CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 22, nº 3 (1981), p. 329-336

<http://www.numdam.org/item?id=CTGDC_1981__22_3_329_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE
Vol. XXII-3 (1981)

3e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Amiens, Juillet 1980

ON THE PRESERVATION OF HOMOTOPY INVARIANCE BY KAN EXTENSIONS

by Armin FREI

1. INTRODUCTION.

In the literature (e.g. in [1, 2, 3, 4, 6, 7]) one considers Kan extensions of functors (usually homology or cohomology functors) defined on a category $\tilde{\mathcal{T}}_0$ of spaces and homotopy classes of maps to another such category $\tilde{\mathcal{T}}_1$, containing $\tilde{\mathcal{T}}_0$ as a full subcategory. The following question arises: (in rough terms) does it matter whether one takes the Kan extension at the level of homotopy classes or at the level of maps? In order to lay out the situation we consider the commutative diagram:

(1.1)
$$\begin{array}{ccccc} \mathcal{T}_0 & & \xrightarrow{\quad E \quad} & & \mathcal{T}_1 \\ {\scriptstyle H_0}\downarrow & \searrow^{F} & & & \downarrow{\scriptstyle H_1} \\ & & \mathcal{K} & & \\ & \nearrow_{\tilde{F}} & & & \\ \tilde{\mathcal{T}}_0 & & \xrightarrow[\quad \tilde{E} \quad]{} & & \tilde{\mathcal{T}}_1 \end{array}$$

where $\mathcal{T}_0$ and $\mathcal{T}_1$ are full subcategories of the category of spaces and continuous maps, E is a full embedding and $\tilde{\mathcal{T}}_0$, $\tilde{\mathcal{T}}_1$, $\tilde{E}$ are their quotients modulo homotopy. H_1 and H_0 are the quotient functors and F is any homotopy invariant functor. Throughout the paper we use the notations G^K, ${}^K G$ for the right respectively the left Kan extension of G along K. As we take Kan extensions along full embeddings we may, and do, always choose them so that the universal transformation is the identity. Thus, in more precise terms, our question is:

When is $\tilde{F}^{\tilde{E}} H_1 = F^E$?

It turns out that this is the case precisely when F^E exists and is homotopy invariant (Theorem 3.4). By duality the same holds when F^E, $\tilde{F}^{\tilde{E}}$ are replaced by the left Kan extensions ${}^E F$, ${}^{\tilde{E}}\tilde{F}$, or when F is a cofunctor. In view of this we investigate when the Kan extension of a homotopy invariant functor is homotopy invariant.

We shall always refer to the situation of diagram (1.1). We assume that $\mathcal{K}$ is sufficiently complete or cocomplete in order to admit the limits or colimits defining the pointwise right or left Kan extensions which we consider.

In Section 2 we treat the case where $\mathcal{T}_0$ contains the cofibration $X \to Zf$ for any map $f: X \to EY$ in $\mathcal{T}_1$ (Zf denotes the mapping cylinder of f) and its dual, that is, the case where $\mathcal{T}_0$ contains the mapping path fibration $Y^I \to X$ for any map $EY \to X$. In the first case F^E is homotopy invariant and in the second case ${}^EF$ is. Both conditions are clearly satisfied when $\mathcal{T}_0$ contains entire homotopy types, as is the case for categories which are admissible in the sense of [1, 6]. Section 3 contains some categorical facts about Kan extensions, which shall be used in the sequel. Although what is said in this section applies to more general situations, for the sake of shortness and ease of reading, we adhere to the situation of diagram (1.1). In Section 4 we show that if $\mathcal{T}_0$ contains Y^I whenever it contains Y, then F^E is homotopy invariant, and, dually, if it contains $Y \times I$ together with Y, then ${}^EF$ is homotopy invariant. The condition that $\mathcal{T}_0$ contain $Y \times I$ along with Y is rather harmless when $\mathcal{T}_0$ is a category of polyhedra or of CW-complexes as $-\times I$ preserves them and also preserves finiteness, local finiteness and finite dimension, thus left Kan extensions of homotopy invariant functors from any reasonable category of polyhedra or CW-complexes are homotopy invariant. Unfortunately the dual is not adequate for such categories; we shall consider right Kan extensions from categories of polyhedra in another paper.

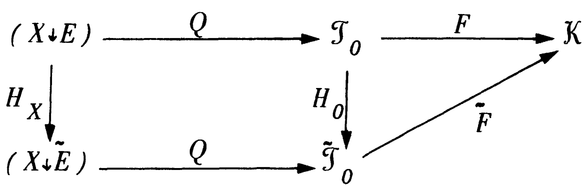
2. THE CASE WHERE $\mathcal{T}_0$ CONTAINS Zf FOR ALL $f: X \to EY$.

THEOREM 2.1. *If the category $\mathcal{T}_0$ contains the mapping cylinder Zf for any map $f: X \to EY$ in $\mathcal{T}_1$, then:*

(i) F^E exists iff $\bar{\bar{F}}^E$ exists.

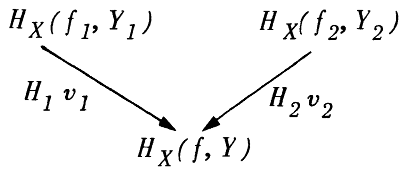
(ii) $\bar{\bar{F}}^E H_1 = F^E$, in particular F^E is homotopy invariant.

PROOF. Let $H_X : (X{\downarrow}E) \to (X{\downarrow}\tilde{E})$ be the functor which takes the object (f, Y) of $(X{\downarrow}E)$, that is a map $f: X \to EY$, to its homotopy class $H_1 f$ and a map g of $(X{\downarrow}E)$ to its homotopy class $H_1 g$. Then the diagram

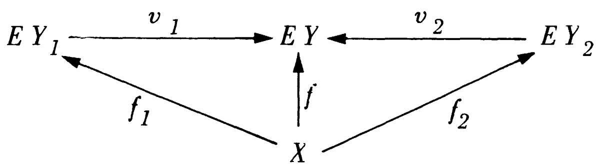


where Q denotes the usual forgetful functor, is commutative. Hence it suffices to show that the functor H_X is initial for all X in $|\mathfrak{T}_1|$. Now, every object in $(X{\downarrow}\tilde{E})$ is clearly of the form $H_X(f, Y)$ and the category $\mathfrak{X} = (H_X{\downarrow}H_X(f, Y))$ is nonempty. In order to show that it is connected we consider two objects

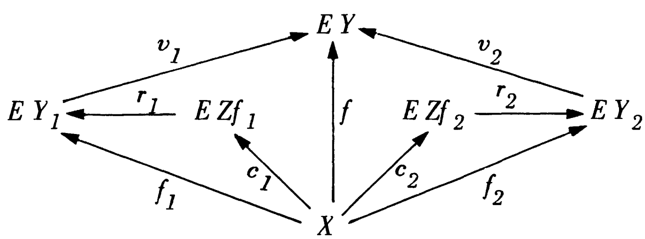
(1.2)



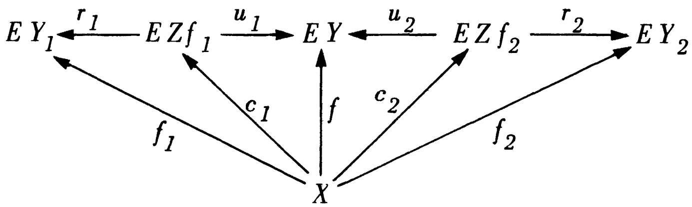
in $\mathfrak{X}$, i.e., a homotopy commutative diagram



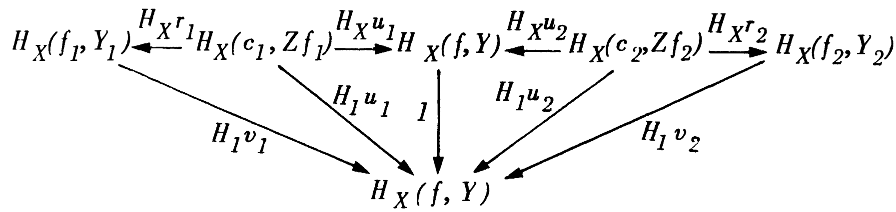
in $\mathfrak{T}_1$. We replace f_1, f_2 by cofibrations c_1, c_2 resp. and obtain a diagram



where r_1 and r_2 are homotopy equivalences and where $v_i r_i c_i \simeq f$. By the homotopy extension property there exist maps $u_i : Zf_i \to Y_i$ with $u_i c_i = f$. Thus we obtain a strictly commutative diagram in $\mathfrak{T}_1$



which gives rise to a commutative diagram in $\mathfrak{X}$:



that is, to a connection between the objects (1.2).

A dual procedure, replacing the appropriate maps by their mapping path fibrations, gives the proof of:

THEOREM 2.2. *If the category $\mathcal{T}_0$ contains the mapping path fibration for any map $EY \to X$ in $\mathcal{T}_1$, then:*

(i) ${}^E F$ exists iff ${}^{\bar{E}}\bar{F}$ exists.

(ii) ${}^{\bar{E}}\bar{F} H_1 = {}^E F$, in particular ${}^E F$ is homotopy invariant.

When $\mathcal{T}_0$ contains entire homotopy types, then the hypotheses of both theorems are satisfied and we immediately have

COROLLARY 2.3. *If $\mathcal{T}_0$ is closed under homotopy types, then the conclusions of both Theorems, 2.1 and 2.2, hold.*

3. SOME CATEGORICAL FACTS.

As the Kan extension of F along a fully faithful functor E can be chosen so that $F^E E = F$, there is a very explicit form of the limiting cone defining $F^E X$.

PROPOSITION 3.1. *For any object X in $\mathcal{T}_1$ the family*

$$\{ F^E f \mid (f, Y) \in |(X{\downarrow}E)| \}$$

is a universal cone.

PROOF. Let X be an object in $\mathcal{T}_1$ and let

$$\{\eta(g,Z) \colon F^E X \to FQ(g,Z)\},$$

where $Q \colon (X \downarrow E) \to \mathcal{T}_1$ is the forgetful functor, be a universal cone. For a map $f \colon X \to EY$, the map $F^E f$ renders the diagram

$$\begin{array}{ccc} F^E X & \xrightarrow{\quad F^E f \quad} & F^E E Y = FY \\ & {\scriptstyle \eta(hf,Z)} \searrow \quad \swarrow {\scriptstyle \lambda(h,Z)} & \\ & FZ & \end{array}$$

commutative for any (h,Z) in $(EY \downarrow E)$, where λ denotes the universal cone defining $F^E EY$. In particular for $(1, Y)$ we have that $\lambda(1, Y) = 1$, thus $\eta(f, Y) = F^E f$.

The proposition above leads to a criterion for F^E to be homotopy invariant. Thus

PROPOSITION 3.2. *If F^E takes homotopic maps $f, g \colon X \to EY$ to the same map, then F^E is homotopy invariant.*

PROOF. Let $h, i \colon X_1 \to X_2$ be homotopic maps in $\mathcal{T}_1$. Using the representation given in Proposition 3.1 of the universal cone defining F^E, the map $F^E h$ is the unique one for which $F^E t . F^E h = F^E th$, while $F^E i$ is the unique one for which $F^E t . F^E i = F^E ti$, for all objects (t, Y) in $(X_2 \downarrow E)$. As $th, ti \colon X_1 \to EY$ are homotopic, one has, by hypothesis, that

$$F^E th = F^E ti, \text{ hence } F^E h = F^E i$$

by universality.

The next proposition points out a nice feature of quotient functors.

PROPOSITION 3.3. *If $G \colon \tilde{\mathcal{T}}_1 \to \mathcal{K}$ is any functor, then G is the right and the left pointwise Kan extension of GH_1 along H_1, with the identity as universal transformation. The same holds, of course, for H_0.*

PROOF. As H_1 is full and onto objects it induces a bijection

$$Nat[G, G'] \to Nat[GH_1, G'H_1]$$

for all functors $G, G' \colon \tilde{\mathcal{T}}_1 \to \mathcal{K}$. Hence G is not only a pointwise, but even an absolute Kan extension.

We now proceed to show that precisely when F^E is homotopy invariant, Kan extending at the map or at the homotopy class level give the same. More precisely:

THEOREM 3.4. *In the situation of diagram (1.1) the following are equivalent:*

(i) F^E *exists and is homotopy invariant.*

(ii) $\tilde{F}^{\tilde{E}}$ *exists and* $\tilde{F}^{\tilde{E}} H_1 = F^E$.

PROOF. (ii) $\Rightarrow$ (i) is trivial. (i) $\Rightarrow$ (ii): As F^E is homotopy invariant there is a unique functor $\bar{F}: \tilde{\mathcal{T}}_1 \to \mathcal{K}$ with $\bar{F} H_1 = F^E$. Then

$$\begin{aligned} \bar{F} &= (F^E)^{H_1} && \text{by Proposition 3.3,} \\ &= F^{H_1 E} && \text{by Lemma 1.2 (i) of [5],} \\ &= \tilde{F}^{\tilde{E} H_0} && \text{as (1.1) is commutative.} \end{aligned}$$

By Proposition 3.3, $\tilde{F} = F^{H_0}$ and by Lemma 1.2 (ii) of [5] there is a unique $\mu: \bar{F}\tilde{E} \to \tilde{F}$ with $\mu H_0 = 1$, thus $\mu = 1$ as H_0 is onto objects, and $\bar{F} = \tilde{F}^{\tilde{E}}$.

4. THE CASE WHERE $\mathcal{T}_0$ IS CLOSED UNDER $(-)^I$ OR $-\times I$.

Suppose that for every Y in $\mathcal{T}_0$ also Y^I lies in $\mathcal{T}_0$. Let $f, g: X \to EY$ be homotopic maps. There is a map $k: X \to Y^I$ such that

$$d_0 k = f \quad \text{and} \quad d_1 k = g,$$

where d_0 and d_1 are the evaluations at 0 and 1 respectively. As d_0 and d_1 are homotopic and lie in $\mathcal{T}_0$ one has, for any functor $\bar{F}: \mathcal{T}_1 \to \mathcal{K}$ with $\bar{F}E$ homotopy invariant, that $\bar{F}f = \bar{F}g$. This holds in particular for the right Kan extension F^E of F. Invoking Proposition 3.2 we then have:

THEOREM 4.1. *If* $\mathcal{T}_0$ *contains* X^I *whenever it contains* X, *then* F^E *is homotopy invariant.*

Dually we have

THEOREM 4.2. *If* $\mathcal{T}_0$ *contains* $X \times I$ *whenever it contains* X, *then* ${}^E F$ *is homotopy invariant.*

We conclude by observing that, if $\mathcal{T}_0$ is closed under homotopy types, one obtains the conclusion (ii) of the theorems in Section 2 via the theorems in this section and Theorem 3.4 (and its dual). Nevertheless the theorems in Section 2 lead a separate life from those in Section 4 as one sees by taking $\mathcal{T}_0$ to be the category of polyhedra, or CW-complexes or of compact spaces, which are closed under $\times I$ but not under homotopy types.

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