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CARTESIAN CLOSED CONCRETE CATEGORIES

by J. ADÁMEK & V. KOUBEK

ABSTRACT. Full extensions of concrete categories $\mathcal{K}$ over a cartesian closed base category $\mathfrak{X}$ are studied. If $\mathcal{K}$ is cartesian closed and its forgetful functor $\mathcal{K} \to \mathfrak{X}$ preserves finite products and hom-objects, then $\mathcal{K}$ is called concretely cartesian closed. We prove that each concrete category has a universal (largest) concretely cartesian closed extension. Furthermore, we prove the existence of a «versatile» concretely cartesian closed category $\mathcal{K}^*$ (i.e. such that each concretely cartesian closed category has a full, finitely productive embedding in $\mathcal{K}^*$).

I. INTRODUCTION.

I, 1. We study universal and versatile concrete categories with a given property. Let us explain first the terms used in the preceding sentence. We start with a (fixed) *base category* $\mathfrak{X}$ and we work with *concrete categories*, i.e. pairs $(\mathcal{K}, |\ |)$, where $\mathcal{K}$ is a category and $|\ | : \mathcal{K} \to \mathfrak{X}$ is a faithful, amnestic functor, denoted on objects by $A \mapsto |A|$, on morphisms by

$$(f: A \to B) \mapsto (f: |A| \to |B|).$$

(Amnesticity means that, whenever $id_{|A|}: A \to B$ is an isomorphism in $\mathcal{K}$, then $A = B$.).

I, 2. By a *property* P of concrete categories we mean a conglomerate of concrete categories (called P-categories, or categories with property P) and a conglomerate of concrete functors (called P-functors, or functors preserving property P); a *concrete functor* is a functor $F: \mathcal{K} \to \mathcal{L}$ between concrete categories with $|\ |_{\mathcal{K}} = |\ |_{\mathcal{L}} \cdot F$ (i.e., on objects $|FA| = |A|$, on morphisms $Ff = f$). The domain and codomain of a P-functor need not be a P-category. Example:

P = concrete completeness.

Here, P-categories are those concrete categories $\mathcal{K}$ which are complete and detect limits in the base category $\mathcal{X}$ (i.e., given a diagram $D: \mathcal{D} \to \mathcal{K}$ and given a limit $\pi_d: X \to |Dd|$ of the underlying diagram $|\ |_{\mathcal{K}} \cdot D$ in $\mathcal{X}$, there exists an object A in $\mathcal{K}$ with $|A| = X$, such that $\pi_d: A \to Dd$ is a limit of D). And P-functors are concrete functors $F: \mathcal{K} \to \mathcal{L}$ which preserve concrete limits in $\mathcal{K}$, no matter whether $\mathcal{K}$ or $\mathcal{L}$ are complete categories or not.

I, 3. A *universal* P-*extension* of a concrete category $\mathcal{K}$ is a P-category $\mathcal{K}^*$ such that

(i) $\mathcal{K}$ is its full subcategory and the embedding $\mathcal{K} \to \mathcal{K}^*$ is a P-functor;

(ii) any P-functor $F: \mathcal{K} \to \mathcal{L}$ into a P-category $\mathcal{L}$ has an extension into a P-functor $F^*: \mathcal{K}^* \to \mathcal{L}$, unique up to natural equivalence.

E.g., if the base category $\mathcal{X}$ is complete, then each concrete category has a universal concrete completion (i.e., a universal P-extension with P = concrete completeness). This has been proved in [1].

I, 4. A *versatile* P-*category* is a P-category $\mathcal{K}$ such that for any P-category $\mathcal{H}$ there exists a full P-embedding $\mathcal{H} \to \mathcal{K}$.

Open problem: Does there exist a versatile concretely complete category, say, over $\mathcal{X} = Set$? Or, over the one-morphism category $\mathcal{X}$? (Here concrete categories are just ordered classes and the open problem is: does there exist a large-complete lattice into which every large-complete lattice can be embedded with all small infima preserved?)

Let us remark that the term «universal» is commonly used in this context, see e.g. [3, 4, 5]. But universality usually means often a different concept in category theory. Therefore we suggest that «versatile» be used for distinction.

I, 5. We are going to prove that every concrete category has a *universal concretely cartesian closed extension*. Here $\mathcal{X}$ is supposed to be cartesian closed. The property P in question consists of concrete categories which are cartesian closed and such that the forgetful functor detects both finite products and hom-objects, and P-functors are concrete functors preserving finite products and hom-objects.

Further, using a general construction of versatile categories of Trnkova [4, 5] we show that there exists a versatile concretely cartesian closed category. In view of the previous result, it suffices to show that there exists a versatile CFP-category $\hat{\mathfrak{K}}$. Here CFP (concrete finite products) is the property of all concrete categories, the forgetful functor of which detects finite products, and CFP-functors are concrete functors, preserving all finite concrete products. Then the universal concretely cartesian closed extension of $\hat{\mathfrak{K}}$ is a versatile concretely cartesian closed category, of course.

These results continue the research of «formal» extensions (complete or cartesian closed) of concrete categories, reported in [1, 2, 4, 5]. In particular in [2] a necessary and sufficient condition for a concrete category is presented to have a fibre small cartesian closed extension which is initially complete. As opposed to the present results, not every concrete category has such an extension.

II. UNIVERSAL CARTESIAN CLOSED EXTENSION.

II, 1. Recall that a finitely productive category is *cartesian closed* if for arbitrary objects A, B a «hom-object» $[A, B]$ is given in such a way that an adjunction takes place:

$$\frac{C \times A \xrightarrow{\ f\ } B}{C \xrightarrow[\hat{f}]{} [A, B]}$$

EXAMPLES. (i) Categories of relational structures are cartesian closed. E.g. the category of graphs is cartesian closed: given graphs $A = (X, \alpha)$ and $B = (Y, \beta)$ (where $\alpha \subset X \times X$ and $\beta \subset Y \times Y$), then

$$[A, B] = (Y^X, \gamma)$$

where

$$\gamma = \{(f, g) \mid f, g \in Y^X \text{ and for each } (x_1, x_2) \in \alpha \text{ we have } (f(x_1), g(x_2)) \in \beta\}.$$

(ii) The category of compactly generated Hausdorff spaces is cartesian closed: $[A, B]$ is the set $hom(A, B)$ endowed with the compact open topology.

(iii) The category of posets is cartesian closed: $[A, B]$ is the set *hom*(A, B), ordered point-wise.

The first example differs basically from the remaining two: all three are concrete categories over *Set* but only for the first one the forgetful functor preserves the hom-objects (i.e., $|[A, B]| = [|A|, |B|]$) plus the adjunction. In the present section we shall concentrate only on the type of concrete, cartesian closed categories represented by this example:

II, 2. DEFINITION. Let $\mathfrak{K}$ be a CFP-category (= concrete, with finite concrete products) over a cartesian closed base category $\mathfrak{X}$. A *concrete hom-object* for a pair of objects A, B of $\mathfrak{K}$ is an object $[A, B]$ in $\mathfrak{K}$ such that

$$|[A, B]| = [|A|, |B|]$$

and, given any object C and any map $f: |A \times C| \to |B|$, then $f: C \times A \to B$ is a morphism in $\mathfrak{K}$ iff the adjoint map $\hat{f}$ (in $\mathfrak{X}$) is a morphism

$$\hat{f}: C \to [A, B] \text{ in } \mathfrak{K}.$$

A CFP-category is said to be *concretely cartesian closed* provided that arbitrary two objects have a concrete hom-object.

II, 3. In [1] (Theorem 8) we have proved the following for an arbitrary base category $\mathfrak{X}$ with finite products: Let $\mathfrak{K}$ be a concrete category and let $\mathfrak{D}$ be a class of finite collections $\{A_i\}_{i=1}^{n} \subset \mathfrak{K}^{\sigma}$ such that a concrete product $A_1 \times \ldots \times A_n$ exists in $\mathfrak{K}$ for each $\mathfrak{D}$-collection. Then $\mathfrak{K}$ has a $\mathfrak{D}$-*universal* CFP-*extension* $\mathfrak{K}^*$. This is a CFP-category in which $\mathfrak{K}$ is a full, concrete subcategory, closed to products of $\mathfrak{D}$-collections with the following universal property:

Given a CFP-category $\mathfrak{L}$, then each concrete functor $F: \mathfrak{K} \to \mathfrak{L}$ preserving products of $\mathfrak{D}$-collections has a CFP-extension $F^*: \mathfrak{K}^* \to \mathfrak{L}$ unique up to natural equivalence.

This result we shall use for the construction of a universal concretely cartesian closed extension. We construct this in two steps. In the first step we assume that a CFP-category $\mathfrak{K}$ is given together with its full CFP-subcategory $\mathcal{H}$ having the property that a concrete hom-object $[A, B]$ exists in $\mathcal{K}$ for arbitrary $A, B \in \mathcal{H}$. We construct a «$\mathcal{H}$-universal» CFP-extension of $\mathcal{K}$, to be made precise below. In the second step, for each CFP-category $\mathcal{K}$ we put

$$\mathcal{K}_0 = \mathcal{K} \text{ and } \mathcal{H}_0 = \{T\}$$

where T is a terminal object; we find a $\mathcal{H}_0$-universal extension $\mathcal{K}_1$ of $\mathcal{K}_0$ and we put $\mathcal{H}_1 = \mathcal{K}_0$, then we find a $\mathcal{H}_1$-universal extension $\mathcal{K}_2$ of $\mathcal{K}_1$, etc. The category $\bigcup_{n=0}^{\infty} \mathcal{K}_n$ is the universal cartesian closed extension of $\mathcal{K}$.

II, 4. CONSTRUCTION. Let $\mathcal{K}$ be a CFP-category and let $\mathcal{H}$ be its full CFP-subcategory such that any pair of objects $A, B \in \mathcal{H}$ has a concrete hom-object $[A, B]$ in $\mathcal{K}$. We shall define a sequence of concrete categories

$$\mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots$$

and we shall prove that their union $\mathcal{L} = \bigcup_{i=-1}^{\infty} \mathcal{L}_i$ is a CFP-extension of $\mathcal{K}$ such that:

(i) each pair of $\mathcal{K}$-objects has a hom-object in $\mathcal{L}$,

(ii) the hom-objects for pairs in $\mathcal{H}$ are preserved, and

(iii) $\mathcal{L}$ is universal with respect to (i) and (ii).

Category $\mathcal{L}_{-1}$. Its objects are all $\mathcal{K}$-objects and all (formal) objects $[A, B]$ such that A, B are $\mathcal{K}$-objects, at least one of which is not in $\mathcal{H}$. (Thus $[A, B]$ denotes, ambiguously, a hom-object in case $A, B \in \mathcal{H}$, and a new object in case $A \notin \mathcal{H}$ or $B \notin \mathcal{H}$. Caution: if, by any chance, a concrete hom-object for a pair $A, B \in \mathcal{K}$ exists though $A \notin \mathcal{H}$ or $B \notin \mathcal{H}$ then we *do not* denote it by $[A, B]$). The underlying objects for $\mathcal{K}$-objects agree with those in $\mathcal{K}$ (i.e.

$$|A|_{\mathcal{K}} = |A|_{\mathcal{L}_{-1}} \quad \text{for } A \in \mathcal{K} \text{);}$$

for each pair A, B not in $\mathcal{H}$ we choose a hom-object $X = [|A|, |B|]$ in $\mathcal{X}$ and we put $|[A, B]|_{\mathcal{L}_{-1}} = X$. Morphisms in $\mathcal{L}_{-1}$ form the least class of $\mathcal{X}$-maps which is closed under composition (so as to make $\mathcal{L}_{-1}$ a category) and such that

(a) Each morphism in $\mathcal{K}$ is a morphism in $\mathcal{L}_{-1}$;

(b) Given a morphism $f: B \to B'$ in $\mathcal{K}$ and an object A in $\mathcal{K}$ then

$$[1_{|A|}, f]: [A, B] \to [A, B']$$

is a morphism of $\mathcal{L}_{-1}$ (no matter whether $[A, B]$ or $[A, B']$ are old objects or new).

(c) Given a morphism $f: C \times A \to B$ in $\mathcal{K}$ then the adjoint map

$$\hat{f}: |C| \to [|A|, |B|]$$

is a morphism $\hat{f}: C \to [A, B]$ in $\mathcal{L}_{-1}$.

Category $\mathcal{L}_0$. Denote by $\mathfrak{D}$ the class of finite collections of objects in $\mathcal{L}_{-1}$ consisting of all finite collections in $\mathcal{K}$ and of all collections $\{[A, B], [A, C]\}$ for A, B, C in $\mathcal{K}$. Then $\mathcal{L}_0$ is a $\mathfrak{D}$-universal CFP-extension of $\mathcal{L}_{-1}$ (see II,3). (It is easy to see that

$$[A, B] \times [A, C] = [A, B \times C]$$

is a concrete product in $\mathcal{L}_{-1}$ for arbitrary A, B, C.)

Categories $\mathcal{L}_{n+1}$. There are three ways in which $\mathcal{L}_{n+1}$ is constructed from $\mathcal{L}_n$, $n \geq 0$ and these are repeated in a cycle. All these categories have the same objects. Morphisms in $\mathcal{L}_{n+1}$ form the least class of $\mathfrak{X}$-maps closed to composition containing all $\mathcal{L}_n$-morphisms and such that

(a) if $n = 0 \bmod 3$: for each $p: C \times A \to B$ in $\mathcal{L}_n$ we have

$$\hat{p}: C \to [A, B] \quad \text{in } \mathcal{L}_{n+1};$$

(b) if $n = 1 \bmod 3$: for each $\hat{p}: C \to [A, B]$ in $\mathcal{L}_n$ we have

$$p: C \times A \to B \quad \text{in } \mathcal{L}_{n+1};$$

(c) if $n = 2 \bmod 3$: for each product $B = \prod_{i=0}^{k} B_i$ in $\mathcal{L}_0$ with projections $\pi_i: B \to B_i$, given an object A and a map $p: |A| \to |B|$ such that all $\pi_i \cdot p: A \to B_i$ are morphisms in $\mathcal{L}_n$, then $p: A \to B$ is a morphism in $\mathcal{L}_{n+1}$.

II, 5. LEMMA. $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$ *is a CFP-extension of* $\mathcal{K}$, *i.e. a CFP-category in which* $\mathcal{K}$ *is a full, concrete CFP-subcategory.*

PROOF. By definition, $\mathcal{L}_0$ is a CFP-category. The step $\mathcal{L}_n \to \mathcal{L}_{n+1}$, for $n = 2 \bmod 3$ «reconstructs» finite products, hence $\mathcal{L}$ is a CFP-category (with finite products agreeing with those in $\mathfrak{L}_0$). Thus it suffices to show that $\mathcal{K}$ is a CFP-subcategory of $\mathfrak{L}_{-1}$ and that $\mathcal{K}$ is full in $\mathfrak{L}$.

(i) $\mathcal{K}$ is full in $\mathfrak{L}_{-1}$. *Proof:* Let $g: A \to B$ be a morphism in $\mathfrak{L}_{-1}$ with $A, B \in \mathcal{K}^o$. By definition of $\mathfrak{L}_{-1}$, we have $g = g_n \cdot \ldots \cdot g_1$, where each of the morphisms $g_k: A_{k-1} \to A_k$ (with $A = A_0$, $B = A_n$) is of one of the types a, b or c. We shall verify that g is a morphism in $\mathcal{K}$, by induction in n. For $n = 1$ this is clear (recall that, if an object $[C, D]$ is in $\mathcal{K}$, then it is actually a concrete hom-object in $\mathcal{K}$ with $C, D \in \mathcal{H}$). For the induction step, we can assume $A_k \notin \mathcal{K}^o$ for $k \neq 0, \ldots, n$ (else we simply use the induction on $g_{k-1} \cdot \ldots \cdot g_1$ and $g_n \cdot \ldots \cdot g_k$). Then the morphisms g_k with $k \neq 1$ must be of type b, i.e. we have objects $C, D_1, \ldots, D_n$ in $\mathcal{K}$ and morphisms $h_k: D_{k-1} \to D_k$ such that

$$A_k = [C, D_k] \quad (k \neq 0) \quad \text{and} \quad g_k = [1_{|C|}, h_k] \quad (k \neq 1).$$

There are two possibilities for g_1:

either it is of type b; then necessarily

$$A_0 = [C, D_0] \quad \text{and} \quad g_1 = [1_{|C|}, h_1]$$

for some D_0, h_1 – this implies

$$g = [C, h_n \cdot \ldots \cdot h_1] : [C, D_0] \to [C, D_n]$$

which is a morphism in $\mathcal{K}$;

or it is of type c, i.e. $g_1 = \hat{f}$ where $f: A_0 \times C \to D_0$ is a morphism of $\mathcal{K}$; then

$$p = h_n \cdot \ldots \cdot h_2 \cdot f: A_0 \times C \to D_n$$

in $\mathcal{K}$ has an adjoint morphism

$$g: A_0 \to [C, D_n] = B$$

since $\hat{p} = g$.

(ii) $\mathcal{K}$ is closed under finite products in $\mathfrak{L}_{-1}$. *Proof:* Let $C \times D$ be a product in $\mathcal{K}$ with projections π_C, π_D. Let $f: X \to C$, $g: X \to D$ be morphisms in $\mathfrak{L}_{-1}$. Then we have a unique map

$$h: |X| \to |C| \times |D| = |C \times D| \quad \text{with } \pi_C \cdot h = f \text{ and } \pi_D \cdot h = g.$$

It is our task to show that $h: X \to C \times D$ is a morphism in $\mathfrak{L}_{-1}$. This is clear if $X \in \mathcal{K}^{\sigma}$, thus we can assume $X = [A\ ,\ B\ ]$. We have

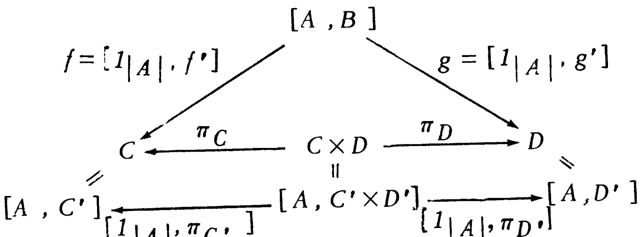
$$f = f_n \cdot \dots \cdot f_1 \quad \text{and} \quad g = g_m \cdot \dots \cdot g_1$$

where each of the morphisms

$$f_i : F_{i-1} \to F_i \quad \text{and} \quad g_j : G_{j-1} \to G_j$$

is of one of the types a, b or c. The proof proceeds by induction in $n+m$.

Let $n+m = 2$, i.e. $f = f_1$ and $g = g_1$. Then necessarily f and g are of type b: we have $C = [A, C'\ ]$ and $f = [1_{|A|}, f'\ ]$



for some morphism $f' : B \to C'$, analogously

$$D = [A\ ,D'\ ] \quad \text{and} \quad g = [A\ , g'\ ]\ .$$

Since $C', D' \in \mathcal{H}$ implies $C' \times D' \in \mathcal{H}$, clearly $C \times D = [A\ , C' \times D'\ ]$, and for the projections $\pi_{C'}$ and $\pi_{D'}$ of $C' \times D'$ we have

$$\pi_C = [\,1_{|A|}, \pi_{C'}\,] \quad \text{and} \quad \pi_D = [1_{|A|}, \pi_{D'}\,].$$

The unique morphism $h' : B \to C' \times D'$ in $\mathcal{K}$ with

$$f' = \pi_{C'} \cdot h' \quad \text{and} \quad g' = \pi_{D'} \cdot h'$$

fulfills $h = [A\ , h'\ ]$. This proves that $h : [A\ , B\ ] \to [A\ , C' \times D'\ ]$ is a morphism in $\mathfrak{L}_{-1}$ (of type b).

Let $n+m = k$ and let the proposition hold whenever $n+m < k$.

A. If all the objects F_i, $i \neq n$, and G_j, $j \neq m$, are outside of $\mathcal{K}$ then necessarily all the morphisms f_i and g_j are of type b. In that case we have

$$f_i = [1_{|A|}, f_i'\,] \quad \text{and} \quad g_j = [\,1_{|A|}, g_j'\,]$$

with $F_i = [A\ , F_i'\ ]$ - particularly $C = [A\ , C'\ ]$, and $G_j = [\ A\ , G_j'\ ]$ - particularly $D = [A,\ D'\ ]$. Then we can proceed as in case $n+m = 2$.

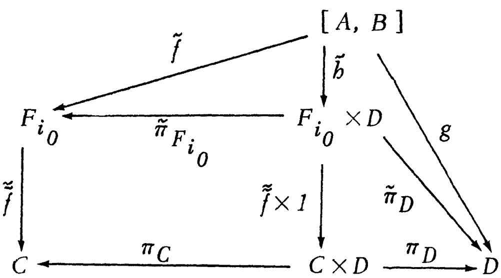
B. Let $F_{i_0} \in \mathcal{K}^\sigma$ for some $i_0 \neq n$ (analogous situation is $G_{j_0} \in \mathcal{K}^\sigma$ for some $j_0 \neq m$). Then $f = \tilde{\tilde{f}} . \tilde{f}$ where

$$\tilde{f} = f_{i_0 - 1} \cdot \ldots \cdot f_1 \quad \text{and} \quad \tilde{\tilde{f}} = f_n \cdot \ldots \cdot f_{i_0} ;$$

by (i) we know that $\tilde{\tilde{f}} : F_{i_0} \to C$ is a morphism in $\mathcal{K}$, hence

$$\tilde{\tilde{f}} \times 1_D : F_{i_0} \times D \to C \times D$$

is a morphism in $\mathcal{K}$. By induction hypothesis on f, g there is a morphism $\tilde{h} : [A, B] \to F_{i_0} \times D$ in $\mathcal{L}_{-1}$ such that, for the projections $\tilde{\pi}_{F_{i_0}}$ and $\tilde{\pi}_D$,



we have $\tilde{f} = \tilde{\pi}_{F_{i_0}} . \tilde{h}$ and $g = \tilde{\pi}_D . \tilde{h}$. And

$$\pi_C . (\tilde{\tilde{f}} \times 1_D) . \tilde{h} = \tilde{\tilde{f}} . \tilde{\pi}_{F_{i_0}} . \tilde{h} = \tilde{\tilde{f}} . \tilde{f} = f ,$$

$$\pi_D . (\tilde{\tilde{f}} \times 1_D) . \tilde{h} = \tilde{\pi}_D . \tilde{h} = g$$

imply $(\tilde{\tilde{f}} \times 1_D) . \tilde{h} = h$. Thus h is a morphism in $\mathcal{L}_{-1}$.

(iii) $\mathcal{K}$ is full in each $\mathcal{L}_n$. Indeed: $\mathcal{L}_{-1}$ is full in $\mathcal{L}_0$ and we shall prove by induction in $n \geq 0$ that any $\mathcal{L}_{n+1}$-morphism $f : D \to C$ with $D \in \mathcal{K}$ is an $\mathcal{L}_0$-morphism as well.

$n = 0 \mod 3$. It clearly suffices to verify that given $\mathcal{L}_n$-morphisms

$$h : D \to C \quad \text{and} \quad f : C \times A \to B \quad \text{with} \quad D, A, B \text{ in } \mathcal{K}$$

also $\hat{f} . h : D \to [A, B]$ is an $\mathcal{L}_0$-morphism. Since $n = 0 \mod 3$, $\mathcal{L}_n$ is a CFP-category, thus $h \times 1 : D \times A \to C \times A$ is a morphism and so is

$$f . (h \times 1) : D \times A \to B .$$

Since both $D \times A$ and B are objects of $\mathcal{K}$, by inductive hypothesis $f . (h \times 1)$ is a $\mathcal{K}$-morphism. By definition of $\mathcal{L}_{-1}$, its adjoint map is an $\mathcal{L}_{-1}$-morphism

- this map is evidently $\hat{f}.h : D \to [A, B]$.

$n = 1 \bmod 3$. It suffices to show that for any pair of $\mathcal{L}_n$-morphisms

$$h : D \to A \times B \quad \text{and} \quad \hat{p} : A \to [B, C] \quad (B, C, D \in \mathcal{K})$$

also $p.h : D \to C$ is an $\mathcal{L}_0$-morphism. Denote by h_A, h_B the components of h. By inductive hypothesis, $\hat{p}.h_A : D \to [B, C]$ is in $\mathcal{L}_0$. This clearly implies that $\hat{p}.h_A = \hat{q}$ for some $q : D \times B \to C$ in $\mathcal{L}_0$. Furthermore, by inductive hypothesis, the morphism $k : D \to D \times B$ with components 1_D, h_B is in $\mathcal{L}_0$ (since $\mathcal{L}_0$ is CFP). Moreover $h = (h_A \times 1_B).k$. Now $q.k : D \to C$ is a morphism in $\mathcal{L}_0$. Since clearly

$$\hat{q} = \hat{p}.h_A = p.\widehat{(h_A \times 1_B)} : D \to [B, C],$$

we get $q = p.(h_A \times 1_B)$ and so

$$q.k = p.(h_A \times 1_B).k = p.h.$$

This proves that $p.h$ is in $\mathcal{L}_0$.

$n = 2 \bmod 3$: clear.

II, 6. LEMMA. *The category* $\mathcal{L}$ *has concrete hom-objects for pairs of* $\mathcal{K}$*-objects and they coincide with those of* $\mathcal{K}$ *for pairs of objects in* $\mathcal{H}$.

$\mathcal{L}$ *is universal in the following sense: given a concretely cartesian closed category* $\mathcal{L}'$, *each CFP-functor* $\Phi : \mathcal{K} \to \mathcal{L}'$ *preserving hom-objects for pairs in* $\mathcal{H}$ *has a CFP-extension* $\Psi : \mathcal{L} \to \mathcal{L}'$ *preserving hom-objects for pairs in* $\mathcal{K}$, *which is unique up to natural equivalence.*

REMARK. For the proof of this lemma it is important that each CFP-category $\mathcal{K}$ has transfer: for each object A in $\mathcal{K}$ and for each isomorphism $i : X \to |A|$ in $\mathcal{X}$ there is a unique object B in $\mathcal{K}$ with $|B| = X$ such that $i : B \to A$ is an isomorphism in $\mathcal{K}$. The (trivial) reason for this is that the product of the singleton collection $|A|$ in $\mathcal{X}$ is e.g. X with projection $i : X \to |A|$; and this product is detected by the forgetful functor.

PROOF. It is evident from the way how $\mathcal{L}_n$ were constructed that the new objects $[A, B]$ in $\mathcal{L}_{-1}$ are hom-objects of A and B in $\mathcal{L}$; for each morphism $f : C \times A \to B$ in $\mathcal{L}$ which lies in $\mathcal{L}_n$, $\hat{f} : C \to [A, B]$ is a morphism in $\mathcal{L}_{n+3}$; for each morphism $\hat{f}: C \to [A, B]$ in $\mathcal{L}_n$, $f: C \times A \to B$ is a morphism in $\mathcal{L}_{n+3}$. For a pair $A, B \in \mathcal{H}$ the same is true with respect to the $\mathcal{K}$-object $[A, B]$. Thus $\mathcal{L}$ has hom-objects for pairs in $\mathcal{K}$, preserved in case of pairs in $\mathcal{H}$.

The universal property of $\mathcal{L}$ readily follows. Given $\Phi : \mathcal{K} \to \mathcal{L}'$ as above, let us extend it to $\Psi_{-1} : \mathcal{L}_{-1} \to \mathcal{L}'$ by choosing a fixed hom-object $[\Phi A, \Phi B]$ for each pair A, B of objects in $\mathcal{K}$, at least one of which is outside of $\mathcal{H}$, and then putting

$$\Psi_{-1}[A, B] = [\Phi A, \Phi B].$$

It is clear that this gives rise to a concrete functor $\Psi_{-1} : \mathcal{L}_{-1} \to \mathcal{L}'$. By definition of universal relative CFP-extensions (II, 3) we have a CFP-extension $\Psi_0 : \mathcal{L}_0 \to \mathcal{L}'$ of Ψ_{-1}. This defines a (concrete) CFP-extension Ψ : $\mathcal{L} \to \mathcal{L}'$ on objects, and, in fact, also on morphisms, because the morphisms $f: A \to B$ added to $\mathcal{L}_{n-1}$ on the n^{th} step have clearly the property that $f: \Psi_0 A \to \Psi_0 B$ is a morphism in $\mathcal{L}'$ (since $\mathcal{L}'$ is concretely cartesian closed). The uniqueness of Ψ is clear.

II, 7. DEFINITION. By a *universal concrete cartesian closed extension* of a concrete category $\mathcal{K}$ is meant its concretely cartesian closed extension $\mathcal{K} \subset \mathcal{K}^*$ in which $\mathcal{K}$ is CFP (closed to concrete finite products) and which has the following universal property:

For each concretely cartesian closed category $\mathcal{L}$ and each CFP-functor $\Phi : \mathcal{K} \to \mathcal{L}$ there exists a CFP-extension $\Phi^* : \mathcal{K}^* \to \mathcal{L}$ preserving hom-objects, which is unique up to natural equivalence.

II, 8. THEOREM. *Every concrete category over a cartesian closed base category has a universal concrete cartesian closed extension.*

PROOF. For a concrete category $\mathcal{K}$ denote by $\mathcal{K}_0$ its universal CFP-extension and put $\mathcal{H}_0 = \{T\}$ where T is the terminal object of $\mathcal{K}_0$. (And $[T, T] = T$ is a concrete hom-object.) By Lemma II, 6 there exists a universal CFP-extension $\mathcal{K}_1$ of $\mathcal{K}_0$ with concrete hom-objects for pairs in $\mathcal{K}_0$. Put $\mathcal{H}_1 = \mathcal{K}_0$. Using Lemma II, 6 again we obtain a universal CFP-extension $\mathcal{K}_2$ of $\mathcal{K}_1$ with concrete hom-objects for pairs in $\mathcal{K}_1$ and preserving hom-objects for pairs in $\mathcal{H}_1$. Put $\mathcal{H}_2 = \mathcal{K}_1$ and proceed in the same way.

The category $\mathcal{K}^* = \bigcup_{i=0}^{\infty} \mathcal{K}_i$ is the universal concrete cartesian closed extension of $\mathcal{K}$. Indeed, its finite products and hom-objects can be computed in $\mathcal{K}_{i+1}$ for collections in $\mathcal{K}_i$; thus $\mathcal{K}^*$ is concretely cartesian closed. Further $\mathcal{K}^*$ is clearly a CFP-extension of $\mathcal{K}$. Let $\Phi : \mathcal{K} \to \mathcal{L}$ be a CFP-functor with $\mathcal{L}$ cartesian closed. Then Φ has a (unique) CFP-extension $\Phi_0 : \mathcal{K}_0 \to \mathcal{L}$. By Lemma II, 6 there exists a (unique) CFP-extension of Φ_0 into a CFP-functor $\Phi_1 : \mathcal{K}_1 \to \mathcal{L}$ preserving hom-objects for pairs in $\mathcal{K}_0 = \mathcal{H}_1$. Again by Lemma II, 6 there exists a (unique) CFP-extension of Φ_1 into a CFP-functor $\Phi_2 : \mathcal{K}_2 \to \mathcal{L}$, preserving hom-objects for pairs in $\mathcal{K}_1 = \mathcal{H}_2$, etc. This defines a (unique) CFP-extension $\Phi^* = \bigcup_{n=0}^{\infty} \Phi_n : \mathcal{K}^* \to \mathcal{L}$ preserving hom-objects.

III. VERSATILE CATEGORIES.

III, 1. A general theorem about versatile categories is proved in [4, 5]. We shall apply this theorem to the property CFP of finite concrete products and we shall derive the existence of a versatile CFP-category which is moreover cartesian closed.

III, 2. DEFINITION. A property P of categories is said to be *canonical* if the following conditions are satisfied:

Categoricity: All isomorphisms of categories and all compositions of P-embeddings are P-embeddings (= full embeddings which are P-functors).

Chain condition: Let $\mathcal{K} = \cup \mathcal{K}_i$ be a union of a chain (= a well ordered set or class) of P-categories such that for each i, $\mathcal{K}_i$ is fully P-embedded into $\mathcal{K}_j$, $i < j$. Then

a) $\mathcal{K}$ is a P-category and each $\mathcal{K}_i$ is P-embedded into $\mathcal{K}$;

b) For each P-category $\mathcal{L}$, an embedding $\mathcal{K} \to \mathcal{L}$ is a P-embedding whenever each of its restriction to $\mathcal{K}_i$ is a P-embedding.

Small character: Every P-category is a union of a chain of small P-embedded P-subcategories.

Amalgam: For arbitrary P-embeddings

$$\Phi_1 : \mathcal{K} \to \mathcal{L}_1 \text{ and } \Phi_2 : \mathcal{K} \to \mathcal{L}_2$$

between P-categories there exists a P-category $\mathfrak{L}$ and P-embeddings

$$\Psi_1 : \mathfrak{L}_1 \to \mathfrak{L} \text{ and } \Psi_2 : \mathfrak{L}_2 \to \mathfrak{L} \text{ with } \Psi_1 . \Phi_1 = \Psi_2 . \Phi_2 .$$

Trivial subcategory: There exists a small P-category which is P-embeddable into any P-category.

III, 3. THEOREM [5]. *For every canonical property of categories there exists a versatile category with this property.*

III, 4. THEOREM. CFP *is a canonical property of concrete categories for each finitely productive base category.*

PROOF. Both categoricity and chain condition are trivial. Small character is also very simple to verify : given a CFP-category $\mathcal{K}$ choose a well order on its objects to obtain a chain $A_0, A_1, \ldots, A_i, \ldots$ such that A_0 is a terminal object of $\mathcal{K}$. Let us define full subcategories $\mathcal{K}_i$ of $\mathcal{K}$ by transfinite induction :

$\mathcal{K}_0$ has one object A_0 ;

given $\mathcal{K}_i$ then objects of $\mathcal{K}_{i+1}$ are $B \times A^n_{i+1}$ where B is an object of $\mathcal{K}_i$ and $n = 0, 1, 2, \ldots$;

for a limit ordinal i we put $\mathcal{K}_i = \bigcup_{j<i} \mathcal{K}_j$.

It is clear that each of the categories $\mathcal{K}_i$ is a small CFP-subcategory of $\mathcal{K}$ (hence also of $\mathcal{K}_{i+1}$) and the union of all of them is $\mathcal{K}$.

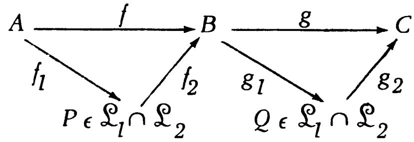
Before turning to the only nontrivial condition, amalgam, let us remark that the trivial subcategory is a category with just one object whose underlying object is terminal in $\mathfrak{X}$.

The proof of amalgam: Without loss of generality we assume that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are CFP-categories with $\mathcal{K} = \mathfrak{L}_1 \cap \mathfrak{L}_2$ a CFP-subcategory of each of them (and Φ_1, Φ_2 are inclusion functors). Let us show that there exists a concrete category $\mathfrak{L}$ (not necessarily finitely productive) containing $\mathfrak{L}_1$ and $\mathfrak{L}_2$ as full subcategories closed to finite products. This will prove the amalgam condition for we can choose a (universal) CFP-extension $\mathfrak{L}^*$ of $\mathfrak{L}$, see I, 4, and then $\mathfrak{L}_1$, $\mathfrak{L}_2$ will be CFP-subcategories of $\mathfrak{L}^*$ (and Ψ_1, Ψ_2 will be the inclusion functors). We define a concrete category $\mathcal{L}$ as follows:

Objects: all $\mathcal{L}_1$-objects and all $\mathcal{L}_2$-objects;

Underlying objects agree with those in $\mathcal{L}_1$ and/or $\mathcal{L}_2$. This leads to no contradiction for $\mathcal{L}_1 \cap \mathcal{L}_2$, since $\mathcal{L}_1 \cap \mathcal{L}_2$ is concretely embedded to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Morphisms: all $\mathcal{L}_1$-morphisms and $\mathcal{L}_2$-morphisms and all maps $f: |A| \to |B|$ for which there exist an object $P \in \mathcal{L}_1 \cap \mathcal{L}_2$ and morphisms $f_1: A \to P$, $f_2: P \to B$ in $\mathcal{L}_1 \cup \mathcal{L}_2$ (i.e. in $\mathcal{L}_1$ or in $\mathcal{L}_2$) with $f = f_2 \cdot f_1$. First, we observe that $\mathcal{L}$ is indeed a category, i.e. closed to composition: given

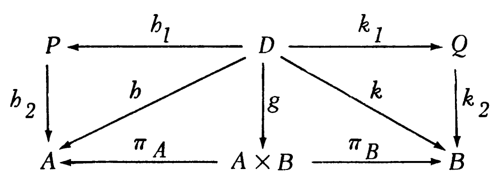


$f_1, f_2, g_1, g_2 \in \mathcal{L}_1 \cup \mathcal{L}_2$, then $g_1 \cdot f_2 : P \to Q$ is a morphism in $\mathcal{L}_1$ (if $B \in \mathcal{L}_1$) or $\mathcal{L}_2$ (if $B \in \mathcal{L}_2$), hence in $\mathcal{L}_1 \cap \mathcal{L}_2$, because $\mathcal{L}_1 \cap \mathcal{L}_2$ is full in $\mathcal{L}_1$ as well as in $\mathcal{L}_2$. Then clearly

$$(g_1 \cdot f_2) \cdot f_1 \in \mathcal{L}_1 \cup \mathcal{L}_2, \quad \text{hence} \quad g . f \in \mathcal{L}.$$

Clearly $\mathcal{L}_1$ and $\mathcal{L}_2$ are full in $\mathcal{L}$.

Second, we shall show that $\mathcal{L}_2$ is closed under finite products in $\mathcal{L}$ (analogously $\mathcal{L}_1$). The terminal object lies in $\mathcal{L}_1 \cap \mathcal{L}_2$ because $\mathcal{L}_1 \cap \mathcal{L}_2$ is closed under finite product in $\mathcal{L}_2$. Let $A \times B$ be a product in $\mathcal{L}_2$. Given $\mathcal{L}$-morphisms $h: D \to A$ and $k: D \to B$ we are to show that the induced map $g: |D| \to |A \times B|$ is a morphism in $\mathcal{L}$. This is clear if $D \in \mathcal{L}_2$; assume $D \in \mathcal{L}_1$. We have a commutative diagram with $P, Q \in \mathcal{L}_1 \cap \mathcal{L}_2$



necessarily $h_1, k_1 \in \mathcal{L}_1$, $h_2, k_2 \in \mathcal{L}_2$. Since $\mathcal{L}_1 \cap \mathcal{L}_2$ is a CFP-subcategory in $\mathcal{L}_1$ as well as in $\mathcal{L}_2$ we have a product $P \times Q \in \mathcal{L}_1 \cap \mathcal{L}_2$. Let $g_1 : D \to P \times Q$ be the $\mathcal{L}_1$-morphism induced by h_1 and k_1; analogously, let

$g_2 \in \mathfrak{L}_2$ be induced by h_2 and k_2. Then by the definition of $\mathfrak{L}$, $g_2 \cdot g_1$: $D \to A \times B$ is an $\mathfrak{L}$-morphism. Clearly $g_2 \cdot g_1 = g$.

III, 5. COROLLARY. *For each cartesian closed base category there exists a versatile concretely cartesian closed category* $\mathfrak{K}^*$. *Every concrete category then has a finitely productive, full, concrete embedding into* $\mathfrak{K}^*$.

PROOF. We have proved the existence of a versatile CFP-category $\mathfrak{K}_0^*$. Let $\mathfrak{K}^*$ be its concretely cartesian closed extension (II, 10). Then $\mathfrak{K}^*$ has all the required properties.

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