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MANFRED B. WISCHNEWSKY

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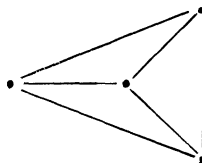
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STRUCTURE FUNCTORS

- Compositions of arbitrary right adjoints with topological functors I -

by Manfred B. WISCHNEWSKY

*Tout cela est fait pour amuser Zévv
dit le Sphinx*).*



INTRODUCTION

In [26] I introduced a new concept for functors which is in fact a class of concepts - called structure functors, resp. more exactly structure functor sequences. Structure functors appear everywhere. Examples are the q-functors in the sense of Ehresmann [6], proclusion functors in the sense of Wyler [29], reflective or coreflective embeddings into categories of sketched structures (A. & C. Ehresmann [7], Gabriel & Ulmer [8]) or into topological categories (Trnkova [24], Wischnewsky [25], Tholen [20]). In [27] I used this concept to characterize completely all full (reflective or) coreflective restrictions of semitopological functors.

In this paper I will use the concept of structure functors in order to characterize all those functors which can be represented as a composition of an arbitrary right adjoint functor with a topological functor. The notion «structure functor» introduced here fills «continuously» the gap between arbitrary topological functors - apparently the best possible type of functor - and arbitrary right adjoint functors. Hence this new notion solves at the same time several long considered open problems. Finally one should mention that the results presented here contain as special instances all fundamental theorems for semitopological functors, given e. g. in Tholen [20] **). In fact more general everything done as yet for semitopological

*) C. Ehresmann, *Catégories et Structures (Prélude)*, Dunod, 1965.

***) The papers [20] and [26] were used as a guideline for this article.

functors has its translation, resp. generalization, for structure functors as it is shown in [28] and several subsequent papers.

0. NOTATIONS.

Let $S: \underline{A} \rightarrow \underline{X}$ be a functor. A *S-cone* is a triple $(X, \psi, D(\underline{A}))$, where X is an \underline{X} -object, $D(\underline{A}): \underline{D} \rightarrow \underline{A}$ is an \underline{A} -diagram (\underline{D} may be void or large) and $\psi: \Delta X \rightarrow SD(\underline{A})$ is a functorial morphism (Δ denotes the «constant» functor into the functor category). We shall abbreviate often $(X, \psi, D(\underline{A}))$ by ψ . $Cone(S)$ denotes the «class» of all *S-cones*. If \underline{D} is the one point category, then ψ is called a *S-morphism* denoted by (A, a) where A is an \underline{A} -object and $a: X \rightarrow SA$ is an \underline{X} -morphism.

The dual notions are *S-cocone* and *S-comorphism*.

The classes of *S-morphisms*, *S-cocones* and *S-comorphisms* are denoted by $Mor(S)$, $Co-cone(S)$, $Co-Mor(S)$. $Epi(S)$ denotes the class of all *S-morphisms* (A, e) with the property:

for all \underline{A} -morphisms $p, q: A \rightarrow B$ the equation $(Sp)e = (Sq)e$ implies $q = p$.

The dual notion is *S-monomorphism*.

1. STRUCTURE FUNCTORS. BASIC DEFINITIONS

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization.

(1.1) DEFINITION (*Semifinal structure functors*).

1) Let $\gamma := (D, \gamma, X)$ be a *Q-cocone*. A (P, Q) -*extension* of γ (or just an extension) consists of a *S-morphism* $(A, f: X \rightarrow SA)$ and a *cocone* $\beta: D \rightarrow \Delta P A$ such that $Q\beta = (\Delta f)\gamma$. Such an extension will be denoted by (A, f, β) .

2) Let (A, f, β) and (A', f', β') be extensions of (D, γ, X) . A *morphism between these extensions* $\alpha: (A, f, \beta) \rightarrow (A', f', \beta')$ is an \underline{A} -morphism $\alpha: A \rightarrow A'$ such that

$$f' = (S\alpha)f \quad \text{and} \quad \beta' = (\Delta P \alpha)\beta .$$

This defines the category of all (P, Q) -extensions of γ .

3) A *semifinal extension* of γ is an initial object in the category of all (P, Q) -extensions of γ .

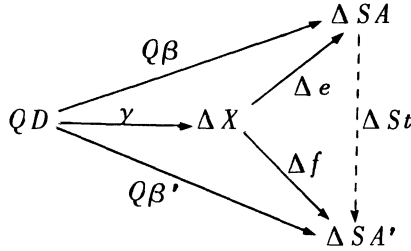


FIGURE 1

4) S is called a *semifinal structure functor**) with respect to (P, Q) or more exactly a *semifinal topologically-right adjoint structure functor*, with respect to (P, Q) if for all diagrams D (\underline{D} may be empty or large) and all Q -cocones (D, γ, X) there exists a semifinal extension.

(1.2) REMARKS. 1) If $P = Id$ and $Q = S$ then the semifinal structure functors (with respect to (Id, S)) are just the semitopological functors.

2) A S -morphism $(A, e: X \rightarrow SA)$ is called a *S-quotient relative to* (P, Q) or just a *S-quotient* (if there is no confusion) if there exist a Q -cocone (D, γ, X) and a cocone $\beta: D \rightarrow \Delta PA$ such that (A, e, β) is a semifinal extension of (D, γ, X) . The class of all S -quotients is denoted by $Quot(S)$.

3) In general $Quot(S)$ does not contain $Iso(S)$, the class of all S -morphisms which are isomorphisms. ($Iso(S) \subset Quot(S)$ if and only if S is semitopological.)

(1.3) DEFINITION (*Semiinitial structure functors*). Let $(D, \phi: \Delta X \rightarrow SD)$ be a S -cone where $D: \underline{D} \rightarrow \underline{A}$ is a diagram in \underline{A} (\underline{D} may be void or large).

1) A *factorization of ϕ* (along S) consists of a S -morphism $(A, e: X \rightarrow SA)$ and an \underline{A} -cone $\alpha: \Delta A \rightarrow D$ with

$$(*) \quad \phi = (S\alpha)(\Delta e).$$

*) Localizable structure functors = multi-structure functors = compositions of arbitrary localizable right adjoints with topological functors are obtained by [26], Appendix A3; details appear somewhere else.

2) The factorization (A, e, α) is called a *semiinitial coextension* (with respect to (P, Q)) of (X, ϕ, D) iff:

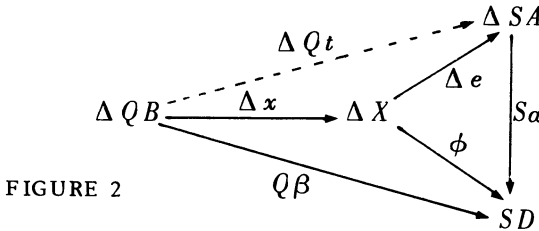
(SI1) For all Q -comorphisms $(B, x: QB \rightarrow X)$, and functorial morphisms $\beta: \Delta B \rightarrow PD$ with $\phi(\Delta x) = Q\beta$ there exists exactly one P -morphism $(A, t: B \rightarrow PA)$ such that

$$Qt = e \cdot x \quad \text{and} \quad \beta = (P\alpha)(\Delta t).$$

(SI2) If $t: A \rightarrow A$ is an \underline{A} -morphism with

$$(St)e = e \quad \text{and} \quad \alpha(\Delta t) = \alpha,$$

then $t = id_A$.



3) A functor S is called a *semiinitial structure functor* relative to (P, Q) , or more exactly a *semiinitial topologically-right adjoint structure functor* relative to (P, Q) , iff for every S -cone (X, ϕ, D) there exists a semiinitial coextension (with respect to (P, Q)).

(1.4) EXAMPLES. The following examples are (semifinal and semiinitial) structure functors:

1) *Topological functors* (take $P = Id$, $Q = S$ and $Quot(S) = Iso(S)$).

2) *Arbitrary right adjoint functors* (take $P = S$, $Q = Id$ and

$$Quot(S) = \text{the class of all units } \eta X: X \rightarrow SA, X \in \underline{X};$$

in particular arbitrary monadic functors.

3) *The composition of structure functors* (let

$$S_1 = Q_1 P_1: \underline{A}_1 \rightarrow \underline{A}_2, \quad S_2 = Q_2 P_2: \underline{A}_2 \rightarrow \underline{X}$$

with $Quot(S_1)$, resp. $Quot(S_2)$, structure functors; then $S_2 S_1 = S$ is a structure functor with respect to $Q = Q_2$, $P = P_2 Q_1 P_1$ and

$$Quot(S) := \{ S_1(e_1) \cdot e_2 \mid e_1 \in Quot(S_1), e_2 \in Quot(S_2) \}$$

and $S_1(e_1)$ and e_2 composable } ;

in particular all functors which can be decomposed into an arbitrary right adjoint and a topological functor.

2. THE DUALITY THEOREM FOR STRUCTURE FUNCTORS

The duality for (topologically-right adjoint) structure functors is a special instance of a much more general duality theorem for structure functor sequences (Wischnewsky [26]). First we need the following

(2.1) LEMMA. *Let*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a semifinal or semiinitial structure functor with respect to (P, Q) . Then Q is faithful with respect to P -morphisms.

PROOF. 1) Let S be a semifinal structure functor with respect to (P, Q) . Let $f, g: B \rightrightarrows PA', A' \in \underline{A}$, be two P -morphisms with $Qf = Qg$ and $f \neq g$. Let I be the class of all \underline{A} -morphisms. Let

$$B_i = B, \gamma_i = Qf: QB_i \rightarrow SA' \text{ for all } i \in I.$$

This defines an I -indexed discrete Q -cocone

$$(B_i, \gamma_i: QB_i \rightarrow QPA')_{i \in I}.$$

Take the (P, Q) -semifinal extension of $(B_i, \gamma_i)_{i \in I}$ denoted by $(A, e, b_i)_{i \in I}$. Let $b'_i := f: B_i \rightarrow PA', i \in I$. Then $Qb'_i = \gamma_i$ for all $i \in I$. Hence there exists a unique morphism $t: A \rightarrow A'$ such that

$$b'_i = (Pt)b_i, i \in I, \text{ and } (St)e = id.$$

This implies that the class

$$J := \{ t \in \underline{A}(A, A') \mid \text{for all } i \in I, (Pt)b_i \in \{f, g\} \}$$

is nonvoid. Hence there exists a surjective mapping $\sigma: I \rightarrow J$. Now define

$$b'_i := \begin{cases} f & \text{if } (P\sigma(i))b_i = g \\ g & \text{if } (P\sigma(i))b_i = f. \end{cases}$$

Then $\gamma_i = Qb'_i$ for all $i \in I$. The universal property delivers an \underline{A} -morphism $t: A \rightarrow A'$ such that $(Pt)b_i = b'_i$ for all $i \in I$. Let $i_0 \in I$ with $\sigma(i_0) = t$.

Then

$$(P\sigma(i_0))b_i = g \Leftrightarrow (P\sigma(i_0))b_i = f.$$

But this is a contradiction. Hence $f = g$.

2) The proof for semiinitial structure functors is done in the same way.

(2.2) COROLLARY (Tholen [20]). *Every semitopological functor is faithful.*

(2.3) THEOREM (Duality for structure functors - Wischnewsky [26]). *Notation as in Section 1. For a factorization*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

there are equivalent:

(i) S is a semiinitial structure functor (for (P, Q)).

(ii) S is a semifinal structure functor (for (P, Q)).

PROOF (Outline [26]). (i) \rightarrow (ii) Let $(D, \phi: QD \rightarrow \Delta X)$ be a Q -cocone. Let C be the full subcategory of the comma-category $(X \downarrow S)$ consisting of all those $A, x: X \rightarrow SA$ for which there exists a cone $\beta: D \rightarrow \Delta PA$ with $Q\beta = (\Delta x)\phi$. β is uniquely determined by (A, x) (Lemma 2.1). Let

$$C: \underline{C} \rightarrow \underline{A}: (A, x) \rightarrow A, \quad \psi: \Delta X \rightarrow SC: \underline{C} \rightarrow \underline{X}: \psi(A, x) := x: X \rightarrow SA, \\ \beta_d(A, x) := \beta(d): QDd \rightarrow SA \quad \text{for } d \in \text{Ob } \underline{D}.$$

Let

$$\psi = (S\alpha)(\Delta e): \Delta X \rightarrow \Delta SA \rightarrow SC$$

be a semiinitial factorization. Since $Q\beta_d = \psi(\Delta\phi(d))$ there exists a unique $\gamma(d): Dd \rightarrow PA$. This defines a functorial morphism $\gamma: D \rightarrow \Delta PA$. Then $Q\gamma = (\Delta e)\phi$ is a semifinal extension for ϕ .

(ii) \rightarrow (i) is proved in a similar way.

(2.4) COROLLARY (Duality for semitopological functors - Tholen [20]). *The following assertions are equivalent for a functor $S: \underline{A} \rightarrow \underline{X}$:*

(i) S is a semifinal functor.

(ii) S is a semiinitial functor.

3. THE LOCALLY-ORTHOGONAL, RESP. THE LEFT-EXTENSION CHARACTERIZATION OF STRUCTURE FUNCTORS

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization of S and $\Pi \subset \text{Mor}(S)$ be a class of S -morphisms.

(3.1) DEFINITION. S is called a Π -locally-orthogonal structure functor with respect to (P, Q) iff every S -cone $(D, \phi: \Delta X \rightarrow SD)$ where $D: \underline{D} \rightarrow \underline{A}$ may be void or large has a factorization

$$\phi = (S\alpha)(\Delta e^*) \text{ where } (A^*, e^*: X \rightarrow SA^*) \in \Pi$$

and $\alpha: \Delta A^* \rightarrow D$ is an \underline{A} -cone such that for every

$$\begin{aligned} x: Y \rightarrow X \text{ in } \underline{X}, (A, e: Y \rightarrow SA) \in \Pi \text{ and} \\ \beta: \Delta P A \rightarrow P D \text{ with } (Q\beta)(\Delta e) = \phi(\Delta x) \end{aligned}$$

there exists exactly one \underline{A} -morphism $t: A \rightarrow A^*$ such that

$$(\Delta e^*)(\Delta x) = (\Delta S t)(\Delta e) \text{ and } \beta = (P\alpha)(\Delta P t).$$

The factorization (A^*, e^*, α) itself is called Π -locally orthogonal.

(3.2) DEFINITION. Let $D(\underline{X}): \underline{D} \rightarrow \underline{X}$ and $D(\underline{A}): \underline{D} \rightarrow \underline{A}$ be diagrams (\underline{D} may be void or large) and $\phi: D(\underline{X}) \rightarrow \Delta X$ and $\pi: D(\underline{X}) \rightarrow SD(\underline{A})$ be functorial morphisms with

$$(D(\underline{A})d, \pi d: D(\underline{X})d \rightarrow SD(\underline{A})d) \in \Pi \text{ for all } d \in \underline{D}.$$

A Π -left-extension of $(D(\underline{A}), \pi, \phi)$ consists of an S -morphism

$$(A^*, e^*: X \rightarrow SA^*) \in \Pi$$

and a functorial morphism

$$\alpha: D(\underline{A}) \rightarrow \Delta A^* \text{ with } (S\alpha)\pi = (\Delta e^*)\phi$$

such that for all S -morphisms $(A, x: X \rightarrow SA)$ and cones

$$\beta: P D(\underline{A}) \rightarrow \Delta P A \text{ with } (Q\beta)\pi = (\Delta x)\phi$$

there exists exactly one \underline{A} -morphism $t: A^* \rightarrow A$ with

$$(S t)e^* = x \text{ and } (\Delta P t)(P\alpha) = \beta.$$

S is called a Π -left-extension structure functor (for (P, Q)) iff, for all «double cones» $(D(\underline{A}), \pi, \phi)$ there exists a Π -left-extension.

Applying again the generalized duality Theorem in [26] we obtain the following

(3.3) THEOREM (*Duality theorem for left-extension structure functors*). Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization and $\Pi \subset \text{Mor}(S)$ a class of S -morphisms. Then the following assertions are equivalent:

- (i) S is a Π -locally orthogonal structure functor (for (P, Q)).
- (ii) S is a Π -left-extension structure functor (for (P, Q)).

In the following we will always assume the following situation:

- (1) The factorizations

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

in question have the property that P has a left-adjoint with unit η .

- (2) The subclasses $\Pi \subset \text{Mor}(S)$ in question are assumed to contain all morphisms $Q\eta(B)$, $B \in \underline{B}$.¹⁾

(3.4) THEOREM. Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization. Then the following assertions are equivalent:

- (i) S is a structure functor (for (P, Q)).
- (ii) S is a $\text{Quot}(S)$ -left-extension functor for (P, Q) .
- (iii) S is a Π -left-extension functor for $\Pi \subset \text{Mor}(S)$.
- (iv) S is a Π -left extension functor for $\Pi \subset \text{Epi}(S)$.

PROOF. (i) \rightarrow (ii). Let

$$(D(\underline{A}), SD(\underline{A}) \xleftarrow{\pi} D(\underline{X}) \xrightarrow{\phi} \Delta X)$$

be a double cone with π being pointwise in Π where $D(\underline{A}): \underline{D} \rightarrow \underline{A}$ and $D(\underline{X}): \underline{D} \rightarrow \underline{X}$. Since $\pi d \in \Pi = \text{Quot}(S)$ for all $d \in \underline{D}$ there exist

1) If S is a structure functor, then P has a left adjoint and $\Pi = \text{Quot}(S)$ always fulfills condition (2).

$$D_d: \underline{D} \rightarrow \underline{B}, \beta_d: D_d \rightarrow \Delta P D(\underline{A})d \text{ and } \phi_d: Q D_d \rightarrow \Delta D(\underline{X})d$$

defining a semifinal extension with $Q\beta_d = (\Delta\pi d)\phi d$. The Q -cocone $Q D_d \rightarrow \Delta X$ has a semifinal extension

$$(A_d, e_d: X \rightarrow SA_d), \alpha_d: D_d \rightarrow \Delta P A .$$

This defines a discrete S -cone $(e_d: \Delta X \rightarrow SA_d)_{d \in D}$. The semifinal factorization delivers a $Quot(S)$ -morphism $(A, e: X \rightarrow SA)$, and an \underline{A} -cone $\alpha: D(\underline{A}) \rightarrow \Delta A$ being the $Quot(S)$ -left-extension looked for. The other implications are left to the reader.

4. THE REPRESENTATION THEOREM FOR STRUCTURE FUNCTORS

(4.1) *The « Canonical Factorization ».* Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a structure functor. I will construct a factorization of S :

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

such that

- (i) $R: \underline{A} \rightarrow \underline{C}$ is a right-adjoint functor, and
- (ii) $T: \underline{C} \rightarrow \underline{X}$ is a topological functor.

The construction is given by the following datas:

- (1) The objects of \underline{C} are all S -morphisms $(A, e: X \rightarrow SA)$ in $Quot(S)$.
- (2) A morphism $(x, f): (A, e) \rightarrow (A', e')$ in \underline{C} is a pair consisting of an \underline{X} -morphism $x: X \rightarrow X'$ and an \underline{A} -morphism $f: A \rightarrow A'$ such that

$$e'x = (Sf)e.$$

- (3) The functor $T: \underline{C} \rightarrow \underline{X}$ is given by the assignments:

$$(A, e: X \rightarrow SA) \mapsto X, (x, f) \mapsto x.$$

- (4) The functor $R: \underline{A} \rightarrow \underline{C}$ is given by the assignments

$A \mapsto (A^*, e^*: SA \rightarrow SA^*)$, where $e^*: SA \rightarrow SA^*$ is the S -quotient of the identity $id: SA \rightarrow SA$,

$$\begin{array}{ccccc}
 A & & SA & \xrightarrow{e^*} & SA^* & & A^* \\
 f \downarrow & \mapsto & Sf \downarrow & & \downarrow Sf^* & & \downarrow f^* \\
 A' & & SA' & \xrightarrow{e'^*} & SA'^* & & A'^*
 \end{array}$$

FIGURE 3

where $f^*: A^* \rightarrow A'^*$ is uniquely determined by the semifinal property of (A^*, e^*) .

(4.2) THEOREM (*Representation Theorem for structure functors*). Let $S: \underline{A} \rightarrow \underline{X}$ be a functor. Then the following assertions are equivalent:

(i) S is a structure functor (with respect to a factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X}).$$

(ii) There exists a factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

where $R =$ right adjoint functor and $T =$ topological functor.

(iii) The canonical factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

fulfills (ii).

PROOF. (i) \rightarrow (iii) Take the factorization $S = TR$ in (4.1).

1) R has a left adjoint $L: \underline{C} \rightarrow \underline{A}$. L is given by the assignments

$$L: \underline{C} \rightarrow \underline{A}: (A, e) \mapsto A, (x, f) \mapsto f.$$

Let (A, e) be a (P, Q) -quotient. Then there exists a P -morphism $(A^*, b: PA \rightarrow PA^*)$ with

$$RL(A, e) = RA = (A^*, Qb: SA \rightarrow SA^*).$$

The semifinal property of (A, e) induces a unique \underline{A} -morphism

$$a: A \rightarrow A^* \text{ such that } (Qb)e = (Sa)e.$$

Now the \underline{C} -morphism $(e, a): (A, e) \rightarrow (A^*, Qb)$ is universal with respect to R . Let $A' \in \underline{A}$ and $(x, f): (A, e) \rightarrow RA'$ be a \underline{C} -morphism. The universal property of $RA' = (A'^*, Qb')$ implies an \underline{A} -morphism $a^*: A'^* \rightarrow A'$ such that $(Sa^*)(Qb') = id$. Then $a^*f: L(A, e) = A \rightarrow A'$ is the unique morphism with $(R(a^*f))(e, a) = (x, f)$.

2) T is topological. Let $D: \underline{D} \rightarrow \underline{C}$ be a diagram and $\phi: TD \rightarrow \Delta X$ be a T -cone over $X \in \underline{X}$. Let

$$D(\underline{X}) := TD: \underline{D} \rightarrow \underline{X}, \text{ resp. } D(\underline{A}) := LD: \underline{D} \rightarrow \underline{A}.$$

Then

$$\pi d := e: D(\underline{X})d = TDd = T(A, e: X \rightarrow SA) \rightarrow SD(\underline{A})d = SA$$

defines a functorial morphism $\pi: D(\underline{X}) \rightarrow SD(\underline{A})$ being pointwise in $Quot(S)$. Hence there exist a functorial morphism $\alpha: D(\underline{A}) \rightarrow \Delta A^*$ and $(A, e^*: X \rightarrow SA^*) \in Quot(S)$ forming a $Quot(S)$ -left-extension of the double cone (π, ϕ) . Then $(\alpha, \phi): D \rightarrow \Delta(A, e)$ is the final structure induced by ϕ .

(ii) \rightarrow (i) follows immediately from (1.4). (iii) \rightarrow (ii) is trivial.

If $P = Id$ then obviously R is a full embedding. Hence we obtain the important

(4.3) COROLLARY (*Tholen & Wischnewsky [20, 25], Representation Theorem for semitopological functors*). *Let $S: \underline{A} \rightarrow \underline{X}$ be a functor. Then there are equivalent:*

- (i) *S is semitopological.*
- (ii) *S is a full reflective restriction of a topological functor.*

By dualizing we obtain

(4.4) THEOREM (*Representation Theorem for co-structure functors*). *Let $S: \underline{A} \rightarrow \underline{X}$ be a functor. Then there are equivalent:*

- (i) *S is a co-structure functor.*
- (ii) *There exists a factorization*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{L} \underline{C} \xrightarrow{T} \underline{X})$$

where $L =$ left adjoint functor and $T =$ topological functor.

- (iii) *The « dual » canonical factorization of S fulfills (ii).*

5. CHARACTERIZATION AND EXISTENCE THEOREMS FOR STRUCTURE FUNCTORS

The results of this paragraph show that a functor is a structure functor iff certain colimits or limits exist. This is in some sense astonishing for the special instance of monadic functors since Adamek gave an example of a non-cocomplete Eilenberg-Moore category over a cocomplete base category [1]. The theorems here show that nevertheless certain (even large) colimits do exist in every Eilenberg-Moore category. The charac-

terization of structure functors by certain «limits» was not known even in the case of semitopological functors. Finally one should mention that these results here give new formal existence criteria for left-adjoints.

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization of the functor S and $\Pi \subset \text{Mor}(S)$.

(5.1) DEFINITION. 1) Given

$$(A, SA \xleftarrow{e} Y \xrightarrow{x} X) \text{ with } (A, e: Y \rightarrow SA) \in \Pi,$$

a Π -left-extension of (A, e, x) is called a Π -semi-pushout of (A, e, x) (relative to (P, Q)).

2) Given a discrete functorial morphism

$$(A_i, e_i: X \rightarrow SA_i)_{i \in I}$$

being pointwise in Π , a Π -left-extension of $(A_i, e_i)_{i \in I}$ is called a Π -multiple pushout of $(A_i, e_i)_{i \in I}$ (relative to (P, Q)).

3) S is called Π -cocomplete (with respect to (P, Q)) if every chain (A, e, x) (as in 1) has a Π -semi-pushout and if every discrete functorial chain $(A_i, e_i)_{i \in I}$ being pointwise in Π has a Π -multiple pushout.

(5.2) DEFINITION. S is called Π -locally orthogonal complete (or just Π -complete if there is no confusion) if:

1) Every S -morphism $(A, x: X \rightarrow SA)$ has a Π -locally orthogonal factorization, and

2) Every discrete S -cone $(A_i, e_i: X \rightarrow SA_i)_{i \in I}$ being pointwise in Π has a Π -locally orthogonal factorization.

(5.3) LEMMA. If S is Π -complete or Π -cocomplete, then $\Pi \subset \text{Epi}(S)$.

PROOF. This is proved in the usual way with the Cantor's diagonalization principle [4].

(5.4) THEOREM (Characterization and existence Theorem for structure functors). Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization and $\Pi \subset \text{Mor}(S)$. The following assertions are equivalent:

- (i) S is a structure functor.
- (ii) S is Π -complete.
- (iii) S is Π -cocomplete.

PROOF. From the results in Section 3 follows that we have only to prove the implications (ii) \rightarrow (i) and (iii) \rightarrow (i).

1) (ii) \rightarrow (i). Let

$$D(\underline{A}): \underline{D} \rightarrow \underline{A}, \quad D(\underline{X}): \underline{D} \rightarrow \underline{X}, \quad \phi: D(\underline{X}) \rightarrow \Delta X \quad \text{and} \quad \pi: D(\underline{X}) \rightarrow SD(\underline{A}),$$

where π is pointwise in Π , be given. Consider the class of all factorizations of the double cone (π, ϕ) , i. e., all

$$(A_i, e_i: X \rightarrow SA_i) \in \Pi \quad \text{and} \quad \alpha_i: D \rightarrow \Delta A_i \quad \text{with} \quad (S\alpha_i)\pi = (\Delta e_i)\phi.$$

The Π -locally orthogonal factorization of the cone $(A_i, e_i: X \rightarrow SA_i)_{i \in I}$ delivers a Π -object $(A^*, e^*: X \rightarrow SA^*)$ and \underline{A} -morphisms $\alpha_i: A^* \rightarrow A_i$ with $e_i = (S\alpha_i)e^*$. Hence we obtain a unique functorial morphism

$$\beta: D \rightarrow \Delta A^* \quad \text{with} \quad (S\beta)\pi = (\Delta e^*)\phi.$$

Let now $(A, x: X \rightarrow SA)$ be an arbitrary S -morphism and $\gamma: PD(\underline{A}) \rightarrow \Delta PA$ be a functorial morphism with $(\Delta x)\phi = (Q\gamma)\pi$. Let $x = (Sf)e$ be the Π -locally orthogonal factorization of x where $(A', e: X \rightarrow SA') \in \Pi$. This induces a functorial morphism

$$\delta: D(\underline{A}) \rightarrow \Delta A' \quad \text{with} \quad (S\delta)\pi = (\Delta e')\phi.$$

Hence there exists an index i with

$$e' = e_i \quad \text{and} \quad \alpha_i = \delta.$$

Then $t = f \cdot \alpha_i: A^* \rightarrow A$ is the unique morphism looked for. Hence S is a Π -left extension structure functor and thus a structure functor.

2) (iii) \rightarrow (i). This is proved in an analogous way. One starts with the class of all factorizations

$$\phi = (S\alpha_i)\Delta e_i \quad \text{with} \quad (A_i, e_i: X \rightarrow SA) \in \Pi \quad \text{and} \quad \alpha_i: \Delta A_i \rightarrow D,$$

$D: \underline{D} \rightarrow \underline{A}$, of a given functorial morphism $\phi: \Delta X \rightarrow SD$. Then take the Π -multiple pushout of $(A_i, e_i: X \rightarrow SA)$. This induces a factorization of

$\phi = (S\alpha)(\Delta e)$ which turns out to be Π -locally orthogonal.

Theorem (5.4) has several important corollaries of which I will state here just two of them.

(5.5) COROLLARY. *Let $S: \underline{A} \rightarrow \underline{X}$ be a functor. Then there are equivalent:*

- (i) *S is semitopological.*
- (ii) *S is Π -complete (for (Id, S)).*
- (iii) *S is Π -cocomplete (for (Id, S)).*

The equivalence (i) \Leftrightarrow (iii) was first proved by Tholen [20]. The equivalence (i) \Leftrightarrow (ii) is new.

(5.6) COROLLARY (Tholen [20]). *The following assertions are equivalent for a category \underline{A} :*

- (i) *\underline{A} is a locally orthogonal $(\Pi, \text{Mono-Cone}(\underline{A}))$ -category.*
- (ii) a) *\underline{A} is Π -cocomplete or Π -complete.*
 b) *\underline{A} has coequalizers.*
 c) *Π is closed under composition with extremal epimorphisms from the left.*

If Π is closed under composition, then \underline{A} is an orthogonal $(\Pi, \text{Mono-Cone}(\underline{A}))$ -category.

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F. B. Mathematik
 Universität Bremen
 Universitätsallee
 D-2800 BREMEN 33. R. F. A.