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IN INTRODUCTION TO THE 3e COLLOQUE SUR LES CATEGORIES, DEDIE A CHARLES EHRESMANN

by Andrée CHARLES EHRESMANN *

I thank you very much for participating to this Colloquium, dedicated to Charles. However, I am somewhat afraid of the responsibility of such a large meeting (more than 70 persons) and I beg your assistance for making this Conference a success both on the categorical and on the human aspects.

1. The first and second «Colloques sur les catégories» were held in Amiens in 73 and 75; Charles and I, we announced the third one for 77; personal problems impeded us from organizing it that year; in 78 and 79, Charles was in too poor health for making its realization possible. So it was postponed up to now.

Charles has always insisted on giving opportunities to young mathematicians; this explains why the audience is very mixed, with beginners among reknown specialists. He also thought it was important for category theory to be linked to other domains (he himself came to categories from differential geometry), so several lectures will be concerned with various applications, as the titles of the parallel afternoon sessions show.

These last years, it was sometimes said that category theory is less vivid than before; Charles always denied it. Since the main schools of categorists are represented in this meeting, the lectures should reflect the actual state of Category theory, some 35 years after Eilenberg and Mac Lane created it; we hope it will not only provide a synthesis of different points of view, but also reveal new trends and give a real impetus to categorical research.

* This is the text of the talk given at the opening session. The numbers between / / refer to the «Liste des Publications de Charles Ehresmann», *Cahiers Topo. et Géom. Diff.* XX-3 (1979), 221-229.

2. This Colloquium is *dedicated to Charles.* I have not said to his memory: for me, Charles is always here and I hope everyone will feel his presence among us, presence revealed by the youthness of his works and the new developments they lead to. Charles has always denied to be rejected in the past, for he looked toward the future and he said it comforted him to think his work would be carried on. So the best way to dedicate this conference to him is to prove his ideas are still actual and stimulating. In fact some of them are probably more up to date now than ever.

In particular his conception of Differential Geometry as the theory of the *differentiable categories of jets and of their actions* /50, 78, 101, 103, 105, 116/, already conceived in the early fifties /32-44/ seems fruitful in Synthetic Differential Geometry. And, in the early sixties (cf. «Œuvres» *) Part III) it led him to internal categories, internal diagrams and their Kan extensions (looked at upside-down), and then general sketched structures /93, 98, 106, 115/ which encompass algebraic and geometric theories (but in a less logical setting), subjects the importance of which is well recognized to-day.

His attempts for axiomatizing the «glueing together» process made him study locales and their associated sheaves in the fifties (cf. /39, 47/ and Section 3); by now, topos theorists have proved every Grothendieck topos may simply be deduced from such a topos of sheaves.

His prevision /94/ that whole classes of functors would be characterized (such as functors of an algebraic type or of a topological type) is better understood now than in 1966, e.g., in the works of the German school (cf. «Oeuvres», Comments Part III-1).

But there are still many of his ideas yet to be exploited. I regret I had not time enough to prepare a lecture about (some of) them. Here are three notions which (in my opinion) deserve to be scrutinized by more recent techniques.

*) «Oeuvres» means «*Charles Ehresmann : Œuvres complètes et commentées*», the four Parts of which (eight volumes) will be published in special volumes of these «*Cahiers*» (Part III, Amiens, 1980).

3. Local categories and structures defined by atlases.

Topological, differentiable, ... manifolds, foliations, fibre bundles are examples of structures defined from more elementary ones by atlases whose changes of charts are in a given pseudogroup. Charles unified these definitions thanks to his theory of local categories (i.e., internal categories in the category of complete distributive meet-lattices) and of *local structures* (cf. /39, 47/ and the papers summarized in the «Guide» /86/).

To get *sheaves over a category* C instead of a topological space, Grothendieck retained the idea of coverings, which he represents as families of morphisms of C with the same codomain E. Charles singled out the open subsets and their order «to be an open subset of», and he equipped C with such a «local» order $\leq$ (the coverings of E are just families of objects lesser than E admitting E as a join). This setting is more general than a Grothendieck topology: Indeed, the «insertion» of a lesser object e into E is not necessarily concretized by a morphism $e \to E$, so that sheaves have to be looked at «upside-down», and they become the *complete local functors over* $(C, \leq)$. It would be interesting to give a Giraud's type theorem to characterize the category of those sheaves over $(C, \leq)$.

In 1957, Charles constructed the *complete enlargement* $\hat{p}: \hat{H} \to C$ *of a local functor* $p: H \to C$, which gives back both the Associated Sheaf Theorem for locales (if H and C are discrete) and the various constructions of structures defined by atlases. $\hat{p}$ is obtained by a two-steps process (cf. /47, 85, 110/):

1° *The enlargement Theorem:* The (local) functor $p: H \to C$ is extended into a (local) functor $\bar{p}: \bar{H} \to C$ with the transport by isomorphisms property, by carrying over the «elementary structures» (which are the objects of H) along isomorphisms of C. This step is akin to a Kan extension, but looked at «upside-down»; it admits several generalizations, e.g. it may be internalized to give Kan extensions of internal diagrams or more general extension theorems for (internal) functors (cf. «Oeuvres», III-2).

2° *The (order-) completion Theorem:* These transported elementary structures (objects of $\bar{H}$) are «glued together»; for this, *maximal descent data* whose images are bounded in C are added to $\bar{H}$, so defining the complete local functor $\hat{p}: \hat{H} \to C$, called the *complete enlargement of* p. A similar construction leads to different (order-) completion theorems for ordered groupoids or categories /68, 76, 85/.

The complete enlargement $\hat{p}: \hat{H} \to C$ may also be constructed /47/ by a one-step method: the objects of $\hat{H}$ are taken as *«complete atlases»* whose charts are p-morphisms and changes of charts are isomorphisms of H.

Notice that a local category is also a special double category C whose horizontal category is an order (cf. «Œuvres», Part III-1). So a generalization of this complete enlargement construction would lead to lax completion theorems for a double functor $P: \mathsf{H} \to \mathsf{C}$, giving a solution to the problem:

How to universally extend P into a double functor $\hat{P}: \hat{\mathsf{H}} \to \mathsf{C}$ creating (a certain kind of C-wise) limits (in the sense of /117, 119/).
In particular completion theorems for functors are found anew when double categories of squares are considered *).

The difficulty is to define complete atlases with non-invertible charts and changes of charts. The «fusées» introduced in /76/ for (order-) completing sub-prelocal categories do not seem adequate enough. One possibility might be to give charts and co-charts (the «curves») as does Frölicher in the case the changes of charts are in a monoid (cf. his paper in this volume). Another one is to take «atlases» whose changes of charts are strings of spans; I'll show elsewhere that all such atlases between functors toward a category C form the universal free completion of C, which admits the category $Pro\,C$ of pro-objects of C as a full sub-category and the universal completion of C as a category of fractions.

4. Germs of categories and of actions.

Locally homogeneous spaces are obtained by glueing together not exactly homogeneous spaces, but topological spaces equipped with a «germ of topological group action». In the same way, germs of topological cat-

*) Herrlich (Proc. Ottawa Conf. 1980) has just given a one-step construction of the initial completion of a concrete functor by a similar method (with complete atlases replaced by complete sources).

egories and germs of actions are obtained by localizing the notion of topological categories (= internal categories in *Top*) and of their actions.

The underlying algebraic structure is a *neocategory* (graph in which some paths of length 2 are assigned a composite with the same ends). Charles introduced them in his study of quotient categories («Oeuvres», Part III-1) and used them for having small presentations of «sketched structures» /93, 106/.

Problems in Analysis prompted us to consider *partial actions of neocategories* (called systèmes de structures in /122/) and their enrichments, and to give extension theorems of (enriched) partial actions into global ones /87/. For instance, Schwartz *distributions* may be constructed as universal solutions to the problem of «making continuous functions differentiable» as follows: the monoid of formal differential operators partially acts on the topological linear space of continuous functions on a bounded open set U ; the globalization of this action gives the space $\mathfrak{D}'_f U$ of finite order distributions on U ; the distributions form the sheaf of complete topological linear spaces associated to the so-defined presheaf $\mathfrak{D}'_f$. This definition is still valid in the infinite-dimensional case. (Cf. «Sur les distributions vectorielles», Caen, 1964.)

Partial internal actions are easily defined and extension theorems are obtained for them («Œuvres», Part III-2). However, these theorems are intricate enough, and it seems worthy to develop a more systematic study of internal partial fibrations.

Germs of categories /92/ are topological neocategories in which the set of composable pairs is open in the pullback of the domain and codomain maps. Examples are provided by neighborhoods of the set of identities in a topological category. Germs of differentiable categories naturally occur in Differential Geometry /78, 101/. Pradines has used them to translate the Lie group structure theorems for Lie groupoids (CRAS Paris 263, 1966 ; 264, 1967 ; 266 et 267, 1968).

Germs of actions or fibrations are similarly defined. They give a generalization of dynamical and semi-dynamical systems. For instance the following example is useful in some control problems: Suppose given a family (E_i) $y' = g_i(y, x)$ of differentiable equations defined on $[t_i', t_i]$, with local existence and unicity of solutions. There exist a germ of category with the reals as objects, in which the arrows from t to t' are the g_i such that $[t', t] \subset [t_i', t_i]$. It partially acts on $B \times R$, where B is the set of possible positions, the composite of $g_i : t \to t'$ and (b, t) being defined as $(z(t'), t')$ iff (E_i) has a unique solution z on $[t', t]$ with $z(t) = b$.

Germs of actions offer a good frame for optimization problems, e. g. for a categorical description of Bellman dynamic programing method (cf. «Systèmes guidables» I-IV, Caen, 1963-5). It would be interesting to study their connections with foliations and their stability problems. They may be adapted to get non-deterministic (germs of) actions (cf. Giry, Proc. Conf. Ottawa 1980), leading to problems similar to those for Automata.

5. Non-abelian cohomology.

In 64, Charles defined the *first-cohomology with coefficients in an indexed category* C. It encompasses e.g. the cohomology of André and (if C is associated to an internal category) the non-abelian cohomology of Lavendhomme-Roisin (Lecture Notes in Math. 753, 1977).

To get *higher order non-abelian cohomology*, he splitted the problem in three parts (cf. «Oeuvres», Part III-2). The ingredients are: a category G, a monoidal concrete category V with an ideal and a V-enriched category A with an ideal J; the problem is to construct the cohomology of an object G of G with coefficients in an object A of A.

1° The first step consists in associating to G its *resolution* $R(G)$, which is a complex of A (= infinite path with its finite composites in J).

2° This complex is carried into the complex $A(R(G), A)$ of V via the enrichment.

3° To this last complex is associated a short «exact sequence» in V for the concrete functor and ideal of V; its target is the *n-th cohomology object* $H^n(G, A)$ (quasi-quotient of the cocycles by the coboundaries).

Abelian cohomology corresponds to A = V = Ab. The general case should be more thoroughly studied, as well as the case /93/ where A is the category of indexed categories and V = Cat, in which the better enrichments known to-day might give interesting results.