CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

S. B. NIEFIELD

Cartesian spaces over T and locales over $\Omega(T)$

Cahiers de topologie et géométrie différentielle catégoriques, tome 23, nº 3 (1982), p. 257-267

<http://www.numdam.org/item?id=CTGDC_1982__23_3_257_0>

CAHIERS DE TOPOLOGIE *Vol. XXIII-3 (1982)*
ET GÉOMÉTRIE DIFFÉRENTIELLE

CARTESIAN SPACES OVER T AND LOCALES OVER $\Omega(T)$

by S. B. NIEFIELD *

ABSTRACT. Recall that an object Y of a finitely complete category $\mathcal{A}$ is *cartesian* if the functor $- \times Y: \mathcal{A} \to \mathcal{A}$ has a right adjoint, denoted $(\ )^Y$. If Y is a space over a sober space T, one can consider the cartesianness of

1. Y in the category Top/T of topological spaces over T,

2. $\hat{Y}$ (the soberification of Y) in the category Sob/T of sober spaces over T, or

3. $\Omega(Y)$ (the locale of opens of Y) in the category $Loc/\Omega(T)$ of locales over the locale $\Omega(T)$ of opens of T.

The goal of this paper is to establish the equivalence of these three conditions.

1. INTRODUCTION.

Recall that a *continuous lattice* is a complete lattice A such that $a = \vee\{ b \mid b << a \}$, for every $a \in A$, where $b << a$ (read «*b is way below a*») if whenever $a < \vee S$ for some directed subset S of A, we have $b < s$ for some $s \in S$.

A space Y is cartesian in Top (by Freyd's Special Adjoint Theorem [5]) iff $- \times Y$ preserves colimits, iff $- \times Y$ preserves coequalizers ($- \times Y$ preserves coproducts in any case) iff $- \times Y$ preserves quotient maps. Such spaces were characterized by Day & Kelly [2] as those spaces Y such that $\Omega(Y)$ is a continuous lattice, or equivalently (cf. 2.4 [19]) $\Omega(Y)$ satisfies

$$U = \vee\{ \wedge H \mid U \in H \subset \Omega(Y),\ H \text{ Scott-open} \}$$

where a subset H of a complete lattice A is *Scott-open* if it is upward closed and meets every directed subset $S \subset A$ such that $\vee S \in H$. Note that

* The research for this paper was supported by a Killam Postdoctoral Fellowship.

a sober space is cartesian iff it is locally compact [8].

Recall that a *frame* [3, 20] (*localic lattice* [1] or *complete Heyting algebra* [4, 16]) is a complete lattice A satisfying the distributive law

$$a \wedge \bigvee S = \bigvee \{ a \wedge s \mid s \in S \} \quad \text{for all} \quad a \in A, \ S \subset A.$$

A *frame homomorphism* is a finite meet and arbitrary sup preserving map. An object of the dual category is called a *locale* [11]. The notation and terminology of this paper is essentially that of [13].

In [9], Hyland shows that a locale A is cartesian in Loc iff it is locally compact (i.e. a continuous lattice). But such locales are necessarily spatial [10]. Thus, a locale is cartesian iff it is isomorphic to $\Omega(Y)$, for some cartesian space Y.

Let T be any space. In [17], we show that a space Y over T is cartesian in Top/T iff

(*) given $y \in U \in \Omega(Y_t)$, there exists $H \subset \Pi_{t \in T} \Omega(Y_t)$ such that $U \in H_t$, H is Scott-open and binding, and $\cap H$ is a neighborhood of y in Y,

where Y_t denotes the *fiber of* Y *over* t (i.e. $p^{-1}t$ with the subspace topology) ; H is *Scott-open* if H_t is for all $t \in T$; H is *binding* if $\{ t \mid U_t \in H_t \}$ is open in T whenever U is open in Y ; and $\cap H$ is the subset of Y whose fiber over t is $\cap H_t$ (i.e. the intersection of the family H_t in the power set of Y_t). Note that the set $\Pi_{t \in T} \Omega(Y_t)$ with the Scott-open binding subsets as opens is the exponential $(T \times 2)^Y$, where 2 denotes the Sierpinski space. Among corollaries we show that a locally compact space over a Hausdorff space T and the inclusion of a locally closed subspace are cartesian in Top/T. Note that although (*) has been useful (as exemplified by the above mentioned corollaries), a less technical condition might also be desirable, for example one that provides some insight into cartesianness in $Loc/\Omega(T)$.

2. CARTESIAN SPACES AND LOCALES.

Throughout this section we shall assume that T is a sober space.

LEMMA 1. *If* Y *is a sober space over* T *such that* $\Omega(Y)$ *is cartesian* *over* $\Omega(T)$, *then* $\Omega(X)\times_{\Omega(T)}\Omega(Y)\approx\Omega(X\times_T Y)$ *for every sober space* X *over* T.

PROOF. It suffices to show that $\Omega(X)\times_{\Omega(T)}\Omega(Y)$ has enough points. Consider the pullbacks

$$\begin{array}{ccccc}
(\amalg_{x\in X}\Omega(1))\times_{\Omega(T)}\Omega(Y) & \xrightarrow{f'} & \Omega(X)\times_{\Omega(T)}\Omega(Y) & \longrightarrow & \Omega(Y)\\
\downarrow & & \downarrow & & \downarrow\\
\amalg_{x\in X}\Omega(1) & \xrightarrow[f]{} & \Omega(X) & \longrightarrow & \Omega(T)
\end{array}$$

Now, $-\times_{\Omega(T)}\Omega(Y)$ preserves coproducts and epimorphisms being left adjoint. Thus, $(\amalg_{x\in X}\Omega(1))\times_{\Omega(T)}\Omega(Y)$ can be expressed as a coproduct of locales of the form $\Omega(1)\times_{\Omega(T)}\Omega(Y)$, and f' is an epimorphism since f is. But, $\Omega(1)\times_{\Omega(T)}\Omega(Y)$ is cartesian in *Loc* (since pulling back along any morphism preserves cartesian objects [17]), and hence spatial. Therefore $\Omega(X)\times_{\Omega(T)}\Omega(Y)$ is spatial, and the desired result follows. □

Suppose Y is a space over T,

$$U\in\Omega(Y),\quad G\in\Omega(T)\quad\text{and}\quad H\subset\amalg_{t\in T}\Omega(Y_t).$$

We shall say that U *is an element of* H *over* G, written $U\in_G H$ if $U_t\in H_t$ for all $t\in G$. We also define ΛH by

$$\Lambda H=Int(\cap H)\quad\text{where}\quad(\cap H)_t=\cap H_t.$$

A continuous map $p: Y\to T$ of spaces induces a geometric morphism $p: Sh\,Y\to Sh\,T$ on the categories of set-valued sheaves on Y and T, respectively. Now, p_* preserves internal locales [16]. In particular, $p_*\Omega_Y$ is an internal locale in $Sh\,T$, where Ω_Y denotes the subobject classifier in $Sh\,Y$. For the basic theory of internal locales in a topos we refer the reader to [14].

THEOREM 2. *The following are equivalent for a continuous map* $p: Y\to T$ *such that the canonical morphism* $\Omega(Y_t)\to\Omega(\hat{Y}_t)$ *is an isomorphism for all* $t\in T$, *where the latter denotes the fiber over* t *of the soberification* $\hat{Y}$ *of* Y.

a) $\Omega(Y)$ *is cartesian in* $Loc/\Omega(T)$.

b) $\hat{Y}$ *is cartesian in* Sob/T.

c) Y *is cartesian in* Top/T.

d) $U = \bigvee\{\bigwedge H \cap p^{-1} G \mid U \underset{G}{\epsilon} H \subset \Pi_{t\epsilon T}\Omega(Y_t)$, H *Scott-open and binding* $\}$,

for all $U \epsilon \Omega(Y)$.

e) $p_*\Omega_Y$ *is locally compact (i. e. a continuous lattice) as an internal locale in* $Sh\,T$.

PROOF. a ⇒ b: Suppose that X and Z are sober spaces over T. Then

$$\begin{aligned} Sob/T(X\times_T \hat{Y}, Z) &\approx Loc/\Omega(T)(\Omega(X\times_T \hat{Y}), \Omega(Z)) \\ &\approx Loc/\Omega(T)(\Omega(X)\times_{\Omega(T)}\Omega(\hat{Y}), \Omega(Z)) \\ &\approx Loc/\Omega(T)(\Omega(X)\times_{\Omega(T)}\Omega(Y), \Omega(Z)) \\ &\approx Loc/\Omega(T)(\Omega(X), \Omega(Z)^{\Omega(Y)}) \\ &\approx Sob/T(X, pt(\Omega(Z)^{\Omega(Y)})), \end{aligned}$$

where the second and fourth isomorphisms follow from Lemma 1 and a respectively, and pt denotes the right adjoint to Ω. Therefore, $\hat{Y}$ is cartesian in Sob/T.

b ⇒ d: Suppose $U \epsilon \Omega(Y)$. Note that condition (*) (see Introduction) holds for $\hat{Y}$ since the proof in [17] that cartesianness implies (*) involves only sober spaces over T. If $\hat{U}$ denotes the image of U under the isomorphism $\Omega(Y) \to \Omega(\hat{Y})$, then (by (*)) given $y \epsilon \hat{U}_t$, there exists

$$\hat{H} \subset \Pi_{t\epsilon T}\Omega(\hat{Y}_t)$$

such that $\hat{U}_t \epsilon \hat{H}_t$, $\hat{H}$ is Scott-open and binding, and $\cap\hat{H}$ is a neighborhood of y in $\hat{Y}$. It easily follows that

(1) $\quad \hat{U} = \bigvee\{\bigwedge \hat{H} \cap \hat{p}^{-1} G \mid \hat{U} \underset{G}{\epsilon} \hat{H},\ \hat{H}$ is Scott-open and binding$\}$

for $\hat{U}$ clearly contains the right hand side, and by the above remark every $y \epsilon \hat{U}$ is contained in $\bigwedge\hat{H} \cap \hat{p}^{-1} G$, for some $\hat{H}$, where $G = \{t \mid \hat{U}_t \epsilon \hat{H}_t\}$. Note that G is open since $\hat{H}$ is binding.

It remains to show that we can remove the ^'s in (1). Using the isomorphisms $\Omega(Y_t) \to \Omega(\hat{Y}_t)$, it suffices to show that for $H \subset \Pi_{t\epsilon T}\Omega(Y_t)$, the soberification of $Int_Y(\cap H)$ is precisely $Int_{\hat{Y}}(\cap \hat{H})$ where $\hat{H}$ is the image of H under the map $\Pi_{t\in T}\Omega(Y_t) \to \Pi_{t\in T}\Omega(\hat{Y}_t)$. But $U \subset H$ iff $U_t \subset V_t$ for each $V_t \in H_t$, iff $\hat{U}_t \subset \hat{V}_t$ for each $\hat{V}_t \in \hat{H}_t$, iff $\hat{U} \subset \cap \hat{H}$.

d $\Longleftrightarrow$ c: This follows easily from Theorem 2.3 of [17], since d is equivalent to (*).

d $\Longrightarrow$ e: First we claim that it suffices to show that for every $U \in \Omega(Y)$ (i.e. a global element of $p_*\Omega_Y$), we have $U = \bigvee\{V \mid V \ll U\}$, where the right hand side is the sup in $p_*\Omega_Y$. To see this we note that if Y is cartesian in Top/T, then $p^{-1}G$ is cartesian in Top/G, for every open subset G of T, and so the desired property also holds for locally defined elements. Recall that if S is a subset of $p_*\Omega_Y$, then

$$\bigvee S = \bigcup\{V \in \Omega(Y) \mid (\exists G \in \Omega(T))(V \in S(G))\}$$

[15]. Thus, we must show that

$$U = \bigcup\{V \in \Omega(Y) \mid (\exists G \in \Omega(T))(V \ll U \cap p^{-1}G \text{ in } p_*\Omega_Y|_G)\}.$$

But, using d, it suffices to show that $\Lambda H \cap p^{-1}G \ll U \cap p^{-1}G$ in $p_*\Omega_Y|_G$, for all $H \subset \Pi_{t\in T}\Omega(Y_t)$ such that $U \in_G H$ and H is Scott-open and binding. Note that $H \cap p^{-1}G \ll U \cap p^{-1}G$ in $p_*\Omega_Y|_G$ if for every globally defined ideal I (i.e. downward closed and directed subset) of $p_*\Omega_Y|_G$,

(2) $\quad U \cap p^{-1}G \subset \bigvee I \Rightarrow$
$(\forall t \in G,\ \exists G' \in \Omega(T))(t \in G' \subset G \wedge \Lambda H \cap p^{-1}G' \in I(G'))$;

for then $V \in I(G)$ (since I is a sheaf), as desired. But then a straightforward «localization» gives the corresponding property for locally defined ideals.

Suppose I is a globally defined ideal of $p_*\Omega_Y|_G$ such that $U \cap p^{-1}G \subset I$. If $t \in G$, then

$$U_t \subset \bigcup\{V_t \mid V \in I(G'),\ t \in G'\}.$$

Now, $U_t \in H_t$ and H_t is Scott-open, so there exists G' containing t such that $V \in I(G')$, and $V_t \in H_t$, since the set of all such V_t is directed. Let $G'' = \{t \mid V_t \in H_t\} \cap G'$. Then $t \in G''$, and

$$\Lambda H \cap p^{-1}G'' \subset V \cap p^{-1}G'' \in I(G'').$$

Therefore, (2) is verified.

e ⇒ a: First, $p_*\Omega_Y$, being locally compact, is cartesian in the category *Loc(Sh T)* of internal locales in *Sh(T)* [9]. But *Loc(Sh T)* is equivalent to $Loc/\Omega(T)$ via an equivalence that identifies $p_*\Omega_Y$ and $\Omega(Y)$. [14]. This completes the proof. □

COROLLARY 3. *If Y is a sober space over T, then Y satisfies the hypothesis, and hence the conclusion of Theorem 2.*

PROOF. This follows immediately since $\hat{Y} \approx Y$. □

COROLLARY 4. *If T is a T_D-space (i.e. points of T are locally closed), then any space Y over T satisfies the hypothesis, and hence the conclusion of Theorem 2.*

PROOF. Suppose $t \in T$. Then the inclusion $t: 1 \to T$ is cartesian in Top/T [17]. To see that $\Omega(Y_t) \approx \Omega(\hat{Y}_t)$ it suffices to show that $(T \times 2)^t$ is sober, where 2 denotes the Sierpinski space, for then

$$Top/T(\hat{Y}_t, T\times 2) \approx Top/T(\hat{Y}, (T\times 2)^t) \approx Top/T(Y, (T\times 2)^t)$$
$$\approx Top/T(Y_t, T\times 2).$$

Now, as a set,

$$(T\times 2)^t = \coprod_{s\in T} \Omega(t_s) = T \amalg 1,$$

where $t: 1 \to T$. The closed subsets F are described as follows. If $1 \in F$ then $t \in F$ (since the fiber over t is Scott-closed). Also, $F \cap T$ is closed in T, and if $1 \notin F$, then $F \setminus \{t\}$ is closed in T (since $\hat{F}$ is binding). Then it is not difficult to show that the irreducible closed subsets are those of the form $F \cup \{1\}$, where $t \in F$ and F is irreducible in T, $F = \overline{\{t\}}$, and F not containing t such that F is irreducible in T. In the former case, the generic point is the generic point of F if $F \neq \overline{\{t\}}$, and 1 if $F = \overline{\{t\}}$. In the latter cases, the generic point is the generic point of F in T. □

Next, we would like to compare the exponentials in the relevant categories when Y is as in the above theorem. We begin with a lemma.

LEMMA 5. *Let Y be a cartesian space over T such that* $\Omega(Y_t) \approx \Omega(\hat{Y}_t)$, *for all t. If X is a space over T, then $\widehat{X\times_T Y} = \hat{X}\times_T \hat{Y}$.*

PROOF. First, we consider the case where T is a one point space. The exponential 2^Y or $2^{\hat{Y}}$ in Top is the lattice of opens $\Omega(Y)$ with the Scott-topology, and hence is sober [7]. Thus, it follows that $Z^Y \approx Z^{\hat{Y}}$ is sober, for all sober space Z since it is a limit of sober spaces. Therefore

$$\begin{aligned} Top(\widehat{X\times Y}, Z) &\approx Top(X\times Y, Z) \approx Top(X, Z^Y) \approx Top(X, Z^{\hat{Y}}) \\ &\approx Top(\hat{X}, Z^{\hat{Y}}) \approx Top(\hat{X}\times\hat{Y}, Z) \end{aligned}$$

for all sober spaces Z, and the desired result follows.

Next, we show that the canonical map $f\colon \widehat{X\times_T Y} \to \hat{X}\times_T\hat{Y}$ is an equalizer in Top, for $\hat{\ } = pt\Omega$, the morphism $\Omega(X\times_T Y) \to \Omega(X)\times_{\Omega(T)}\Omega(Y)$ is an equalizer in Loc (its inverse image is clearly a surjection), and pt preserves finite limits being a right adjoint. Thus, it suffices to show that f is an epimorphism. Consider the following commutative diagram

$$\begin{array}{ccccc}
\amalg_{t\in T}\widehat{X_t\times Y_t} & \longrightarrow & \widehat{X\times_T Y} & & \\
\wr\| & & \Big\downarrow f & & \\
\amalg_{t\in T}\hat{X}_t\times\hat{Y}_t & & & & \\
\wr\| & & & & \\
\amalg_{t\in T}\hat{X}_t\times_T\hat{Y} & \xrightarrow{\;g'\;} & \hat{X}\times_T\hat{Y} & \longrightarrow & \hat{Y} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\amalg_{t\in T}\hat{X}_t & \xrightarrow{\;g\;} & \hat{X} & \longrightarrow & T
\end{array}$$

where the bottom squares are pullbacks, the top isomorphism follows from the first part of the proof (since $\hat{Y}_t$ is cartesian in Top being the pullback of a cartesian object over T and $\hat{Y}_t \approx \widehat{Y_t}$) and the bottom isomorphism follows from the commutativity of

$$\begin{array}{ccc}
\widehat{X_t} & \longrightarrow & \hat{X} \\
\Big\downarrow & & \Big\downarrow \\
1 & \longrightarrow & T
\end{array}$$

and the pullback

$$\begin{array}{ccccc}
\widehat{X_t}\times\widehat{Y_t} & \longrightarrow & \widehat{Y_t} & \longrightarrow & \hat{Y} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\hat{X}_t & \longrightarrow & 1 & \longrightarrow & T
\end{array}\;.$$

Note that $\hat{Y}_t \approx Y_t$ since $\hat{Y}_t$ is sober (Sob is closed to pullbacks) and $\Omega(Y_t) \approx \Omega(\hat{Y}_t)$. Now, g is an epimorphism since $\amalg_{t\in T} X_t \to X$ is, and $\hat{}$ preserves coproducts and epimorphisms. But $\hat{X} \times_T \hat{Y}$ is cartesian over $\hat{X}$ (being the pullback of a cartesian object over T), and so g' is also an epimorphism. Therefore, f is an epimorphism, and the proof is complete. □

COROLLARY 6. *If Y is as in Theorem 2, and Z is a sober space over T, then the exponential Z^Y in Top/T is sober, and hence isomorphic to the exponential $Z^{\hat{Y}}$ in Sob/T. Moreover, $\Omega(T\times 2)^{\Omega(Y)} \approx \Omega((T\times 2)^Y)$.*

PROOF. First, we note that $Z^Y = Z^{\hat{Y}}$ as exponentials in Top/T. Thus, it suffices to show that Z^Y is sober. But if X is any space over T we have

$$Top/T(X, Z^{\hat{Y}}) \approx Top/T(X, Z^Y) \approx Top/T(X\times_T Y, Z)$$
$$\approx Top/T(\widehat{X\times_T Y}, Z) \approx Top/T(\hat{X}\times_T \hat{Y}, Z) \approx Top/T(\hat{X}, Z^{\hat{Y}})$$

where the third isomorphism holds since Z is sober. Therefore, $Z^{\hat{Y}}$ is sober. When $Z = T\times 2$, we know $\Omega(T\times 2)^{\Omega(Y)}$ has enough points [9] and $pt(\Omega(T\times 2)^{\Omega(Y)}) \approx (T\times 2)^{\hat{Y}}$. Therefore,

$$\Omega(T\times 2)^{\Omega(Y)} \approx \Omega((T\times 2)^Y). \qquad \square$$

Note that we do not know whether Ω preserves exponentials in general.

COROLLARY 7. *The following are equivalent for a locale A over $\Omega(T)$:*

a) A is cartesian in $Loc/\Omega(T)$.

b) $A \approx \Omega(Y)$ for some cartesian space Y over T.

c) A is locally compact as an internal locale in $Sh\,T$.

PROOF. a ⇒ b: Consider the pullback

$$\begin{array}{ccc} \amalg_{t\in T}(\Omega(1)\times_{\Omega(T)} A) & \xrightarrow{f'} & A \\ \downarrow & & \downarrow \\ \amalg_{t\in T}\Omega(1) & \xrightarrow{f} & \Omega(T) \end{array}$$

where f' is an epimorphism since A is cartesian over $\Omega(T)$ and f is an epimorphism. Thus, it suffices to show that $\Omega(1)\times_{\Omega(T)}A$ has enough points, for all $\Omega(1)\to\Omega(T)$. But, $\Omega(1)\times_{\Omega(T)}A$ is cartesian in *Loc* (it is the pullback of a cartesian locale over $\Omega(T)$) and the desired result follows.

b $\Rightarrow$ c: follows from c $\Rightarrow$ e of Theorem 2.

c $\Rightarrow$ a: Note that the proof of e $\Rightarrow$ a of Theorem 2 does not use the assumption that the locale in question is spatial. Thus, the same proof applies. □

COROLLARY 8. *If T is a Hausdorff space and A is a locally compact locale over $\Omega(T)$, then A is cartesian in $Loc/\Omega(T)$.*

PROOF. We know that $A \approx \Omega(Y)$ for some locally compact sober space Y over T. But, such a space is cartesian over T [17], and the result follows from Corollary 7. □

COROLLARY 9. *The inclusion of a sublocale A of $\Omega(T)$ is cartesian iff it is locally closed (i.e. the intersection of an open and a closed sublocale).*

PROOF. This follows immediately from Corollary 7, a $\Leftrightarrow$ b, and the analogous result for spaces [17]. □

Note that Corollary 9 is proved in [18] for an arbitrary base locale.

Let $\underline{\underline{Top}}$ denote the 2-category of toposes, geometric morphisms, and natural transformations between their inverse images [12]. The following proposition relates the above results to exponentials in $\underline{\underline{Top}}/Sh\,T$.

PROPOSITION 10. *Let A be a locale over $\Omega(T)$. Then $Sh\,B^{Sh\,A}$ exists in $\underline{\underline{Top}}/Sh\,T$ for all locales B over $\Omega(T)$ iff A is cartesian in $Loc/\Omega(T)$. Moreover, $Sh\,B^{Sh\,A} \approx Sh(B^A)$.*

PROOF. Recall that $Loc/\Omega(T)$ is equivalent to the category $\underline{\underline{L\,Top}}/Sh\,T$ of localic toposes over $Sh\,T$ [14]. Moreover, the latter is a reflective subcategory of $\underline{\underline{Top}}/Sh\,T$ [14], via a reflection R which satisfies

$$R(\underline{E}\times_{Sh\,T}Sh\,A) \approx R(\underline{E})\times_{Sh\,T}Sh\,A$$

for all toposes $\underline{E}$ over $Sh\,T$ [18].

If A is cartesian over $\Omega(T)$, then

$$\underline{\underline{Top}}/Sh\,T(\underline{E}\times_{Sh\,T}Sh\,A,\ Sh\,B) \approx \underline{\underline{L\ Top}}/Sh\,T(R(\underline{E}\times_{Sh\,T}Sh\,A),\ Sh\,B)$$
$$\approx \underline{\underline{L\ Top}}/Sh\,T(R(\underline{E})\times_{Sh\,T}Sh\,A,\ Sh\,B)$$
$$\approx \underline{\underline{L\ Top}}/Sh\,T(R(\underline{E}),\ Sh(B^A))$$
$$\approx \underline{\underline{Top}}/Sh\,T(\underline{E},\ Sh(B^A))$$

where the third isomorphism holds since $\underline{\underline{L\ Top}}/Sh\,T$ is equivalent to $Loc/\Omega(T)$.

The converse follows from an appropriate 2-categorical version 1.31 of [6]. □

Department of Mathematics
Union College
SCHENECTADY, N. Y. 12308
U. S. A.

REFERENCES.

1. J. BENABOU, Treillis locaux et paratopologie, *Cahiers Topo. et Géom. Diff.* I (1957/58), Exposé 2.

2. B.J. DAY & G.M. KELLY, On topological quotients preserved by pullbacks and products, *Proc. Cambridge Phil. Soc.* 67 (1970), 553-558.

3. C.H. DOWKER & D. PAPERT, Quotient frames and subspaces, *Proc. London Math. Soc.* 16 (1966), 275-296.

4. M.P. FOURMAN & D.S. SCOTT, Sheaves and logic, *Lecture Notes in Math.* 753 (1979), 302-401.

5. P. FREYD, *Abelian categories*, Harper & Row, New York 1964.

6. P. FREYD, Aspects of topoi, *Bull. Austral. Math. Soc.* 7 (1972), 1-76.

7. G. GIERZ & al, *A compendium of continuous lattices*, Springer, 1980.

8. K.H. HOFFMANN & J.D. LAWSON, The spectral theory of distributive continuous lattices, *Trans. A.M.S.* 246 (1978), 285-310.

9. J.M.E. HYLAND, Function spaces in the category of locales, Continuous lattices, *Lecture Notes in Math.* 871, Springer (1981), 264-281.

10. J.R. ISBELL, Function spaces and adjoints, *Math. Scand.* 36 (1975), 317-339.

11. J.R. ISBELL, Atomless parts of spaces, *Math. Scand.* 31 (1972), 5-32.

12. P.T. JOHNSTONE, *Topos Theory*, L.M.S. Math. Mono. 0, Acad. Press 1977.

13. P.T. JOHNSTONE, Tychonoff's Theorem without the axiom of choice, *Fund. Math.*

14. P.T. JOHNSTONE, Factorization and pullback theorems for localic geometric morphisms, *Inst. Math. Pures Univ. Cath. Louvain*, Rapport 79 (1979).

15 P.T. JOHNSTONE, The Gleason cover of a topos, II, *J. Pure and Appl. Alg.* to appear.

16. C.J. MIKKFLSEN, Lattice-theoretic and logical aspects of elementary topoi, *Aarhus Univ. Various Publ. Series* 25 (1976).

17. S.B. NIEFIELD, Cartesianness: topological spaces, uniform spaces and affine schemes, *J. Pure and Appl. Alg.* 23 (1982), 147-167.

18. S.B. NIEFIELD, Cartesian inclusions: locales and toposes, *Comm. in Alg.* 9 (16) (1981), 1639-1671.

19. D.S. SCOTT, Continuous lattices, *Lecture Notes in Math.* 274, Springer (1972) 97-137.

20. H. SIMMONS, A framework for topology, *Proc. Wroclaw logic Conference* 1977 North-Holland (to appear).