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# LOCALLY INJECTIVE G-SHEAVES OF ABELIAN GROUPS

*by Roswitha HARTING*

The problem of the existence of enough injective abelian group objects in an elementary topos with a natural number object leads to the construction of the internal (parametrized) coproduct of abelian group objects [4]. From certain properties of this parametrized coproduct we earlier derived some further consequences [5], among them the surprising result that «all internal notions of injectivity for abelian group objects are equivalent».

In the following summary we shall apply this result to $Shv(X)^{G^{op}}$, the topos of set-valued sheaves on a topological space $X$ with a left action of a group-valued sheaf $G$.

We require the following results and definitions [4, 5] (where $\mathfrak{S}$ denotes an elementary topos with natural number object and $Ab(\mathfrak{S})$ the category of abelian group objects in $\mathfrak{S}$ ).

(0.1) THEOREM AND DEFINITION. *For any object $X$ in $\mathfrak{S}$ the functor $X^*: Ab(\mathfrak{S}) \to Ab(\mathfrak{S}/X)$ has a left adjoint $\oplus_X: Ab(\mathfrak{S}/X) \to Ab(\mathfrak{S})$ which respects monomorphisms and is faithful.*

*For $A(x) \in Ob\, Ab(\mathfrak{S}/X)$ the abelian group object $\oplus_X A(x)$ in $\mathfrak{S}$ is called* parametrized coproduct *of $A(x)$.* (We use «parametrized» to emphasize that the indexing object is in general not just a set but for example a set with an action of a group on it.)

A consequence of this theorem is the following proposition [5]:

(0.2) PROPOSITION. *If $Ab(\mathfrak{S})$ has enough injectives, then so does $Ab(\mathfrak{S}^{\underline{A}})$ for any internal category $\underline{A}$ in $\mathfrak{S}$.*

In the following the internal Hom-functor $Ab(\mathfrak{S})^{op} \times Ab(\mathfrak{S}) \to Ab(\mathfrak{S})$

is denoted by $Hom(-,-)$. For $A$, $B$ abelian group objects in $\mathcal{E}$, $Hom(A, B)$ is the abelian group object in $\mathcal{E}$ that internalizes the abelian group of group-morphisms from $A$ to $B$.

(0.3) DEFINITION. *An abelian group object $B$ in $\mathcal{E}$ is called*

*(i)* internally injective *if for every monomorphism $A \rightarrowtail C$ in $Ab(\mathcal{E})$, $Hom(C, B) \twoheadrightarrow Hom(A, B)$ is an epimorphism in $\mathcal{E}$.*

*(ii)* locally injective *if for every diagram of the form*

$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

*in $Ab(\mathcal{E})$ there exists a cover $U \twoheadrightarrow 1$ of $\mathcal{E}$ and a morphism $g: U^*C \to U^*B$ in $Ab(\mathcal{E}/U)$ such that*

$$\begin{array}{ccc} U^*A & \xrightarrow{U^*m} & U^*C \\ {\scriptstyle U^*f}\downarrow & \swarrow{\scriptstyle g} & \\ U^*B & & \end{array}$$

*commutes.*

(0.4) PROPOSITION. *The following conditions on an abelian group object $B$ in $\mathcal{E}$ are equivalent (for (iii) we suppose that $Ab(\mathcal{E})$ has enough injectives):*

*(i) $B$ is locally injective.*

*(ii) For every diagram of the form*

$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

*in $Ab(\mathcal{E})$ there exist a cover $U \twoheadrightarrow 1$ of $\mathcal{E}$ and a morphism*

$$g: U \to Hom(C, B) \text{ in } \mathcal{E}$$

*such that the diagram*

$$\begin{array}{ccc} Hom(C,B) & \xrightarrow{\;Hom(m,B)\;} & Hom(A,B) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \ulcorner f \urcorner} \\ U & \twoheadrightarrow & 1 \end{array}$$

*commutes in* $\mathcal{E}$.

*(iii) There exists a cover* $U \twoheadrightarrow 1$ *of* $\mathcal{E}$ *such that* $\eta_B : B \rightarrowtail B^U$ *is an injective effacement* [2]. *Here* $\eta : id\, Ab(\mathcal{E}) \to \pi_U . U^*$ *denotes the unit of the adjunction* $U^* \dashv \pi_U$, *and the abelian group object structure of* $B^U$ *is induced by that of* $B$.

(A monomorphism $m: A \rightarrowtail C$ in $Ab(\mathcal{E})$ is an injective effacement iff for every monomorphism $f: A \rightarrowtail B$ in $Ab(\mathcal{E})$ there exists a morphism $g: B \to C$ in $Ab(\mathcal{E})$ such that $g.f = m$.)

(0.5) LEMMA. *An abelian group object* $B$ *in* $\mathcal{E}$ *is internally injective iff for every monomorphism* $m: A \rightarrowtail C$ *in* $Ab(\mathcal{E})$ *and every generalized element* $f: V \to Hom(A,B)$ *in* $\mathcal{E}$ *there exist an object* $U$, *an epimorphism* $h: U \twoheadrightarrow V$ *and a morphism* $g: U \to Hom(C,B)$ *in* $\mathcal{E}$ *such that*

$$\begin{array}{ccc} Hom(C,B) & \xrightarrow{\;Hom(m,B)\;} & Hom(A,B) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f} \\ U & \overset{h}{\twoheadrightarrow} & V \end{array}$$

*commutes.*

(0.6) THEOREM. *An abelian group object* $B$ *in* $\mathcal{E}$ *is locally injective iff* $B$ *is internally injective.*

In the following we shall study the meaning of this result in the topos $\mathcal{E} = Shv(X)^{G^{op}}$, where $X$ denotes a topological space (resp. a locale [7, 9]) and $G$ a group-valued sheaf on $X$. Then $Ab(Shv(X)^{G^{op}})$ is the category of abelian group-valued sheaves on $X$ equipped with a left action of $G$ compatible with the abelian group structure. So $Ab(Shv(X)^{G^{op}})$ is the category of $G$-modules on $X$, and will be denoted from now on by $G\text{-}Mod(X)$.

(1.1) SOME REMARKS.

(i) The following diagram of forgetful functors commutes:

$$\begin{array}{ccc} G\text{-}Mod(X) & \xrightarrow{V''} & Shv(X)^{G^{op}} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle \bar{V}} \\ Ab(Shv(X)) & \xrightarrow{V'} & Shv(X) \end{array}$$

All these forgetful functors create epimorphisms and monomorphisms, they all have a left adjoint, and they respect injectives [10].

(ii) In $Ab(Shv(X))$ the notions of injectivity and internal injectivity coincide [5]. For $A$, $B$ in $Ab(Shv(X))$ the internal Hom is obtained as follows: for $U$ open in $X$, $Hom(A,B)(U)$ is defined to be

$$Hom_{Ab(Shv(U))}(A/U, B/U).$$

(iii) $G$-$Mod(X)$ has enough injectives. (This follows immediately from (0.2).)

(iv) In $G$-$Mod(X)$ the internal Hom is obtained as follows: For $A$, $B$ $G$-modules on $X$, $Hom(A,B)$ is defined to be

$$Hom(VA, VB) \quad \text{in} \quad Ab(Shv(X))$$

equipped with the following action of $G$: for $U$ open in $X$,

$$\begin{array}{ccc} GU \times Hom_{AbShv(U)}(VA/U, VB/U) & \to & Hom_{AbShv(U)}(VA/U, VB/U) \\ (s, h) & \longmapsto & s \circ h \end{array}$$

is defined by:

$$(s \circ h)_W(x) := (s/W).h_W((s^{-1}/W).x),$$

where $W \subset U$, $W$ open in $X$ and $x \in AW$.

(1.2) PROPOSITION. *Let $B$ be a $G$-module on $X$.*

*(i) $B$ is internally injective iff $VB$ is injective in $Ab(Shv(X))$.*

*(ii) If $B$ is internally injective, then $B^G$ is injective.*

PROOF. (i): Suppose $B$ to be internally injective. To show that $B$ is injective in $Ab(Shv(X))$, it is sufficient to show that $B$ is internally injective in $Ab(Shv(X))$ (cf. (1.1)(ii)). So let $m: A \rightarrowtail C$ be a monomorphism in $Ab(Shv(X))$. Let $G$ operate trivially on $A$ and $C$; then $m: A \rightarrowtail C$ becomes a monomorphism in $G\text{-}Mod(X)$. Since $B$ is internally injective it follows that $Hom(C, B) \twoheadrightarrow Hom(A, B)$ is an epimorphism in $Shv(X)^{G^{op}}$, and hence an epimorphism in $Shv(X)$ (cf. (1.1) (i) and (iv)). So $B$ is injective in $Ab(Shv(X))$.

The other implication is equally easy.

(ii): Let

$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & C \\ {\scriptstyle f}\downarrow & & \\ B^G & & \end{array}$$

be a diagram in $G\text{-}Mod(X)$ and suppose $B$ to be internally injective. We have a sequence of natural bijections between the following sets:

$$\begin{array}{ll}
A \to B^G \text{ in } G\text{-}Mod(X) & \\
\hline
G \to V''(Hom(A, B)) \text{ in } Shv(X)^{G^{op}} & \bar{F} \dashv \bar{V},\ \bar{F}(1) = G \\
\hline
1 \to \bar{V}V''(Hom(A, B)) \text{ in } Shv(X) & \bar{V}V'' = V'V,\ (1.1)\,(iv) \\
\hline
1 \to V'(Hom(VA, VB)) \text{ in } Shv(X) & \\
\hline
VA \to VB \text{ in } Ab(Shv(X)). &
\end{array}$$

So $f: A \to B^G$ determines, and is determined by, a morphism $\bar{f}: VA \to VB$ in $Ab(Shv(X))$. $B$ is supposed to be internally injective, so, by (i), $VB$ is injective in $Ab(Shv(X))$. Hence there is a morphism $h$ in $Ab(Shv(X))$ such that

$$\begin{array}{ccc} VA & \overset{Vm}{\rightarrowtail} & VC \\ {\scriptstyle \bar{f}}\downarrow & \swarrow {\scriptstyle h} & \\ VB & & \end{array}$$

commutes. As above $h$ determines a morphism $\hat{h}: C \to B^G$ in $G\text{-}Mod(X)$, and it is easy to verify that $\hat{h}m = f$. So $B^G$ is injective in $G\text{-}Mod(X)$.

(1.3) REMARK. Let $\Delta Z$ be the ring-valued sheaf on $X$ associated to the constant presheaf with value Z. Then $U \mapsto \Delta Z(U)[GU]$ defines an abelian group-valued presheaf with the usual left action of $G$, where we denote by $\Delta Z(U)[GU]$ the group-ring over $GU$. The associated sheaf is a $G$-module on $X$ and is denoted by $Z[G]$. Some obvious calculations show that there is a natural isomorphism

$$Hom(Z[G], A) \approx A^G .$$

In the following the composite

$$A \xrightarrow{\eta_A} A^G \xrightarrow{\approx} Hom(Z[G], A)$$

is again denoted by $\eta_A$ (cf. (0.4) (iii)).

(1.4) PROPOSITION. *Let $B$ be a G-module on $X$. The following conditions on $B$ are equivalent:*

*(i) $Hom(Z[G], B)$ is an injective G-module on $X$.*

*(ii) There exists an epimorphism $D \twoheadrightarrow 1$ in $Shv(X)^{G^{op}}$ such that $B^D$ is an injective G-module on $X$.*

*(iii) $\eta_B : B \rightarrowtail Hom(Z[G], B)$ is an injective effacement in G-Mod(X).*

*(iv) There exists an epimorphism $D \twoheadrightarrow 1$ in $Shv(X)^{G^{op}}$ such that $\eta_B : B \rightarrowtail B^D$ is an injective effacement in G-Mod(X).*

*(v) $B$ is injective in Ab(Shv(X)).*

*(vi) $B$ is internally injective in G-Mod(X).*

*(vii) There exists an open cover of $X$, $X = \bigcup_{i \in I} U_i$, such that $B/U_i$ is injective in $Ab(Shv(U_i))$ for every $i \in I$.*

PROOF. $(iii) \xRightarrow{(1.3)} (iv) \xRightarrow{(1.1)(iii),\ (0.4)(iii),\ (0.6)} (vi) \xRightarrow{(1.2)(ii)} (i) \Longrightarrow (iii)$.

$(ii) \Longrightarrow (iv) \xRightarrow{\text{above}} (i) \xRightarrow{(1.3)} (ii)$.

$(vi) \xLeftrightarrow{(1.2)(i)} (v) \xLeftrightarrow{[6]} (vii)$.

So they are all equivalent.

(1.5) REMARK. If $X$ is reduced to a point, then $Shv(X)^{G^{op}}$ is the topos of $G$-sets, $Ab(Shv(X)^{G^{op}})$ is the category of left $Z[G]$-modules, and $Ab(Shv(X))$ is the category of abelian groups. For two $Z[G]$-modules $A$, $B$ the internal Hom, $Hom(A,B)$, is the abelian group of all Z-linear maps from $A$ to $B$ equipped with the following $G$-action:

$$(s \circ f)(x) = sf(s^{-1}x) \quad \text{for } s \in G,\ f \in Hom_Z(A,B),\ x \in A.$$

$\eta_B \colon B \rightarrowtail Hom(Z[G],B)$ is here defined by

$$\eta_B(b)(s) = b \quad \text{for every } s \in G.$$

We remark that in the above case the Proposition (1.4) remains true if $G$ is replaced by a monoid $M$ and consequently $Hom(Z[G],B)$ by $B^M$ (cf. [3]).

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