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ON TOPOLOGICAL MODULES AND DUALITY

by Brian J. DAY

INTRODUCTION.

If R is a commutative limitspace ring with identity, then the category V of limitspace modules over R is symmetric monoidal closed; that is $V = (V, R, \otimes, [-,-], \dots)$. This follows from the fact that we are here dealing with a finitary commutative algebraic limitspace theory on the cartesian closed category of all limitspaces and continuous maps. Thus it makes sense to talk about duality in V with respect to the internal Hom of V (that is, the structure of continuous convergence on the function spaces) and some basic dualising object Q for which the canonical map $Q \to [[Q, Q], Q]$ is an isomorphism. An object $A \in V$ for which the canonical map $A \to [[A, Q], Q]$ is an isomorphism is called *Q-reflexive.*

In the first instance we are interested in studying Q-reflexive topological R-modules in the case where Q itself is a topological R-module with no small submodules. This is done by considering Q to be just one Q-reflexive model in a class $M \subset V$ of Q-reflexive topological R-modules. We then form the epireflexive hull H of M in V. It is seen that, if Q is an injective cogenerator in H, then the limit closure $\hat{M}$ of M in V is a reflective full subcategory of H, hence of V. The main result here is that all the objects of $\hat{M}$ are then Q-reflexive topological R-modules.

Up to this point we consider only the Ens-based epireflective hull of M in V. However, we could also consider the V-based epireflective hull V' of M in V. The reason why we do not consider the V-hull V' in the first instance is that we do not know that Q is an injective cogenerator of V'.

Next we form the epireflective V-hull V" of M in V' and obtain a symmetric monoidal closed category (namely V'') in which Q is a strong V-cogenerator. This then leads to a duality $\mathfrak{R}^{op} \approx \mathfrak{R}$ where $\mathfrak{R}$ denotes the category of all Q-reflexive limitspace R-modules. This extends known dualities since $\mathfrak{R}$ contains $\hat{M}$ under the above condition on M and Q.

Some references to material which has already appeared on the topic may be found in Binz [2 and 3]. We shall assume some familiarity with Binz and Keller [4] and with the method used in Day [8]. For general category theory we refer the reader to Mac Lane [11].

The contents of this article are:

1. THE BASIC SITUATION.

The basic situation for our investigation is the following. Let Top denote the category of all topological spaces and continuous maps and suppose that Top is epireflectively embedded in some cartesian closed extension U. We shall suppose that U is complete and cocomplete, has 1 (the terminal object) as a generator, and has canonical {epi; strong-mono} factorisations. We shall also suppose that U is cowellpowered (that is, each object of U has only a small set of epi-images) and is weakly wellpowered (that is, each object of U has only a small set of strong subobjects).

$$\begin{tikzcd} \mathrm{Top} \arrow[r, hook] \arrow[dr] & \mathrm{U} \arrow[d, "U = \mathrm{U}(1,-)"] \\ & \mathrm{Ens} \end{tikzcd}$$

Given U (which the reader can most conveniently take to be the category of limitspaces or the category of Choquet pseudotopologies), we form the symmetric monoidal closed category

$$\mathrm{V} = (\mathrm{V}, R, \otimes, [-,-], \ldots)$$

of R-modules in U for some fixed commutative ring object R (with an identity) in U; see, for example, Borceux and Day [5]. Then V is complete and cocomplete, has R as a generator, and has canonical {epi; strong-mono} factorisations. It is also cowellpowered and weakly wellpowered. In fact, V is U-monadic over U for a finitary commutative algebraic U-theory.

Among the objects of V there are the topological R-modules; that is, the R-modules over U whose underlying U-space structure lies in Top. The category of topological R-modules and continuous R-module homomorphisms is closed in V under set-indexed products and strong subobjects, and consequently is epireflexive in V.

We shall denote set-indexed powers by $\{X, A\}$ and copowers by $X . A$, the context making it clear which category we are forming these in.

2. DUALITY OF STRONG PROJECTIVE LIMITS.

A limit $\lim A_\lambda$, $\lambda \epsilon \Lambda$, of topological R-modules is said to be *strong* if Λ is a small cofiltered category and each projection $p_\lambda : \lim A_\lambda \to A_\lambda$ is an identification map. For example, a topological product $\prod_{\gamma \epsilon \Gamma} A_\gamma$ may be regarded as a strong limit indexed by the finite subsets of Γ.

LEMMA 2.1. *Given a strong limit* $\lim A_\lambda$ *of topological R-modules, the collection* $\{ker p_\lambda \mid \lambda \epsilon \Lambda\}$ *is a filter base on* $\lim A_\lambda$, *which converges to* 0.

PROOF. Since the limit is strong, the collection

$$\{p_\lambda^{-1}(V) \mid V \text{ open in } A_\lambda\}$$

is a *base* for the topology on $\lim A_\lambda$. For suppose

$$p_{\lambda_1}^{-1}(V_1) \cap \ldots \cap p_{\lambda_n}^{-1}(V_n)$$

is a sub-basic open set in $\lim A_\lambda$. Because Λ is cofiltered, there exists an $\alpha \epsilon \Lambda$ and connecting maps

$$f_{\alpha\lambda_1} : A_\alpha \to A_{\lambda_1}, \ \ldots, \ f_{\alpha\lambda_n} : A_\alpha \to A_{\lambda_n}.$$

Thus

$$p_{\lambda_1}^{-1}(V_1) \cap \ldots \cap p_{\lambda_n}^{-1}(V_n) = p_\alpha^{-1}(V_\alpha),$$

where

$$V_\alpha = f_{\alpha\lambda_1}^{-1}(V_1) \cap \ldots \cap f_{\alpha\lambda_n}^{-1}(V_n).$$

Hence the standard sub-base is a base; and $ker p_\lambda$ approaches 0. //

We now fix in V a topological R-module Q which has no small submodules.

LEMMA 2.2. *Given a strong limit* $\lim A_\lambda$ *of topological R-modules, the canonical map from* $colim[A_\lambda, Q]$ *to* $[\lim A_\lambda, Q]$ *is a surjection in* V.

PROOF. By Lemma 2.1, $ker p_\lambda$ approaches 0. Thus each map factors through some projection because Q has no small submodules. Thus the canonical map from $colim[A_\lambda, Q]$ to $[\lim A_\lambda, Q]$ is a surjection. //

THEOREM 2.3. *If* $\{A_\lambda, \lambda \epsilon \Lambda\}$ *is a family of topological R-modules, each* *of which is Q-reflexive (i.e.,* $A_\lambda \approx [[A_\lambda, Q], Q]$ *for all* $\lambda \in \Lambda$*), then any strong limit* $\lim A_\lambda$ *is Q-reflexive.*

PROOF. The surjection of Lemma 2.2 yields a mono m in the diagram:

$$\begin{tikzcd}
{[[\lim A_\lambda, Q], Q]} \arrow[r, "m"] \arrow[d, shift left, "r"] & {[\operatorname{colim}[A_\lambda, Q], Q]} \arrow[d, "\wr"] \\
\lim A_\lambda \arrow[u, shift left, "\eta"] \arrow[r, "\approx"] & \lim [[A_\lambda, Q], Q]
\end{tikzcd}$$

defining r. But it is easily seen that $r\eta = 1$. //

As a corollary we have that a product of Q-reflexive topological R-modules is Q-reflexive. It is of interest here to see that there are simple examples of strong limits other than products.

PROPOSITION 2.4. *Let* $\lim A_\lambda$ *be a cofiltered limit of topological R-modules, each of whose connecting maps is an open surjection. Then each projection* $p_\lambda : \lim A_\lambda \to A_\lambda$ *is an open mapping provided it is a surjection.*

PROOF. As already established, the sub-base

$$\{ p_\lambda^{-1}(V) \mid V \text{ open in } A_\lambda \}$$

is a base for the topology on $\lim A_\lambda$. Also

$$p_\lambda\left(\bigcup_\alpha p_\alpha^{-1}(V_\alpha)\right) = \bigcup_\alpha p_\lambda p_\alpha^{-1}(V_\alpha),$$

so, for each $\alpha \in \Lambda$, choose $\beta(\alpha) \in \Lambda$ and morphisms

$$f_{\lambda\beta(\alpha)} : A_{\beta(\alpha)} \to A_\lambda, \quad f_{\beta(\alpha)\alpha} : A_{\beta(\alpha)} \to A_\alpha .$$

Then

$$\bigcup_\alpha p_\lambda p_\alpha^{-1}(V_\alpha) = \bigcup_\alpha f_{\lambda\beta(\alpha)} p_{\beta(\alpha)} p_\alpha^{-1}(V_\alpha) = \bigcup_\alpha f_{\lambda\beta(\alpha)} f_{\beta(\alpha)\alpha}^{-1}(V_\alpha)$$

which is open in A_λ. //

COROLLARY 2.5. *If* $\{ f_n : A_{n+1} \to A_n \mid -\infty \le n \le \infty \}$ *is a chain of open surjections, then* $\lim A_n$ *is strong.*

PROOF. Clearly each projection $\lim A_\lambda \to A_\lambda$ is here a surjection. //

3. PRELIMINARIES ON INJECTIVITY.

For this section we fix in V a class M of topological R-modules which is closed under finite products. We suppose that M contains an R-module Q which cogenerates M.

The module Q is said to be M-*injective* if, given any submodule $A \leq \prod_{\lambda \in \Lambda} M_\lambda$ in V, $M_\lambda \in$ M, and any homomorphism $f: A \to Q$, then exists a homomorphism $g: \prod M_\lambda \to Q$ which extends f.

The module Q is said to be M-*separating* if, given any closed submodule $A \leq \prod_{\lambda \in \Lambda} M_\lambda$ and any $x \in \prod M_\lambda$ with $x \notin A$, there exists a homomorphism $g: \prod M_\lambda \to Q$ such that $g(A) = 0$ and $g(x) \neq 0$.

Let H denote the epireflective hull of M in V.

PROPOSITION 3.1. *(i) If Q is* M*-injective, then it is an injective cogenerator in* H *and all strong monics in* H *are regular.*

(ii) If, in addition to (i), Q is M*-separating, then the closed submodules are precisely the strong monics in* H.

The proof of this proposition is exactly the same as the proof given in Day [8], Section 2.

It is a simple consequence of the fact that all strong monics in H are regular (so that the regular monics in H are closed under composition) that the limit closure of M (denoted $\hat{\text{M}}$) in H (hence in V) exists and is reflective in H (hence in V). Here we use the fact that Q cogenerates M, hence H, and the special adjoint-functor theorem.

We now seek special conditions under which Q is M-injective and M-separating. On having fixed M and Q, we call a neighborhood V of $0 \in Q$ M-*effective* if it contains no non-zero submodules and, for each map $f: M \to Q$, $M \in$ M, f is continuous iff $f^{-1} V$ is a neighborhood of $0 \in M$.

PROPOSITION 3.2. *The module Q is* M*-injective if it contains an* M*-effective neighborhood and each submodule (subspace topology) $A \leq M$, $M \in$* M, *allows the lifting of homomorphisms into Q.*

The proof is directly inspired from that of Kaplan [9] Theorem 1.

For example, let $R = Q = \mathrm{R}$ or C, and suppose that each $M \in \mathrm{M}$ is a normed linear space. Then Q is M-injective by Proposition 3.2 together with the Hahn-Banach Theorem.

For another example, let $R = Q = K$ be any topological field containing a non-trivial open neighborhood of $0 \in K$. Let

$$\mathrm{M} = \{0, K, K^2, \dots, K^n, \dots\}.$$

Then K is M-injective by an adaptation of the proof of Kaplan [9] Theorem 1.

Let $Q = Hom_{\mathrm{Z}}(R, \mathrm{R}/\mathrm{Z})$ and suppose each $M \in \mathrm{M}$ is locally compact and Hausdorff. Then Q is M-injective since we have:

$$[A, Hom_{\mathrm{Z}}(R, \mathrm{R}/\mathrm{Z})] \approx Hom_{\mathrm{Z}}(A, \mathrm{R}/\mathrm{Z});$$

see Kaplan [9] Theorem 1 for the case $R = \mathrm{Z}$. For the case of R discrete, duality has been studied in this context by Stöhr [12].

In order to examine the M-separating condition we first suppose that each $M \in \mathrm{M}$ has the property that the closure of an R-submodule $A \leq M$ is again an R-module. Next we suppose that, given any closed submodule $A \leq M$, $M \in \mathrm{M}$, and any $x \in M$, $x \notin A$, there exists a homomorphism $f: M \to Q$ such that $f(A) = 0$ and $f(x) \neq 0$. Under these assumptions on M and Q it is a simple matter to check the proof of Kaplan [9] Theorem 2 and see that Q is M-separating.

4. DUALITY OF COREPRESENTABLE MODULES.

We again choose in V a full subcategory $\mathrm{M} \subset \mathrm{V}$ of Q-reflexive topological R-modules such that $Q \in \mathrm{M}$. It is easily seen that Q is then cogenerating in M. The family M is thought of as a collection of *models* from which we can construct such objects as M-corepresentable R-modules and also pro-M-objects. From Section 2 we know that

$$\Pi M_\lambda \approx [[\Pi M_\lambda, Q], Q] \quad \text{if all } M_\lambda \in \mathrm{M}.$$

Let A be an M-corepresentable R-module in the sense that there exists an equaliser presentation:

$$A \longrightarrow \Pi M_\lambda \rightrightarrows \Pi N_\mu , \qquad M_\lambda , N_\mu \in \mathrm{M} .$$

Then we have the following:

PROPOSITION 4.1. *If* $\mathrm{V}(\Pi M_\lambda , Q) \to \mathrm{V}(A, Q)$ *is a surjection, then* A *is* Q*-reflexive.*

PROOF. Consider the diagram:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & \Pi M_\lambda \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Pi N_\mu \arrow[d] \\ A^{**} \arrow[r, tail] \arrow[d, dashed] & (\Pi M_\lambda)^{**} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & (\Pi N_\mu)^{**} \arrow[d] \\ A \arrow[r, tail] & \Pi M_\lambda \arrow[r, shift left] \arrow[r, shift right] & \Pi N_\mu \end{tikzcd}$$

where A^* denotes $[A, Q]$. The dashed arrow is a mono since the map $A^{**} \to (\Pi M_\lambda)^{**}$ is a mono, and the result follows. //

COROLLARY 4.2. *If* Q *is injective in* H, *then* A *is* Q*-reflexive.*

The injectivity of Q in H is sometimes an unnecessary assumption since it may happen that the limit closure of M in H (hence in V) consists entirely of pro-M-objects (see Day [7] for the definition of a pro-M-object) and that each pro-M-object is Q-reflexive. This happens, for example, when $R = Q$ is a discrete principal ideal domain and M consists of the (discrete) free R-modules of finite rank.

In order to generalise this example, suppose E denotes the category of topological identification maps and let $\mathrm{D} = \mathrm{E} \cap \mathrm{M}$. Then

$$\lim_{\mathrm{E}(A,M)} M = \int_{\mathrm{D}} \{ \mathrm{E}(A,M), M \}$$

is a strong projective limit, for each $A \in \mathrm{H}$, if each map $f: A \to M$, $A \in \mathrm{H}$, $M \in \mathrm{M}$, factors as

$$A \xrightarrow[e]{} N \xrightarrow[g]{} M$$

where $e \in \mathrm{E}$ and $N \in \mathrm{M}$. Note that this limit is cofiltered since any pair $e_1 : A \to M_1$ and $e_2 : A \to M_2$ yields a canonical map $A \to M_1 \oplus M_2$ which factors as an $e \in \mathrm{E}$ followed by an $N \to M_1 \oplus M_2$, $N \in \mathrm{M}$, by hypothesis on M.

PROPOSITION 4.3. *The canonical map*

$$\int^{\mathrm{D}} \mathrm{E}(A,N) \times \mathrm{H}(N,M) \to \mathrm{H}(A,M)$$

is a bijection.

PROOF. It is a surjection because each $f \in \mathrm{H}(A,M)$ factors appropriately. It is an injection since any two factorisations of $f \in \mathrm{H}(A,M)$ are related by identification maps to a third factorisation:

$$\begin{tikzcd}
 & N_1 \arrow[dr] & \\
A \arrow[ur, "e_1"] \arrow[r, two heads] \arrow[dr, "e_2"'] & N \arrow[u] \arrow[d] \arrow[r, dashed] & M \\
 & N_2 \arrow[ur] &
\end{tikzcd}$$

Now M cogenerates H so Q cogenerates H if it cogenerates M, which it does if each object of M is Q-reflexive. Thus, if Q has no small submodules, each map $\lim\limits_{\mathrm{E}(A,M)} M \to Q$ factors through some projection for all $A \in \mathrm{H}$. This, in turn, means that the canonical map

$$\operatorname*{colim}_{\mathrm{E}(A,M)} \mathrm{H}(M,Q) \to \mathrm{H}(\lim_{\mathrm{E}(A,M)} M, Q)$$

is a surjection for all $A \in \mathrm{H}$.

PROPOSITION 4.4. *The canonical map* $A \to \lim\limits_{\mathrm{E}(A,M)} M$ *is an epi in* H *for all* $A \in \mathrm{H}$.

PROOF. Consider the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{\mathrm{E}(A,M)} \mathrm{H}(M,Q) \arrow[rr, no head, "\approx"] \arrow[dr, "\approx"'] & & \mathrm{H}(A,Q) \\
 & \mathrm{H}(\lim_{\mathrm{E}(A,M)} M, Q) \arrow[ur] &
\end{tikzcd}$$

and use the fact that Q cogenerates H. //

Thus $A \approx \lim_{E(A,M)} M$ in H iff $A \to \lim_{E(A,M)} M$ is a strong monic in H. Note that, by Kelly [10], H has canonical {epi; strong-mono} factorisations, so we can consider the epireflective hull P of M in H.

PROPOSITION 4.5. $P \approx \hat{M}$.

PROOF. If $A \in P$ then the canonical map $A \to \int_M \{H(A,M),M\}$ is a strong mono in H. Thus we can consider the following diagram:

$$\begin{tikzcd} A \arrow[rr, tail] \arrow[d] & & \int_M \{H(A,M),M\} \arrow[d, "\wr"] \\ \int_D \{E(A,N),N\} \arrow[rr, "\approx"] & & \int_M \{\int^D E(A,N) \times H(N,M), M\} \end{tikzcd}$$

and obtain that $A \to \int_D \{E(A,N),N\}$ is a strong mono in H, hence is an isomorphism. //

5. THE V-HULL OF Q.

So far we have considered only the ordinary Ens-based epireflective hull of M in V. However, we could also consider the V-based epireflective hull (which we shall denote by V') although we do not know that Q is injective in V'.

First let us consider the V-hull in V of a class $M \subset V$. The objects B in this hull V' are defined as the objects of V which satisfy the condition that there exists a strong mono

$$B \to \prod_{\lambda \in \Lambda} [A_\lambda, M_\lambda] \quad \text{in } V$$

where $M_\lambda \in M$ and Λ is a small set. If M happens to be *small*, then it is equivalent to require that the canonical map

$$B \to \prod_{M \in M} [[B,M],M]$$

be a strong mono in V.

PROPOSITION 5.1. *If each $M \in$ M is Q-reflexive, then the* V*-hull of* M *in*

V *coincides with the* V*-hull of* Q *in* V .

PROOF. Simply consider the strong mono:

$$B \to \prod_{\lambda \in \Lambda} [A_\lambda, M_\lambda] \approx \prod_{\lambda \in \Lambda} [A_\lambda, [[M_\lambda, Q], Q]] \approx$$
$$\approx [\sum_{\lambda \in \Lambda} A_\lambda \otimes [M_\lambda, Q], Q] . \ //$$

Thus we need only consider the V-hull V' of Q in V. This hull is a reflective symmetric monoidal closed subcategory of V (it is closed under exponentiation in V so is symmetric monoidal closed by Day [6]).

Next we consider the V-hull V" of Q in V'. It is straightforward to verify that V' has canonical {epi; strong-mono} factorisations (see Kelly [10]) and that Q is a *strong* V-cogenerator of V". Also, again by Day [6], V" is a symmetric monoidal closed category.

PROPOSITION 5.2. *In* V" *an object* A *is* Q*-reflexive iff* $[A, Q]$ *is* Q*-reflexive.*

PROOF. This follows from the triangle identity:

$$\begin{tikzcd} {[A,Q]} \arrow[r, "\approx"] \arrow[dr, "1"'] & {[[[A,Q],Q],Q]} \arrow[d, "\wr"] \\ & {[A,Q]} \end{tikzcd}$$

and the fact that Q is a strong V-cogenerator. //

Thus if we consider the full subcategory $\mathfrak{R}$ of V" consisting of the Q-reflexive objects, then $\mathfrak{R}^{op} \approx \mathfrak{R}$. This duality extends known dualities for various instances of R and Q, since we already know that $\mathfrak{R}$ contains, for example, all strong projective limits. The category V" also enables us to express the dual of a strong projective limit as a colimit of the duals; more precisely:

PROPOSITION 5.3. *The dual of a* Q*-reflexive limit* $\lim_{\lambda \in \Lambda} A_\lambda$ *of* Q*-reflexive objects* A_λ *is isomorphic to* $\operatorname{colim}_{\lambda \in \Lambda} [A_\lambda, Q]$ *in* V" .

PROOF. We have

$$\lim [[A_\lambda, Q], Q] \approx [\operatorname{colim} [A_\lambda, Q], Q]$$

so the canonical map

$$[[\lim A_\lambda, Q], Q] \to [\operatorname{colim} [A_\lambda, Q], Q]$$

is an isomorphism. The result then follows from the fact that Q strongly V-cogenerates V". //

6. SOME REMARKS ON POINTWISE STRUCTURES.

Given a class M of topological R-modules and a $Q \in$ M which cogenerates M, the ordinary Ens-based hull H of M in V is equipped with a *pointwise* internal-hom (A, B) obtained by topologising H(A, B) as a subspace of $\{UA, B\}$. If we now suppose that $R \approx (Q, Q)$, then H becomes a symmetric monoidal closed category (H, R, $\otimes$, $(-,-)$, ...) (cf. Barr [1]). A Q-reflexive R-module is now taken to mean an R-module $A \in$ H such that $A \approx ((A, Q), Q)$.

PROPOSITION 6.1. *If Q has no small submodules, then $\{X, Q\}$ is Q-reflexive for all $X \in$ Ens. If, in addition, Q is* M*-injective, then (A, Q) is Q-reflexive for all $A \in$ H.*

PROOF. Since Q has no small submodules we have an epi

$$X.(Q, Q) \to (\{X, Q\}, Q).$$

Thus $\{X, Q\}$ is Q-reflexive (see the proof of Theorem 2.3). Secondly, for any $A \in$ H, consider the diagram

$$\begin{tikzcd} UA.(Q,Q) \approx UA.R \arrow[r] \arrow[d] & A \arrow[d] \\ (\{UA, Q\}, Q) \arrow[r, two heads] & ((A,Q),Q) \end{tikzcd}$$

The diagonal is an epi, so the canonical map $A \to ((A, Q), Q)$ is an epi for all $A \in$ H. Thus, by the triangle identity:

$$\begin{tikzcd} (A,Q) \arrow[r] \arrow[dr, "1"'] & (((A,Q),Q),Q) \arrow[d] \\ & (A,Q) \end{tikzcd}$$

we have $(A, Q) \approx (((A, Q), Q), Q)$ as required. //

Next consider $\mathbb{W} \subset \mathrm{H}$; the objects B of $\mathbb{W}$ are those of H which satisfy one of the following equivalent conditions:

(i) There exists an $A \in \mathrm{H}$ and a strong mono $B \to (A, Q)$ in H ;

(ii) The canonical map $B \to ((B, Q), Q)$ is a strong mono in H .

The category $\mathbb{W}$ is epireflective in H and is closed under (pointwise) exponentiation in H . This implies that each $B \in \mathbb{W}$ is Q-reflexive, since each $B \in \mathbb{W}$ admits an equaliser presentation of the form:

$$B \longrightarrow (A, Q) \rightrightarrows (X, Q) .$$

Thus, to summarise, $\mathbb{W}$ is a symmetric monoidal closed category and each object of $\mathbb{W}$ is Q-reflexive, so the tensor product, duality and internal-hom are related by:

$$(A, B) \approx (A \otimes (B, Q), Q) = (A \otimes B^*)^* .$$

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