CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 24, nº 1 (1983), p. 47-56

<http://www.numdam.org/item?id=CTGDC_1983__24_1_47_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIII-4 (1982)

SPECTRA OF DIFFERENTIAL RINGS

by William F. KEIGHER

1. INTRODUCTION.

In this paper we propose to compare the various local spectra of differential rings which have been defined recently in the general setting of a topos in the sense of [4]. In particular, there have been two different notions of the local spectrum of a differential ring A in topos $\mathcal{E}$ which have been considered, one by the author in [9] (which we shall refer to as the full spectrum) and the other by M. Bunge in [2] (called the differential spectrum). In this paper we propose a third spectrum which in a certain sense is intermediate to the other two and has certain advantages which the other two lack.

We recall that the (local) spectrum of a commutative ring A in the classical sense (i.e., the topos $\mathcal{E}$ is the category $\mathcal{S}$ of sets) is the pair $(Spec\, A, \tilde{A})$ where $Spec\, A$ is the set of prime ideals of A with the Zariski topology and $\tilde{A}$ is the usual structural sheaf on $Spec\, A$, so that $\tilde{A}$ is a local ring in the topos $Shv(Spec\, A)$. The importance of the spectrum is that it serves as a model for the geometric objects one considers in algebraic geometry, since any scheme in the sense of Grothendieck is locally isomorphic to the spectrum of some commutative ring A. In this way global properties of schemes are related to local properties of the spectra, and it is this global-to-local relationship which has yielded many fruitful results in the last twenty years.

Ideally, the spectrum of a differential ring should play the same role in the generalized sense of considering differential equations as does the usual spectrum of a commutative ring when one considers algebraic equations in the same generalized setting.

In this vein, differential algebra may be considered either as commutative algebra together with the added «differential» structure of derivations, or as an extension of commutative algebra by viewing commutative algebra as a special case of differential algebra, in which the derivations are trivial. It seems reasonable, therefore, that the spectrum of a differential ring should both reflect the differential structure present and reduce to the usual spectrum of a commutative ring in the case of trivial derivations. To a certain extent, both the full spectrum and the differential spectrum satisfy these criteria.

When one considers differential algebra as commutative algebra with addition differential structure, it is apparent that the differential versions of many of the standard results in commutative algebra no longer are valid. For example, the radical of a differential ideal is not necessarily a differential ideal [6, Example page 12], and more importantly for our purposes, a maximal differential ideal is not necessarily a prime differential ideal [1, page 310]. More generally, one observes that certain results in differential algebra are valid only in characteristic zero and fail in positive characteristic (e.g., the counterexamples cited above exist in positive characteristic). However, if one assumes in addition that the differential ring is an algebra over the field Q of rational numbers or more generally is special in the sense of [8], then the differential versions of these and many other results from commutative algebra are valid.

Therefore, it is apparent that, unless one restricts attention to differential rings over Q, the notion of prime differential ideal is unnecessarily restrictive (excluding as it does maximal differential ideals). On the other hand, the notion of prime ideal is too general in the sense that it does not reflect the differential structure at hand. For example, the quotient of a differential ring by a prime ideal which is not differential is not a differential ring. With these shortcomings in mind, it is clear that the generalization to differential algebra of the basic ingredient of the spectrum, i.e., the prime ideal, should include the maximal differential ideals, should reflect the differential structure and should reduce to the usual prime ideal in the case of a trivial derivation. These considerations lead to the notion of a quasi-prime ideal as defined in [8], i.e., an ideal is quasi-prime if it is maximal among differential ideals which are disjoint from a multiplicative subset of the differential ring. Since the full spectrum is defined by taking all prime ideals, and the differential spectrum is defined by taking only prime differential ideals (which are quasi-prime), the quasi-spectrum (to be defined by taking quasi-prime ideals) will be intermediate to the other two spectra. In particular the differential spectrum of a non-trivial differential ring may be vacuous (consider, for example, the quotient of a differential ring by a maximal (but not prime) differential ideal); the quasi-spectrum is free from this sort of pathology. On the other hand, the full spectrum may contain non-differential ideals, while the quasi-spectrum is more closely related to the differential ideal structure of the differential ring.

2. PRELIMINARIES.

Throughout we assume that $\mathfrak{S}$ is a topos with a natural numbers object N. We recall from [9] that a differential ring A in $\mathfrak{S}$ consists of an object A in $\mathfrak{S}$ and morphisms

$$+: A\times A \to A, \quad \times: A\times A \to A, \quad o: 1\to A, \quad e: 1\to A \text{ and } d: A\to A$$

such that $(A, +, \times, o, e)$ is a commutative ring in $\mathfrak{S}$ and d is a derivation on A, i.e.,

$$d(a+b) = da + db \text{ and } d(ab) = dab + adb$$

for any $a, b \in A$. If A and B are differential rings in $\mathfrak{S}$ then a morphism of differential rings $f: A \to B$ is a ring morphism which commutes with the derivations. The category of differential rings in $\mathfrak{S}$ will be denoted by $Diff(\mathfrak{S})$.

In [9] it was shown that the forgetful functor $Diff(\mathfrak{S}) \to Ann(\mathfrak{S})$ is comonadic, where $Ann(\mathfrak{S})$ denotes the category of commutative rings in $\mathfrak{S}$. As a consequence, there is a bijection between the set of derivations on A and the set of ring morphisms $A \to A^N$ which are costructure morphisms. Specifically, any derivation $d: A \to A$ determines the costructure morphism $\alpha: A \to A^N$ where for any $a \in A$ and $n \in N$, $\alpha(a)(n) = d^{(n)}a$ is the n-th derivative of a, i.e.,

$$d^{(o)}a = a \text{ and } d^{(n+1)}a = d(d^{(n)}a) \text{ for } n > o.$$

Conversely, any costructure morphism $\alpha: A \to A^N$ determines the derivation $d: A \to A$ given as the composite

$$A \xrightarrow{\alpha} A^N \xrightarrow{A^s} A^N \xrightarrow{A^o} A^1 \xrightarrow{\approx} A$$

where $o: 1 \to N$ and $s: N \to N$ are the given morphisms of N.

We also note that the ring A^N is very closely related to the ring of formal power series in one variable over A, and the costructure morphism $\alpha: A \to A^N$ corresponding to the derivation $d: A \to A$ is likewise related to a «Taylor series expansion»; for details in the case $\mathcal{E} = \mathcal{S}$, cf. [7].

Now suppose that A is a differential ring in $\mathcal{E}$. Since $(-)^N$ has a right adjoint, it induces a morphism $\Omega^A \to \Omega^{A^N}$, namely

$$(X \rightarrowtail A) \mapsto X^N \rightarrowtail A^N .$$

We define the differential of any subobject $X \rightarrowtail A$ to be the pullback

$$\begin{array}{ccc} X_\Delta & \longrightarrow & X^N \\ \downarrow & & \downarrow \\ A & \xrightarrow{\alpha} & A^N \end{array}$$

and we will say that a subobject $X \rightarrowtail A$ is differential if the diagram

$$\begin{array}{ccc} X & \longrightarrow & X^N \\ \downarrow & & \downarrow \\ A & \xrightarrow{\alpha} & A^N \end{array}$$

is a pullback. If $X \rightarrowtail A$ and $Y \rightarrowtail A$ are subobjects of A, we write $X \leqslant Y$ to mean there is a (unique) monomorphism $X \to Y$ in $\mathcal{E}/A$. The basic properties of the differential are given in the following

PROPOSITION. *Let A be a differential ring in $\mathcal{E}$.*

1. *If $X \rightarrowtail A$, then $X_\Delta \leqslant X$ and $(X_\Delta)_\Delta = X_\Delta$.*
2. *$X_\Delta = X$ iff $X \rightarrowtail A$ is differential.*
3. *If $X \rightarrowtail A$ and $Y \rightarrowtail A$ are such that $X \leqslant Y$, then $X_\Delta \leqslant Y_\Delta$.*
4. *If $\{X_i \rightarrowtail A\}$ is a family of subobjects of A such that $\cap X_i \rightarrowtail A$ exists, then $(\cap X_i)_\Delta = \cap (X_i)_\Delta$.*
5. *If $f: A \to B$ is a morphism of differential rings in $\mathcal{E}$ and the diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback, then the diagram

$$\begin{array}{ccc} X_\Delta & \longrightarrow & Y_\Delta \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

PROOF. The proof is elementary and follows immediately from the definitions.

We recall from [8] that in the case $\mathcal{E} = \mathcal{S}$, the differential of a subset of a differential ring has many of the same properties as the subset. In particular, the differential I_Δ of an ideal I is a differential ideal, namely the largest differential ideal contained in I. However, the differential P_Δ of a prime ideal P is not, in general, a prime differential ideal, but instead a quasi-prime ideal. In particular, we recall the following, in which $r(I)$ denotes the radical of I.

PROPOSITION. *Let A be a differential ring in $\mathcal{S}$. The following are equivalent:*

(1) Q is a quasi-prime ideal in A.

(2) Q is a primary ideal in A and $Q = r(Q)_\Delta$.

(3) $r(Q)$ is a prime ideal in A and $Q = r(Q)_\Delta$.

(4) There is a prime ideal P in A such that $Q = P_\Delta$.

Hence we will say that an ideal I in a differential ring A in $\mathcal{E}$ is a *quasi-prime ideal* if $r(I)$ is a prime ideal in A and $I = r(I)_\Delta$, where $r(I)$ denotes the radical of I as in [11]. We note that if $\mathcal{E}$ is a topos satisfying De Morgan's law as in [5] and if Zorn's Lemma holds in $\mathcal{E}$ (for example if $\mathcal{E}$ is generated by subobjects of 1 and defined over a topos of sets in which the Axiom of Choice holds), then the proof of the equivalence of (1)-(4) in the above proposition can be carried over in $\mathcal{E}$. However, we do not need these extra hypotheses on $\mathcal{E}$, as it is only the properties that $r(I)$ be prime and $I = r(I)_\Delta$ which we need to obtain the quasi-spectrum.

3. THE QUASI-SPECTRUM.

Following the notation of [2], we let DR denote the theory of differential rings. As in [2], if DLR denotes the quotient theory of differential local rings (i.e., differential rings which have a unique maximal ideal which is differential), then the differential local spectrum is obtained from Theorem 6.58 of [4, page 206] by taking the class of admissible morphisms to be that of all differential local morphisms of differential local rings. In particular, for a differential ring A, $D\,Spec\,A$ consists of the set of all differential prime ideals in A with the Zariski topology, and the structural sheaf is the usual structural sheaf $\tilde{A}$ restricted to $D\,Spec\,A$ (which is a differential local ring, since its stalk at any $P \in D\,Spec\,A$ is A_P, a differential local ring).

If we let L+DR denote the quotient theory of DR obtained by adding the sequent

(Local) $\quad ((a+b) \in U) \Rightarrow ((a \in U) \vee (b \in U)),$

so that a local+differential ring is a differential ring having a unique maximal ideal (which is not necessarily differential), then the full local spectrum is obtained as above by taking the same class of admissible morphisms. In particular, for a differential ring A, $Spec\,A$ consists of the set of all prime ideals of A with the Zariski topology, with the usual structural sheaf $\tilde{A}$ (which is a local+differential ring). The full spectrum may also be obtained as a right adjoint to the forgetful functor

$$\text{DLR-top} \rightarrow \text{DR-top}$$

as in [9]. One advantage the full spectrum enjoys over the differential spectrum is that the canonical map $A \to \Gamma(\tilde{A})$ is an isomorphism for any differential ring A.

Now let QLDR be the quotient theory of DR obtained by adding the sequents

(Local) $\quad ((a+b \in U) \Rightarrow ((a \in U) \vee (b \in U))$

and (DSUn) $\quad ((\exists m \in N)(D^m(a^n) \in U)) \Rightarrow (a \in U)$

for each $n > o$. Hence a *q-local differential ring* is a differential ring having a unique maximal ideal which is the radical of a quasi-prime ideal. The class of admissible morphisms is again that of all local differential morphisms of q-local differential rings. Observe, however, that QLDR, while being a geometric theory, is not finitely presented, and hence Theorem 6.58 of [4, page 206] does not apply. In fact, it is an open question whether the quasi-spectrum exists in an arbitrary topos $\mathcal{E}$. However, in the case $\mathcal{E} = \mathcal{S}$, the quasi-spectrum does exist, as we shall see presently. We first make several observations.

By a *differential q-prime filter* S of a differential ring A we shall mean that S is the complement of the radical of a quasi-prime ideal in A. Equivalently, S is a prime filter which satisfies the condition

$$((\exists m \in N)(D^m(a^n) \in S)) \Rightarrow (a \in S)$$

for every $n > o$.

PROPOSITION. *If S is a saturated multiplicative subobject of A, then S is a differential q-prime filter of A iff $A[S^{-1}]$ is a q-local differential ring.*

PROOF. It was shown in [11] that $A[S^{-1}]$ is a local ring iff S is a prime filter of A. Hence it remains to show that in $A[S^{-1}]$ the condition

(1) $\quad ((\exists m \in N)(D^m((a/s)^n) \in U)) \Rightarrow (a/s \in U)$

for every $n > o$ is equivalent to the condition on S

(2) $\quad ((\exists m \in N)(D^m(a^n) \in S)) \Rightarrow (a \in S)$.

If we assume (1), then U is a differential q-prime filter on $A[S^{-1}]$. But since S is saturated, the diagram

$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & A[S^{-1}] \end{array}$$

is a pullback, where $A \to A[S^{-1}]$ is the canonical differential ring morphism. It follows easily that S is a differential q-prime filter on A. Conversely, assuming that S satisfies (2), let $a/s \in A[S^{-1}]$ and let $m > o$ be minimal such that $D^m((a/s)^n) \in U$. Since $a/1 = (a/s).(s/1)$ in $A[S^{-1}]$, it follows from the Axiom (Diff 2) of [2, page 89] that

$$D^m((a/s)^n) = D^m(a^n/s^n) =$$
$$= (1/s^n)[D^m(a^n)/1 - \sum_{k=o}^{m-1} \binom{m}{k} D^k(a^n/s^n).D^{m-k}(s^n)/1].$$

Now since U is saturated, we see that

$$D^m(a^n)/1 - \sum_{k=o}^{m-1} \binom{m}{k} D^k(a^n/s^n).D^{m-k}(s^n)/1 \in U$$

and since $m \in N$ was taken minimal so that $D^m(a^n/s^n) \in U$, it follows that $D^m(a^n)/1 \in U$, and hence $D^m(a^n) \in S$. By (2), it follows that $a \in S$, and hence $a/s \in U$ as desired.

PROPOSITION *(Factorization Lemma). Let $f: A \to L$ be a differential ring morphism with L q-local. Then there exists a factorization*

$$A \xrightarrow{p} A_f \xrightarrow{\hat{f}} L$$

of f where A_f is q-local and $\hat{f}$ is local which is « best possible » in the sense that for any other factorization

$$A \xrightarrow{q} B \xrightarrow{g} L$$

of f with B q-local and g local, there is a unique (necessarily local) $s: A_f \to B$ such that $sp = q$ and $gs = \hat{f}$.

PROOF. The proof is similar to the proof of the factorization Lemma in the commutative case, i.e., Lemma 4.1 of [3, page 249], and depends primarily upon the observation that if the diagram

$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & L \end{array}$$

is a pullback, then S is a differential q-prime filter of A and hence by the preceding proposition, $A_f = A[S^{-1}]$ is q-local.

COROLLARY. *A morphism $f: A \to L$ of differential rings, where L is q-local, is extremal iff $L \cong A[S^{-1}]$ for some differential q-prime filter of A.*

PROOF. This is immediate from the factorization Lemma.

Let A be a differential ring in $\mathcal{S}$. Then the quasi-spectrum of A exists and consists of the set $Q\,Spec\,A$ of q-prime ideals in A (i. e., those prime ideals P in A which satisfy $P = r(P_\Delta)$, or equivalent the image of the mapping $r: Quas\,A \to Spec\,A$ where $Quas\,A$ denotes the set of quasi-prime ideals of A) with the Zariski topology, and the structural sheaf $\hat{A}$ is the restriction of $\bar{A}$ to $Q\,Spec\,A$. If the characteristic of A is positive then it follows from [10] that $r: Quas\,A \to Spec\,A$ is a bijection, and hence $A \to \Gamma\bar{A}$ is an isomorphism. On the other hand, if A is an algebra over the rational numbers Q, then A is special in the sense of [8] and hence every quasi-prime ideal is a prime differential ideal, so that the quasi-spectrum of A coincides with the differential spectrum of A.

In both [2] and [3], spectra other than local spectra are considered. In particular, integral spectra and field spectra are defined in the commutative case in [3] and in the differential case in [2]. In the case of quasi-prime ideals, the corresponding types of differential rings are called quasi-domains and quasi-fields and are considered in some detail in [10]. However, we will not consider the «quasi-versions» of these spectra at this time, as there seems to be little to be gained in doing so at present.

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