0, C)$$

so C is a model for $\mathcal{K}$ by commutativity of the diagram:

$$\begin{array}{ccc}
\mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, K) & & \\
\downarrow\downarrow & & \\
\mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, NA_0) & \xrightarrow{\quad M \quad} & \mathcal{C}(NA_1, C) \\
\Big\downarrow 1 \otimes \mathcal{C}(1, \eta_{A_0}) & & \Big\uparrow M \\
\mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, IRA_0) & \xrightarrow[\mathcal{C}(\eta_{A_0}, 1)^{-1} \otimes 1]{\cong} & \mathcal{C}(IRA_0, C) \otimes \mathcal{C}(NA_1, IRA_0).
\end{array}$$

Conversely, if C is a model for $\mathcal{K}$, then

$$\mathcal{C}(IRA, C) \sim \mathcal{C}(NA, C) \quad \text{for all} \quad A \in \mathfrak{A}$$

so $C \in \mathfrak{M}$. To establish this isomorphism let $\alpha, \beta : K \to NA_0$ be the kernel pair of η_{A_0} for each $A_0 \in \mathfrak{A}$. Then we obtain the dashed arrow in the following diagram from the fact that $\mathcal{C}$ has N-absolute regular factorisations:

$$\begin{array}{ccc}
\mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, K) & & \\
{\scriptstyle 1 \otimes \mathcal{C}(1,\alpha)} \downarrow \; \downarrow {\scriptstyle 1 \otimes \mathcal{C}(1,\beta)} & & \\
\mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, NA_0) & & \\
{\scriptstyle M} \downarrow & \searrow {\scriptstyle 1 \otimes \mathcal{C}(1, \eta_{A_0})} & \\
\mathcal{C}(NA_1, C) & \dashleftarrow & \mathcal{C}(NA_0, C) \otimes \mathcal{C}(NA_1, IRA_0).
\end{array}$$

This dashed arrow then transforms into the dashed arrow in the following diagram:

$$\begin{array}{ccc}
\mathcal{C}(IRA_0, C) & \xrightarrow{\sim} & \int_{A_1} [\mathcal{C}(NA_1, IRA_0), \mathcal{C}(NA_1, C)] \\
{\scriptstyle \mathcal{C}(\eta_{A_0}, C)} \downarrow & \nearrow & \downarrow {\scriptstyle \int_{A_1} [\mathcal{C}(NA_1, \eta_{A_0}), \mathcal{C}(NA_1, C)]} \\
\mathcal{C}(NA_0, C) & \xrightarrow{\sim} & \int_{A_1} [\mathcal{C}(NA_1, NA_0), \mathcal{C}(NA_1, C)].
\end{array}$$

Because $\int_{A_1} [\mathcal{C}(NA_1, \eta_{A_0}), \mathcal{C}(NA_1, C)]$ is both a monomorphism and a retraction, it is an isomorphism; thus $\mathcal{C}(\eta_{A_0}, C)$ is an isomorphism. //

3. CHARACTERISATION THEOREMS.

The remaining theory is analogous to that of Diers [5, 4], but we shall give a brief outline for completeness.

Suppose that $\mathcal{C}$ is complete, has N-absolute regular factorisations and each object of $\mathcal{C}$ has only a set of regular quotient objects.

THEOREM 3.1. *A full subcategory of $\mathcal{C}$ is an N-variety if and only if it is closed under products, cotensoring, subobjects and N-absolute colimits.*

PROOF. If $\mathfrak{M}$ is a full subcategory of $\mathcal{C}$ with the required properties, then $I : \mathfrak{M} \to \mathcal{C}$ has a left adjoint S (say) by the adjoint functor theorem. Since

$\mathfrak{M}$ is closed under N-absolute colimits in $\mathcal{C}$, it is thus a properly N-reflective subcategory of $\mathcal{C}$ (by Proposition 2.1) and this is an N-variety (by Proposition 2.2). //

Let $\mathfrak{D}$ be a class of mean tensor products in $\mathcal{C}$ and let $\mathfrak{L}$ be a class of spans

$$\mathcal{V}^{op} \longleftarrow \mathcal{H} \longrightarrow \mathfrak{A}$$

such that $\mathcal{C}$ has colimits of type $\mathfrak{L}$ and each colimit in $\mathfrak{D}$ is of type $\mathfrak{L}$.

THEOREM 3.2. *If $N: \mathfrak{A} \to \mathcal{C}$ is dense by colimits of class $\mathfrak{D}$ then a full subcategory of $\mathcal{C}$ which is closed under products, cotensoring, subobjects and colimits of type $\mathfrak{L}$, is an N-variety.*

PROOF. If $I: \mathfrak{M} \to \mathcal{C}$ is closed under colimits of type $\mathfrak{L}$ then IS preserves colimits of type $\mathfrak{L}$ and thus is a Kan extension by N (see [4] Proposition 2.2). Thus IS preserves N-absolute colimits, so $\mathfrak{M} \subset \mathcal{C}$ is closed under N-absolute colimits. //

COROLLARY 3.3. *If $N: \mathfrak{A} \to \mathcal{C}$ is dense by colimits of type $\mathfrak{L}$ then a full subcategory of $\mathcal{C}$ is an N-variety if and only if it is closed under products, cotensoring, subobjects and colimits of type $\mathfrak{L}$.*

PROOF. By Theorem 3.2 and the fact that colimits of type $\mathfrak{L}$ are now N-absolute. //

4. EXAMPLES.

EXAMPLE 4.1. Suppose $\mathcal{V}$ is a complete and cocomplete *π-category* in the sense it satisfies the following axioms (cf. Borceux and Day [1]):

π1. For any small category $\mathcal{P}$ with finite products, any product-preserving functor $G: \mathcal{P} \to \mathcal{V}$ and any functors $H, K: \mathcal{P}^{op} \to \mathcal{V}$, the canonical morphism

$$\int^P (H \times K)(P) \otimes GP \to (\int^P HP \otimes GP) \times (\int^P KP \otimes GP)$$

is an isomorphism.

π2. For any object $X \in \mathcal{V}$ the functor $- \times X: \mathcal{V} \to \mathcal{V}$ preserves coequalisers of reflective pairs.

Let $\mathcal{A}$ be a small category with finite coproducts and call a functor $F: \mathcal{A}^{op} \to \mathcal{V}$ *a sheaf* if it preserves finite products. Thus we have the Yoneda embedding

$$N: \mathcal{A} \to \mathcal{F} \quad \text{and} \quad \mathcal{F} \subset [\mathcal{A}^{op}, \mathcal{V}] .$$

Now let $\mathcal{L}$ be the class of spans of the form

$$\mathcal{V}^{op} \xleftarrow{\ G\ } \mathcal{H} \xrightarrow{\ H\ } \mathcal{A}$$

where $\mathcal{H}$ is small and has finite coproducts preserved by G. Then $N: \mathcal{A} \to \mathcal{F}$ is dense by colimits of type $\mathcal{L}$ since, by the representation theorem, we have

$$F \simeq \textstyle\int^A FA \otimes \mathcal{A}(-, A): \mathcal{A}^{op} \to \mathcal{V} \quad \text{for each} \quad F \in [\mathcal{A}^{op}, \mathcal{V}] ;$$

but the right-hand side is a sheaf if F is a sheaf, by axiom π 1. By axiom π 2, $\mathcal{F}$ has N-absolute regular factorisations if $\mathcal{V}$ has regular factorisations. Also each sheaf in $\mathcal{F}$ has only a set of regular quotients if this is so for $\mathcal{V}$. Thus, under these conditions on $\mathcal{V}$, a full subcategory of $\mathcal{F}$ is an N-variety if and only if it is closed under limits, subobjects and colimits of type $\mathcal{L}$ (Corollary 3.3).

Note that if $\mathcal{A}$ is a finitary algebraic theory (see Borceux and Day [1]) then $\mathcal{F}$ is the category of $\mathcal{A}$-algebras. Thus any variety $I: \mathcal{M} \to \mathcal{F}$ of $\mathcal{A}$-algebras contains an abstractly finite projective generator and is thus a category of algebras for another tneory, in fact for a quotient theory of $\mathcal{A}$ (see Borceux and Day [0]).

EXAMPLE 4.2. Consider a monad $\mathcal{T} = (T, \mu, \eta)$ on a category $\mathcal{C}$ and let

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{T}} & \xrightarrow{\quad N \quad} & \mathcal{C}^{\mathcal{T}} \\ & {\scriptstyle U_{\mathcal{T}}} \searrow \quad \swarrow {\scriptstyle U = U^{\mathcal{T}}} & \\ & \mathcal{C} & \end{array}$$

be the standard resolution of $\mathcal{T}$ into a Kleisli category $\mathcal{C}^{\mathcal{T}}$ and an Eilenberg-Moore category $\mathcal{C}_{\mathcal{T}}$ (we usually omit the forgetful functor U from the notation). An identity (ω_1, ω_2) of $\mathcal{T}$ is a pair of morphisms

$$(\omega_1, \omega_2): X \to \mathcal{C}(C_1, TC_0).$$

A $\mathcal{T}$-algebra (C, ζ) *is a model* for the identity (ω_1, ω_2) if the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}(C_0,C)\otimes\mathcal{C}(C_1,TC_0) & \xrightarrow{(\mathcal{C}(1,\zeta).T)\otimes 1} & \mathcal{C}(TC_0,C)\otimes\mathcal{C}(C_1,TC_0) \\
\Big\uparrow{\scriptstyle 1\otimes\omega_1} & & \Big\downarrow{\scriptstyle M} \\
\mathcal{C}(C_0,C)\otimes X & & \mathcal{C}(C_1,C) \\
\Big\downarrow{\scriptstyle 1\otimes\omega_2} & & \Big\uparrow{\scriptstyle M} \\
\mathcal{C}(C_0,C)\otimes\mathcal{C}(C_1,TC_0) & \xrightarrow[(\mathcal{C}(1,\zeta).T)\otimes 1]{} & \mathcal{C}(TC_0,C)\otimes\mathcal{C}(C_1,TC_0).
\end{array}$$

Now suppose that $\mathcal{C}$ has regular factorisations, is complete, and each object has only a *set* of regular quotient objects. Suppose also that $T: \mathcal{C} \to \mathcal{C}$ preserves coequalisers of reflective pairs: It is clear that the varieties of $\mathcal{T}$-algebras are precisely the N-varieties, and it is the case that $\mathcal{C}^{\mathcal{T}}$ has N-absolute regular factorisations, is complete, and each object has only a set of regular quotient objects (since T preserves coequalisers of reflective pairs).

Finally, $N: \mathcal{C}_{\mathcal{T}} \to \mathcal{C}^{\mathcal{T}}$ is dense by coequalisers of U-contractible pairs (see [4] Example 4.3). Thus a class of $\mathcal{T}$-algebras is a variety of $\mathcal{T}$-algebras if and only if it is closed under limits, subobjects and U-split quotient objects.

REFERENCES

0. F. BORCEUX and B. J. DAY, On characterisation of finitary algebraic categories, *Bull. Australian Math. Soc.* 18 (1978), 125-135.

1. F. BORCEUX and B. J. DAY, Universal algebra in a closed category, Preprint, *Inst. de Math. Pure et Appl., Univ. Cath. Louvain* (1977).

2. F. BORCEUX and G.M. KELLY, A notion of limit for enriched categories, *Bull. Austral. Math. Soc.* 12 (1975), 49-72.

3. B. J. DAY and G.M. KELLY, Enriched functor categories, Reports of the Midwest Cat. Seminar, *Lecture Notes in Math.* 106, Springer (1969), 178-191.

4. B. J. DAY, Density presentations of functors, *Bull. Australian Math. Society,* 16 (1977), 427-448.

5. Y. DIERS, Variétés d'une catégorie, *Ann. Soc. Sci. Bruxelles* Sér. I, 90 (1976), 159-172.

6. E. J. DUBUC, Kan extensions in enriched category theory, *Lecture Notes in Math.* 145, Springer (1970).

7. S. EILENBERG and G.M. KELLY, Closed categories, *Proc. Conf. categ. algebra* La Jolla 1965, Springer 1966, 421-562.

8. F. E. J. LINTON, Some aspects of equational categories, *Proc. Conf. categ. alg.* La Jolla 1965, Springer 1966, 84-94.

9. S. MACLANE, *Categories for the working mathematician*, Grad. Texts in Math. 5, Springer, 1971.

Department of Pure Mathematics
University of Sydney
SYDNEY, N.S.W. 2006
AUSTRALIE