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## SYNTACTICAL THEORY OF FUNCTORS

by Axel MÖBUS

A logical theory is regarded as a category. This motivates a syntactical definition of a functor between theories. The definition given here covers only those functors in the usual sense that preserve monomorphisms. As a compensation we need less data to describe a functor.

### 1. FUNCTORS.

If  $L$  is a language and  $\Gamma$  is a theory in  $L$ , then  $(L, \Gamma)$  is a category in the following sense [MR, Ch. 8]:

The objects of  $(L, \Gamma)$  are equivalence classes of formulae  $\phi(\vec{x})$ , the morphisms are equivalence classes of functional formulae, i. e. formulae

$$\alpha(\vec{x}, \vec{y}): \phi(\vec{x}) \rightarrow \psi(\vec{y}),$$

such that

$$\Gamma \vdash \alpha(\vec{x}, \vec{y}) \Rightarrow \phi(\vec{x}) \wedge \psi(\vec{y}) \quad (\text{F1})$$

$$\Gamma \vdash \alpha(\vec{x}, \vec{y}) \wedge \alpha(\vec{x}, \vec{y}') \Rightarrow \vec{y} = \vec{y}' \quad (\text{F2})$$

$$\Gamma \vdash \phi(\vec{x}) \Rightarrow \exists \vec{y}: \alpha(\vec{x}, \vec{y}) \quad (\text{F3})$$

(If the ranges of the sequences  $\vec{x}, \vec{y}$  are not disjoint, the definition has to be modified accordingly.)

A functor between theories  $(L, \Gamma) \rightarrow (L', \Gamma')$  will be given by its operation on formulae and the values it takes at «projections». So a functor  $(T, \pi): (L, \Gamma) \rightarrow (L', \Gamma')$  consists of:

i) an assignment: to each formula  $\psi(\vec{x})$  in  $L$  we assign a formula  $\{T(\psi(\vec{x}))\}(t_{\vec{x}})$  in one variable in  $L'$ . If  $\text{type}(\vec{x}) = \text{type}(\vec{x}')$ , then  $\text{type}(t_{\vec{x}}) = \text{type}(t_{\vec{x}'})$  is required. (The sequence  $\vec{x}$  may be the empty sequence. In this case  $\vec{x} = \vec{x}' \equiv \text{true}$ .)

ii) a family of formulae  $\pi_{\vec{x}}^{\vec{x}\vec{y}}(t_{\vec{x}\vec{y}}, t_{\vec{x}})$  in  $L'$ .

To increase legibility the arrow indicating a sequence of variables is omitted in the sequel.

$$\{T(\psi(x_1, \dots, x_n, y))\} \langle t_{x_1}, \dots, t_{x_n} \rangle$$

is to be an abbreviation for

$$\begin{aligned} \exists t'_{x_1 \dots x_n y} : \{T(\psi(x_1, \dots, x_n, y))\} \langle t'_{x_1 \dots x_n y} \rangle \\ \wedge \bigwedge_{i=1 \dots n} \pi_{x_i}^{x_1 \dots x_n y} (t'_{x_1 \dots x_n y}, t_{x_i}). \end{aligned}$$

DEFINITION 1.  $(T, \pi): (L, \Gamma) \rightarrow (L', \Gamma')$  is a functor if (A1)-(A4) are derivable in  $\Gamma'$  and the deduction rules (R1)-(R3) are valid (i.e., whenever the hypothesis is derivable in  $\Gamma$ , then the conclusion is derivable in  $\Gamma'$ ).

(A1)  $\pi_x^{xy}(t_{xy}, t_x)$  is functional with domain  $\{T(x=x \wedge y=y)\} \langle t_{xy} \rangle$  and codomain  $\{T(x=x)\} \langle t_x \rangle$ .

$$(A2) \{T(x=x)\} \langle t_x \rangle \Rightarrow \pi_x^x(t_x, t_x).$$

$$(A3) \pi_{xyz}^{xyz}(t_{xyz}, t_{xy}) \wedge \pi_x^{xy}(t_{xy}, t_x) \Rightarrow \pi_x^{xyz}(t_{xyz}, t_x).$$

$$(A4) \{T(\psi(x))\} \langle t_x \rangle \wedge t_x = t_x \Rightarrow \{T(\psi(x) \wedge x=x')\} \langle t_x, t_x \rangle.$$

$$(R1) \frac{\beta(x, y) \Rightarrow \psi(y)}{\{T(\beta(x, y))\} \langle t_y \rangle \Rightarrow \{T(\psi(y))\} \langle t_y \rangle}.$$

$$(R2) \frac{\beta(x, y) \wedge \beta(x, y') \Rightarrow y=y'}{\exists t_x : \{T(\beta(x, y))\} \langle t_x, t_y \rangle \wedge \{T(\beta(x, y'))\} \langle t_x, t'_y \rangle \Rightarrow t_y = t'_y}.$$

$$(R3) \frac{\text{same hypothesis as in (R2)}}{\{T(\exists y : \beta(x, y))\} \langle t_x \rangle \Rightarrow \exists t_y : \{T(\beta(x, y))\} \langle t_x, t_y \rangle}.$$

EXAMPLES.

1)  $(Q, \pi)$  «object representing partial morphisms» [JTT, 1.26]:

$$\{Q(\phi(x))\} \langle q_x \rangle \equiv (x \in q_x \wedge x' \in q_x) \Rightarrow (\phi(x) \wedge x = x'),$$

$$\pi_x^{xy}(q_{xy}, q_x) \equiv x \in q_x \Leftrightarrow \exists y : (x, y) \in q_{xy}.$$

2)  $(F, \rho)$  «filter functor»:

$$\{F(\phi(x))\}(f_x) \equiv \begin{cases} [(x \in A \wedge A \in f_x) \Rightarrow \phi(x)] \\ \wedge [(A \in f_x \wedge B \in f_x) \Rightarrow A \cap B \in f_x] \\ \wedge [A \in f_x \wedge A \subset B \wedge B \subset \{x \mid \phi(x)\} \Rightarrow B \in f_x]. \end{cases}$$

3) Let  $\lambda(y)$  be a formula of  $L$ . The «comma theory»  $(L, \Gamma)/\lambda(c)$  is formed by adding a constant « $c$ » to  $L$  and  $\lambda(c)$  to the axioms of  $\Gamma$ . If  $\Gamma \vdash \alpha(y, z) \Rightarrow \lambda(y) \wedge \kappa(z)$ , then  $\alpha$  induces a functor

$$\alpha_* = (A, \underline{\alpha}): (L, \Gamma)/\lambda(c) \rightarrow (L, \Gamma)/\kappa(d)$$

where

$$\{A(\phi(x, c))\}(a_x) \equiv \exists x \exists y: a_x = (x, y) \wedge \phi(x, y) \wedge \alpha(y, d)$$

and

$$\underline{\alpha}_x^{xv}(a_{xv}, a_x) \equiv \exists x \exists y \exists v: a_{xv} = (x, v, y) \wedge a_x = (x, y) \wedge \alpha(y, d).$$

Another functor

$$\alpha^* = (A^*, \bar{\alpha}): (L, \Gamma)/\kappa(d) \rightarrow (L, \Gamma)/\lambda(c)$$

is defined in the same way with  $y$  and  $z$  exchanged.

PROPOSITION 1. Let  $(T, \pi)$  be a functor.

i) If  $\alpha(x, y): \phi(x) \rightarrow \psi(y)$  is functional, then

$$\{T(\alpha(x, y))\}\langle t_x, t_y \rangle: \{T(\phi(x))\}(t_x) \rightarrow \{T(\psi(y))\}(t_y)$$

is functional.

ii)  $T$  preserves composition of functional formulae.

iii)  $T$  preserves identities.

iv)  $T$  preserves monos (i. e., 1-1 functional formulae).

PROOF.

$$\begin{aligned} \text{i) (F1): } \{T(\alpha(x, y))\}\langle t_x, t_y \rangle &\Rightarrow \{T(\alpha(x, y))\}\langle t_x \rangle \\ &\Rightarrow \{T(\phi(x))\}(t_x) \quad (\text{R1}). \end{aligned}$$

(F2): this is just an instance of (R2).

(F3):  $\phi(x) \Rightarrow \exists y: \alpha(x, y)$ . Hence

$$\{T(\phi(x))\}(t_x) \Rightarrow \{T(\exists y: \alpha(x, y))\}(t_x)$$

by (R1). Now apply (R3).

ii) Let  $\alpha(x, y)$  and  $\beta(x, y)$  be functional. Then

$$\begin{aligned} & \{ T(\exists y: \alpha(x, y) \wedge \beta(y, z)) \} \{ t_{xz} \} \\ & \Rightarrow \exists t_y: \{ T(\alpha(x, y) \wedge \beta(y, z)) \} \{ t_{xz}, t_y \}, \end{aligned}$$

$$\begin{aligned} \text{i. e., } & \exists t_{xyz} \exists t_y \{ T(\alpha(x, y) \wedge \beta(y, z)) \} \{ t_{xyz} \} \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \\ & \Rightarrow \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y \{ T(\alpha(x, y) \wedge \beta(y, z)) \} \{ t_{xyz} \} \\ & \quad \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \text{ (A1)} \\ & \Rightarrow \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y \{ T(\alpha(x, y)) \} \{ t_{xy} \} \wedge \{ T(\beta(y, z)) \} \{ t_{yz} \} \\ & \quad \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \text{ (R1)}. \end{aligned}$$

Therefore

$$\begin{aligned} & \{ T(\exists y: \alpha(x, y) \wedge \beta(y, z)) \} \{ t_x, t_z \} \quad (*) \\ & \Rightarrow \exists t_{xz} \exists t_{xyz} \exists t_{xy} \exists t_{yz} \exists t_y: \\ & \quad \{ T(\alpha(x, y)) \} \{ t_{xy} \} \wedge \{ T(\beta(y, z)) \} \{ t_{yz} \} \wedge \pi(t_{xyz}, t_{xz}) \wedge \pi(t_{xyz}, t_y) \\ & \quad \wedge \pi(t_{xyz}, t_{xy}) \wedge \pi(t_{xyz}, t_{yz}) \wedge \pi(t_{xz}, t_x) \wedge \pi(t_{xz}, t_z). \end{aligned}$$

One can add

$$\pi(t_{xy}, t_x), \quad \pi(t_{xy}, t_y), \quad \pi(t_{yz}, t_y) \quad \text{and} \quad \pi(t_{yz}, t_z)$$

by (A1) and (A3). Thus

$$(*) \Rightarrow \exists t_y: \{ T(\alpha(x, y)) \} \{ t_x, t_y \} \wedge \{ T(\beta(y, z)) \} \{ t_y, t_z \} \quad (**)$$

The converse implication  $(**) \Rightarrow (*)$  need not be shown, since both  $(*)$  and  $(**)$  are functional.

iii) Since  $\phi(x) \wedge x = x': \phi(x) \rightarrow \phi(x')$  is functional, so is

$$\{ T(\phi(x) \wedge x = x') \} \{ t_x, t_x \}.$$

The statement is then a consequence of (A4).

iv) This follows from (R2).

**PROPOSITION 2.** *If  $U$  is a monomorphism preserving functor in the usual sense between the categories associated to  $(L, \Gamma)$  and  $(L', \Gamma')$ , there is a functor  $(T, \pi)$  in the sense of Definition 1 whose interpretation according to Proposition 1 (i) is equivalent to  $U$ .*

PROOF. For each type  $X$  choose an appropriate sequence  $x$ . Then, if

$U(x=x) \equiv \xi(y)$  set

$$\{T(x=x)\}(t_x) \equiv \exists y: \xi(y) \wedge t_x=y$$

( $t_x$  of suitable type). If

$$U(\phi(x') \xrightarrow{\phi(x') \wedge x'=x} x=x) \equiv \kappa(w, y),$$

set

$$\{T(\phi(x'))\}(t_{x'}) \equiv \exists w \exists y: \kappa(w, y) \wedge t_{x'}=y.$$

And if

$$U(x=x \wedge z=z) \equiv \lambda(v), \quad U(x=x \wedge z=z \xrightarrow{pr_0} x=x) \equiv \mu(v, y),$$

set

$$\pi_x^{xz}(t_{xz}, t_x) \equiv \exists v \exists y: \mu(v, y) \wedge t_{xz}=v \wedge t_x=y.$$

Then (A1)-(A3) are obviously fulfilled by  $(T, \pi)$ .

Now apply  $U$  to the following diagrams:

$$(A4) \quad \begin{array}{ccc} \phi(x) & \longrightarrow & x=x \\ \downarrow & & \uparrow pr_0 \uparrow pr_1 \\ \phi(x) \wedge x=x' & \longrightarrow & x=x \wedge x'=x' \end{array}$$

$$(R1) \quad \begin{array}{ccc} \phi(x, y) & \longrightarrow & x=x \wedge y=y \\ \downarrow & & \downarrow \\ \psi(y) & \longrightarrow & y=y \end{array}$$

$$(R2) \quad \begin{array}{ccc} \beta(x, y) & \longrightarrow & x=x \wedge y=y \\ & \searrow m & \downarrow \\ & & y=y \end{array}$$

using  $U m$  mono,

$$(R3) \quad \begin{array}{ccc} \exists y: \beta(x, y) \wedge \beta(x', y) & \longrightarrow & x=x \wedge x'=x' \\ \downarrow j & & \uparrow \\ \beta(x, y) \wedge \beta(x', y) & \longrightarrow & x=x \wedge x'=x' \wedge y=y \end{array}$$

where  $j$  sends  $(x, x')$  to the unique corresponding  $y$ .

If application of  $U$  to the first diagram yields

$$\begin{array}{ccc}
 \alpha(a) & \xrightarrow{\kappa(a, c)} & \gamma(c) \\
 \downarrow \lambda(a, b) & & \mu(d, c) \uparrow \uparrow \nu(d, c) \\
 \beta(b) & \xrightarrow{\rho(b, d)} & \delta(d)
 \end{array}$$

then

$$\begin{aligned}
 \{T(\phi(x))\}(t_x) &\equiv \exists a \exists c: \kappa(a, c) \wedge c = t_x \quad (*) \\
 \{T(\phi(x) \wedge x = x')\}(t_{xx'}) &\equiv \exists b \exists d: \rho(b, d) \wedge d = t_{xx'}, \\
 \{T(\phi(x) \wedge x = x')\}(t_x, t_x') &\equiv \\
 &\equiv \exists b \exists d \exists c \exists c': \rho(b, d) \wedge \mu(d, c) \wedge \nu(d, c') \wedge c = t_x \wedge c' = t_x'.
 \end{aligned}$$

But (\*) implies

$$\exists b \exists d \exists c: \rho(b, d) \wedge \mu(d, c) \wedge \nu(d, c) \wedge c = t_x$$

by functionality of  $\lambda(a, b)$  and commutativity of the diagram.

The verification of (R1)-(R3) proceeds in a similar way.

DEFINITION 2. If

$$(T, \pi): (L, \Gamma) \rightarrow (L', \Gamma') \quad \text{and} \quad (S, \sigma): (L', \Gamma') \rightarrow (L'', \Gamma'')$$

are functors, the *composite*  $(ST, \sigma \cdot \pi)$  is defined as follows:

$$\begin{aligned}
 \{ST(\psi(x))\}(st_x) &\equiv \{S(\{T(\psi(x))\}(t_x))\}(st_x), \\
 \sigma \cdot \pi^{xy}(st_{xy}, st_x) &\equiv \{S(\pi^{xy}(t_{xy}, t_x))\}(st_{xy}, st_x).
 \end{aligned}$$

The *identity* functor is given by

$$\begin{aligned}
 \{I(\psi(x))\}(i_x) &\equiv \exists x: i_x = x \wedge \psi(x), \\
 \iota_x^{xy}(i_{xy}, i_x) &\equiv \exists x \exists y: i_{xy} = (x, y) \wedge i_x = x.
 \end{aligned}$$

The identity functor is a neutral element with respect to composition, and composition is associative up to provable equivalence and exchange of variables.

## 2. NATURAL TRANSFORMATIONS.

DEFINITION 3. If  $(T, \pi), (S, \sigma): (L, \Gamma) \rightarrow (L', \Gamma')$  are functors, a *nat-*

ural transformation  $\eta: S \rightarrow T$  is a family of  $\Gamma'$ -functional formulae

$$\eta_x(s_x, t_x): \{S(x=x)\}(s_x) \rightarrow \{T(x=x)\}(t_x)$$

such that

$$\begin{aligned} \Gamma' \vdash \exists s_y: \{S(a(x, y))\} \langle s_x, s_y \rangle \wedge \eta_y(s_y, t_y) \\ \Rightarrow \exists t_x: \eta_x(s_x, t_x) \wedge \{T(a(x, y))\} \langle t_x, t_y \rangle \end{aligned}$$

for any formula  $\alpha$  and any partition  $(\vec{x}; \vec{y})$  of its sequence of free variables.

EXAMPLES.

4)  $\kappa: (I, \iota) \rightarrow (Q, \pi)$  (Example 1),

$$\kappa_x(i_x, q_x) \equiv \exists x: i_x = x \wedge q_x = \{x\};$$

$\mu: (QQ, \pi \cdot \pi) \rightarrow (Q, \pi)$ ,

$$\mu_x(qq_x, q_x) \equiv q_x = \{x \mid qq_x = \{\{x\}\}\}$$

$((Q, \kappa, \mu)$  is a monad, the definition of monad being straightforward).

5) Let  $\alpha_*$  and  $\alpha^*$  be defined as in Example 3; then

$$\begin{aligned} (*) \quad \{AA^*(\phi(x, d))\}(aa_x^*) \\ \equiv \exists x \exists y \exists z: aa_x^* = (x, z, y) \wedge \phi(x, z) \wedge \alpha(y, z) \wedge \alpha(y, d). \end{aligned}$$

If  $\alpha$  is a partial function,  $(*) \Rightarrow \phi(x, d)$ , and so

$$\lambda_x(aa_x^*, i_x) \equiv \exists x \exists y \exists z: aa_x^* = (x, z, y) \wedge i_x = x$$

is a natural transformation  $\lambda: \alpha_* \alpha^* \rightarrow Id$ .

If  $\alpha$  is functional, there is an analogous  $\mu: Id \rightarrow \alpha^* \alpha_*$  and  $\lambda$  and  $\mu$  fulfill the usual conditions of a counit and unit.

## LITERATURE.

JTT. JOHNSTONE, P., *Topos Theory*, Academic Press, London, 1977.

MR. MAKKAI, M & REYES, G. E., First order categorical logic, *Lecture Notes in Math.* 611, Springer (1978).

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