

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome
24, n° 2 (1983), p. 145-154

http://www.numdam.org/item?id=CTGDC_1983__24_2_145_0

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**ON A FAMILY OF VARIETIES NOT SATISFYING
STOKA'S MEASURABILITY CONDITION**

by Leonardo CIRLINCIONE

RÉSUMÉ. On donne un premier exemple de famille mesurable de variétés dont le groupe attaché au groupe maximal d'invariance n'est pas mesurable. On prouve, ainsi, que la condition suffisante de mesurabilité, donnée par M.I. Stoka [3], n'est pas nécessaire.

Let \mathcal{F}_q be a family of varieties, depending on the (essential) parameters a_1, a_2, \dots, a_q , of an homogeneous n -dimensional space \mathcal{H}_n and let G_r be an r -dimensional Lie group of transformations of \mathcal{H}_n . Assume that \mathcal{F}_q is G_r -invariant and no element of G_r (except the identity map) fixes every variety of \mathcal{F}_q . In case G_r is contained in no other group having the same properties, G_r is called *the maximal group of invariance of \mathcal{F}_q* and its subgroups are the *groups of invariance of \mathcal{F}_q* .

Let $T \in G_s$, where G_s is an s -dimensional group of invariance of \mathcal{F}_q . If (a_i) and (β_i) ($i = 1, \dots, q$) are the parameters characterizing the varieties \mathcal{V} and $T(\mathcal{V})$ of \mathcal{F}_q , the map $(a_i) \mapsto (\beta_i)$ defines an s -dimensional Lie group H_s (of the parameter space \mathcal{H}_q of \mathcal{F}_q), isomorphic to G_s ([4], page 33) (*the associated group of G_s with respect to \mathcal{F}_q*). H_s can be either a measurable Lie group ([4], page 12) or not. We say that \mathcal{F}_q has the (elementary) *measure $\phi(a_i) da_i$* , with respect to G_s , if H_s is measurable, admitting ϕ as measure. According to M.I. Stoka, \mathcal{F}_q is *measurable* if ϕ is also the measure for each measurable Lie group which is associated to the same group of invariance of \mathcal{F}_q .

M.I. Stoka proves (see [3], [4] page 40) that if the group associated to the maximal group of invariance of \mathcal{F}_q is a measurable Lie group, then \mathcal{F}_q is measurable. This condition is also necessary if $q = 1, 2, 3$ ([4] page 41). Nevertheless this is not generally true; in fact we give an

example of a 5-dimensional measurable family \mathcal{F}_5 of varieties of $\mathcal{U}_3(\mathbb{R})$ such that the group associated to the maximal group of invariance of \mathcal{F}_5 is not measurable.

The first section of this article is devoted to the determination of the subgroups of G_7 , the similarity transformation group of $\mathcal{U}_3(\mathbb{R})$, depending on 5 or 6 parameters. This is helpful in order to give the mentioned example.

1. THE SIMILARITY TRANSFORMATION GROUPS OF $\mathcal{U}_3(\mathbb{R})$ DEPENDING ON 5 OR 6 PARAMETERS.

Let G_n be an n -dimensional Lie group of $\mathcal{U}_3(\mathbb{R})$ generated by the infinitesimal transformations X_1, \dots, X_n . The operators

$$(1) \quad Y_i = a_i^j X_j \quad (i = 1, \dots, m; m < n, a_i^j \in \mathbb{R}, \text{rank}(a_i^j) = m)$$

define an m -dimensional subgroup G_m of G_n iff

$$(2) \quad (Y_s, Y_t) = k_{st}^i Y_i \quad (s, t = 1, \dots, m; k_{st}^i \in \mathbb{R}).$$

Let $G_m^1 = \langle Y_1, \dots, Y_m \rangle$ and $G_m^2 = \langle Z_1, \dots, Z_m \rangle$ be two m -dimensional subgroups of G_n . G_m^1 is conjugate to G_m^2 (in G_n) iff there exist

$$T \in G_n \quad \text{such that} \quad T(Y_i) = b_i^s Z_s \quad (b_i^s \in \mathbb{R}). \quad (2)$$

By supposing in (1) $a_s^t \neq 0$, one can change Y_i into $(a_s^t)^{-1} Y_s$. Thus we may assume $a_s^t = 1$. Then change Y_i ($i \neq s$) into $Y_i - a_i^t Y_s$; we see that it is not restrictive to put $a_i^t = 0$.

Let G_7 be the similarity transformation group of $\mathcal{U}_3(\mathbb{R})$. We can write the equations of G_7 in the following way (see [4] page 154)

$$(3) \quad \begin{cases} x = b((1+l^2 \cdot m^2 - n^2)x' + 2(lm-n)y' + 2(ln+m)z') + t_1, \\ y = b(2(lm+n)x' + (1 \cdot l^2 + m^2 - n^2)y' + 2(mn-l)z') + t_2, \\ z = b(2(ln \cdot m)x' + 2(mn+l)y' + (1 \cdot l^2 - m^2 + n^2)z') + t_3, \end{cases}$$

$b, l, m, n, t_1, t_2, t_3 \in \mathbb{R}, b \neq 0$ and $b(1+l^2+m^2+n^2)$ is the homothetic ratio.

- (1) We use the Einstein's convention.
- (2) More information can be found in [1].

G_7 is generated by the infinitesimal transformations

$$(4) \begin{cases} X_1 f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}, & X_2 f = y \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial y}, \\ X_3 f = -x \frac{\partial f}{\partial z} + z \frac{\partial f}{\partial x}, & X_4 f = x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x}, \\ X_5 f = \frac{\partial f}{\partial x}, & X_6 f = \frac{\partial f}{\partial y}, & X_7 f = \frac{\partial f}{\partial z}, \end{cases}$$

and its group structure is given by

$$(5) \begin{cases} (X_1, X_2) = (X_1, X_3) = (X_1, X_4) = 0, & (X_1, X_5) = \cdot X_5, \\ (X_1, X_6) = \cdot X_6, & (X_1, X_7) = \cdot X_7; & (X_2, X_3) = X_4, \\ (X_2, X_4) = X_3, & (X_2, X_5) = 0, & (X_2, X_6) = \cdot X_7, \\ (X_2, X_7) = X_6; & (X_3, X_4) = -X_2, & (X_3, X_5) = X_7, \\ (X_3, X_6) = 0, & (X_3, X_7) = \cdot X_5; & (X_4, X_5) = \cdot X_6, \\ (X_4, X_6) = X_5, & (X_4, X_7) = 0; \\ (X_5, X_6) = (X_5, X_7) = (X_6, X_7) = 0. \end{cases}$$

REMARK 1. We pointed out that in (4) $X_1 f$ generates the dilatation group; $\langle X_2 f \rangle$, $\langle X_3 f \rangle$, $\langle X_4 f \rangle$ are (resp.) the three rotation groups on the axes x, y, z and $X_5 f, X_6 f, X_7 f$ generate the translation group.

In order to determine the subgroups of G_7 of dimension 5, we consider the transformations

$$Y_i = a_i^j X_j \quad (i = 1, 2, 3, 4, 5, \quad a_i^j \in \mathbb{R}, \quad \text{rank}(a_i^j) = 5),$$

where X_j ($j = 1, 2, 3, 4, 5, 6, 7$) is given by (4). Assume

$$a_5^1 = a_5^2 = a_5^3 = a_5^4 = a_5^5 = a_5^6 = 0;$$

then $a_5^7 \neq 0$. Thus, we may put $a_5^7 = 1, a_i^7 = 0, i \neq 5$, obtaining

$$A: Y_i = a_i^j X_j \quad (i = 1, 2, 3, 4), \quad j \neq 7, \quad Y_5 = X_7.$$

Under the hypothesis

$$a_5^1 = a_5^2 = a_5^3 = a_5^4 = a_5^5 = 0, \quad a_5^6 \neq 0$$

we have

$$B: Y_i = a_i^j X_j, \quad j \neq 6, \quad Y_5 = X_6 + a_5^7 X_7.$$

Likewise one can infer the remaining cases :

$$C: Y_i = a_i^j X_j \quad j \neq 5, \quad Y_5 = X_5 + a_5^6 X_6 + a_5^7 X_7,$$

$$D: Y_i = a_i^j X_j \quad j \neq 4, \quad Y_5 = X_4 + a_5^5 X_5 + a_5^6 X_6 + a_5^7 X_7,$$

$$E: Y_i = a_i^j X_j \quad j \neq 3, \quad Y_5 = X_3 + a_5^4 X_4 + a_5^5 X_5 + a_5^6 X_6 + a_5^7 X_7,$$

$$F: Y_i = a_i^j X_j \quad j \neq 2, \quad Y_5 = X_2 + a_5^3 X_3 + a_5^4 X_4 + a_5^5 X_5 + a_5^6 X_6 + a_5^7 X_7$$

$$G: Y_i = a_i^j X_j \quad j \neq 1,$$

$$Y_5 = X_1 + a_5^2 X_2 + a_5^3 X_3 + a_5^4 X_4 + a_5^5 X_5 + a_5^6 X_6 + a_5^7 X_7.$$

In case A (as well as in the remaining cases), we can again put the same assumption on the coefficients of Y_4 . Thus one finds

$$A_a: Y_i = a_i^j X_j \quad (i = 1, 2, 3), \quad j \neq 6, 7, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_b: Y_i = a_i^j X_j \quad j \neq 5, 7, \quad Y_4 = X_5 + a_4^6 X_6, \quad Y_5 = X_7,$$

$$A_c: Y_i = a_i^j X_j \quad j \neq 4, 7, \quad Y_4 = X_4 + a_4^5 X_5 + a_4^6 X_6, \quad Y_5 = X_7,$$

$$A_d: Y_i = a_i^j X_j \quad j \neq 3, 7, \quad Y_4 = X_3 + a_4^4 X_4 + a_4^5 X_5 + a_4^6 X_6, \quad Y_5 = X_7,$$

$$A_e: Y_i = a_i^j X_j \quad j \neq 2, 7, \quad Y_4 = X_2 + a_4^3 X_3 + a_4^4 X_4 + a_4^5 X_5 + a_4^6 X_6, \\ Y_5 = X_7,$$

$$A_f: Y_i = a_i^j X_j, \quad j \neq 1, 7,$$

$$Y_4 = X_1 + a_4^2 X_2 + a_4^3 X_3 + a_4^4 X_4 + a_4^5 X_5 + a_4^6 X_6, \quad Y_5 = X_7.$$

From A_a it follows

$$A_{a_1}: Y_i = a_i^j X_j \quad (i = 1, 2), \quad j \neq 5, 6, 7, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_2}: Y_i = a_i^j X_j \quad j \neq 4, 6, 7, \quad Y_3 = X_4 + a_3^5 X_5, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_3}: Y_i = a_i^j X_j \quad j \neq 3, 6, 7, \quad Y_3 = X_3 + a_3^4 X_4 + a_3^5 X_5, \\ Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_4}: Y_i = a_i^j X_j \quad j \neq 2, 6, 7, \quad Y_3 = X_2 + a_3^3 X_3 + a_3^4 X_4 + a_3^5 X_5, \\ Y_4 = X_6, \quad Y_5 = X_7.$$

$$A_{a_5}: Y_i = a_i^j X_j \quad j \neq 1, 6, 7, \quad Y_3 = X_1 + a_3^2 X_2 + a_3^3 X_3 + a_3^4 X_4 + a_3^5 X_5, \\ Y_4 = X_6, \quad Y_5 = X_7.$$

Now A_{a_1} splits again in four cases. Consider the first

$$A_{a_1}(I): Y_1 = a_1^1 X_1 + a_1^2 X_2 + a_1^3 X_3, \quad Y_2 = X_4, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7$$

In view of (5), (2) yields

$$(Y_1, Y_2) = -a_1^3 X_2 + a_1^2 X_3 = k_{12}^i Y_i,$$

whence

$$a_1^1 k_{12}^1 = 0, \quad a_1^2 k_{12}^1 = -a_1^3, \quad a_1^3 k_{12}^1 = a_1^2, \quad k_{12}^2 = k_{12}^3 = k_{12}^4 = k_{12}^5 = 0.$$

These equations are compatible if

$$\text{rank} \begin{pmatrix} a_1^1 & 0 \\ a_1^2 & -a_1^3 \\ a_1^3 & a_1^2 \end{pmatrix} < 2.$$

Therefore $(a_1^2)^2 + (a_1^3)^2 = 0$, i. e. $a_1^1 \neq 0$. As it is not restrictive to set $a_1^1 = 1$, we have the first 5-dimensional subgroup of G_7

$$(6) \quad G_5^1 = \langle X_1, X_4, X_5, X_6, X_7 \rangle.$$

In the second case, A_{a_1} (II), we have

$$Y_1 = a_1^1 X_1 + a_1^2 X_2 + a_1^4 X_4, \quad Y_2 = X_3 + a_2^4 X_4, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7.$$

From $(Y_1, Y_2) = k_{12}^i Y_i$ it follows that

$$a_1^1 k_{12}^1 = 0, \quad a_1^2 k_{12}^1 = a_1^4, \quad a_1^4 k_{12}^1 + a_2^4 k_{12}^2 = -a_1^2, \\ k_{12}^2 = k_{12}^3 = k_{12}^4 = k_{12}^5 = 0.$$

A simple computation leads to

$$(a_1^2)^2 + (a_1^4)^2 = 0, \quad \text{i. e.} \quad a_1^2 = a_1^4 = 0.$$

Thus one finds the family of 5-dimensional groups

$$(7) \quad \{G_5^2(a) = \langle X_1, X_3 + a X_4, X_5, X_6, X_7 \rangle\}_{a \in \mathbb{R}}.$$

In the case A_{a_1} (III) we may assume $a_1^1 \neq 0$. For supposing $a_1^1 = 0$ and $a_1^3 = 0$ (then it must be necessarily $a_1^4 \neq 0$) we obtain a special case of A_{a_1} (I) (since it is not restrictive to put also $a_2^4 = 0$), while if $a_1^3 \neq 0$ a particular case of A_{a_1} (II) occurs. Therefore

$$Y_1 = X_1 + a_1^3 X_3 + a_1^4 X_4, \quad Y_2 = X_2 + a_2^3 X_3 + a_2^4 X_4, \\ Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7.$$

Compute as usual (Y_1, Y_2) ; then (2) and (5) imply

$$k_{12}^1 = 0, \quad k_{12}^2 = a_1^4 a_2^3 - a_1^3 a_2^4, \quad a_1^3 k_{12}^1 + a_2^3 k_{12}^2 = -a_1^4,$$

$$a_1^4 k_{12}^1 + a_2^4 k_{12}^2 = a_1^3, \quad k_{12}^3 = k_{12}^4 = k_{12}^5 = 0.$$

Thus

$$(1 + (a_2^3)^2 + (a_2^4)^2) k_{12}^2 = 0, \quad \text{whence } a_1^3 = a_1^4 = 0.$$

Hence we have the family of subgroups of G

$$(8) \quad \{G_5^3(b, c) = \langle X_1, X_2 + bX_3 + cX_4, X_5, X_6, X_7 \rangle\}_{b, c \in \mathbb{R}}.$$

A_{a_1} (IV) splits further in three cases: yet they fall under those above examined. The study of the remaining cases is a routine computation. One finds that the early groups fill up the class of subgroups of G_7 depending on five parameters.

THEOREM 1. *Let G_5 be a 5-dimensional subgroup of the similarity transformation group G_7 of $\mathcal{U}_3(\mathbb{R})$. Then G_5 is conjugate in G_7 to $G_5^3(b, c)$, for suitable $b, c \in \mathbb{R}$.*

PROOF. The change of coordinates $T(x, y, z) = (z, x, y)$ induces the operator permutation

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \\ X_1 & X_4 & X_2 & X_3 & X_7 & X_5 & X_6 \end{pmatrix}.$$

Thus we note that $(G_5^1)^T = G_5^2(0)$. Likewise one checks that $G_5^2(a)$ is conjugate to $G_5^3(a, 0)$.

We can make use of the same processes of Theorem 1 to classify the 6-dimensional subgroups of G_7 . The following theorem is the result we obtain:

THEOREM 2. *There exists exactly one 6-dimensional subgroup of G_7 the orthogonal group $G_6 = \langle X_2, X_3, X_4, X_5, X_6, X_7 \rangle$.*

2. A 5-DIMENSIONAL MEASURABLE FAMILY OF VARIETIES OF $\mathcal{U}_3(\mathbb{R})$

Let \mathcal{U} be the variety of $\mathcal{U}_3(\mathbb{R})$

$$(9) \quad (a_1 x + a_2 y + a_3 z + 1)((a_2 a_4 + a_3 a_5) x - a_1 a_4 y - a_1 a_5 z - a_1) = 0$$

where $a_i \in \mathbb{R}$ and $a_1(a_4^2 + a_5^2) \neq 0$. Clearly (9) is a pair of orthogonal

planes of $\mathcal{U}_3(\mathbb{R})$.

Let $\mathcal{F}_5 = \mathcal{U}^{G_7}$ be the family of varieties obtained from \mathcal{U} through the transformations of G_7 . The maximal group of invariance of \mathcal{F}_5 is, of course, G_7 . We prove

THEOREM 3. *The group associated to G_7 , with respect to \mathcal{F}_5 , is a non-measurable group.*

PROOF. Let T be the transformation (3) of G_7 . The parameters β_i of the variety $T(\mathcal{U})$ define the associated group H_7 :

$$(10) \quad \left\{ \begin{aligned} \beta_1 &= b \frac{(1+l^2-m^2-n^2)a_1 + 2(lm+n)a_2 + 2(ln-m)a_3}{t_1 a_1 + t_2 a_2 + t_3 a_3 + 1} \\ \beta_2 &= b \frac{2(lm-n)a_1 + (1-l^2+m^2-n^2)a_2 + 2(mn+l)a_3}{t_1 a_1 + t_2 a_2 + t_3 a_3 + 1} \\ \beta_3 &= b \frac{2(ln+m)a_1 + 2(mn-l)a_2 + (1-l^2-m^2+n^2)a_3}{t_1 a_1 + t_2 a_2 + t_3 a_3 + 1} \\ \beta_4 &= b \frac{2(lm-n)(a_2 a_4 + a_3 a_5) - (1-l^2+m^2-n^2)a_1 a_4 - 2(mn+l)a_1 a_5}{t_1(a_2 a_4 + a_3 a_5) - t_2 a_1 a_4 - t_3 a_1 a_5 - a_1} \\ \beta_5 &= b \frac{2(ln+m)(a_2 a_4 + a_3 a_5) - 2(mn-l)a_1 a_4 - (1-l^2-m^2+n^2)a_1 a_5}{t_1(a_2 a_4 + a_3 a_5) - t_2 a_1 a_4 - t_3 a_1 a_5 - a_1} \end{aligned} \right.$$

The coefficients of the infinitesimal transformations generating H_7 are

$$\begin{aligned} \xi_1^1 &= a_1, \quad \xi_1^2 = a_2, \quad \xi_1^3 = a_3, \quad \xi_1^4 = a_4, \quad \xi_1^5 = a_5, \\ \xi_2^1 &= 0, \quad \xi_2^2 = 2a_3, \quad \xi_2^3 = -2a_2, \quad \xi_2^4 = 2a_5, \quad \xi_2^5 = -2a_4, \\ \xi_3^1 &= -2a_3, \quad \xi_3^2 = 0, \quad \xi_3^3 = 2a_1, \quad \xi_3^4 = 0, \quad \xi_3^5 = -2 \frac{a_2 a_4 + a_3 a_5}{a_1}, \\ \xi_4^1 &= 2a_2, \quad \xi_4^2 = -2a_1, \quad \xi_4^3 = 0, \quad \xi_4^4 = 2 \frac{a_2 a_4 + a_3 a_5}{a_1}, \quad \xi_4^5 = 0, \\ \xi_5^1 &= -a_1^2, \quad \xi_5^2 = -a_1 a_2, \quad \xi_5^3 = -a_1 a_3, \quad \xi_5^4 = a_4 \frac{a_2 a_4 + a_3 a_5}{a_1}, \\ &\quad \xi_5^5 = a_5 \frac{a_2 a_4 + a_3 a_5}{a_1}, \\ \xi_6^1 &= -a_1 a_2, \quad \xi_6^2 = -a_2^2, \quad \xi_6^3 = -a_2 a_3, \quad \xi_6^4 = -a_4^2, \quad \xi_6^5 = -a_4 a_5, \\ \xi_7^1 &= -a_1 a_3, \quad \xi_7^2 = -a_2 a_3, \quad \xi_7^3 = -a_3^2, \quad \xi_7^4 = -a_4 a_5, \quad \xi_7^5 = -a_5^2. \end{aligned}$$

H_7 is a measurable group if the following system of equations

$$(11) \left\{ \begin{aligned}
 & a_1 \frac{\partial \phi}{\partial a_1} + a_2 \frac{\partial \phi}{\partial a_2} + a_3 \frac{\partial \phi}{\partial a_3} + a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5} = -5\phi \\
 & a_3 \frac{\partial \phi}{\partial a_2} - a_2 \frac{\partial \phi}{\partial a_3} + a_5 \frac{\partial \phi}{\partial a_4} - a_4 \frac{\partial \phi}{\partial a_5} = 0 \\
 & a_1 (a_3 \frac{\partial \phi}{\partial a_1} - a_1 \frac{\partial \phi}{\partial a_3}) + (a_2 a_4 + a_3 a_5) \frac{\partial \phi}{\partial a_5} = -a_3 \phi \\
 & a_1 (a_2 \frac{\partial \phi}{\partial a_1} - a_1 \frac{\partial \phi}{\partial a_2}) + (a_2 a_4 + a_3 a_5) \frac{\partial \phi}{\partial a_4} = -a_2 \phi \\
 & a_1^2 (a_1 \frac{\partial \phi}{\partial a_1} + a_2 \frac{\partial \phi}{\partial a_2} + a_3 \frac{\partial \phi}{\partial a_3}) - (a_2 a_4 + a_3 a_5) (a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5}) = \\
 & \quad = (3(a_2 a_4 + a_3 a_5) - 4a_1^2) \phi \\
 & a_2 (a_1 \frac{\partial \phi}{\partial a_1} + a_2 \frac{\partial \phi}{\partial a_2} + a_3 \frac{\partial \phi}{\partial a_3}) + a_4 (a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5}) = -(4a_2 + 3a_4) \phi \\
 & a_3 (a_1 \frac{\partial \phi}{\partial a_1} + a_2 \frac{\partial \phi}{\partial a_2} + a_3 \frac{\partial \phi}{\partial a_3}) + a_5 (a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5}) = -(4a_3 + 3a_5) \phi
 \end{aligned} \right.$$

admits exactly one non-trivial solution (up to a constant), see Deltheil [2] page 28.

The last three equations of (11) are equivalent to

$$(12) \left\{ \begin{aligned}
 & a_1 \frac{\partial \phi}{\partial a_1} + a_2 \frac{\partial \phi}{\partial a_2} + a_3 \frac{\partial \phi}{\partial a_3} = -4\phi \\
 & a_4 \frac{\partial \phi}{\partial a_4} + a_5 \frac{\partial \phi}{\partial a_5} = -3\phi .
 \end{aligned} \right.$$

Clearly the equations (12) and the first equation of (11) form a system of equations admitting no solution. Therefore H_7 is not measurable.

REMARK 2. The first equation of (11) arises from the dilatation parameter, while the remaining equations correspond, in order of sequence, to the rotation and translation parameters.

Let G_n be an n -dimensional group of G_7 , where $n \leq 5$ and let H_n be the associated group with respect to \mathcal{F}_5 . If $n \leq 4$, then H_n is intransitive, whence it is not measurable ([4] page 15).

In case $n = 5$, from Theorem 1 it follows that G_n is conjugate to $G_5^3(b, c)$, for suitable parameters $b, c \in \mathbb{R}$. In view of Remarks 1 and 2, we note that the Deltheil's system of H_5 contains the first and the last

three equations of (11). But we have already seen in the proof of Theorem 3 that they admit no solutions. Therefore :

PROPOSITION 1. \mathcal{F}_5 admits no measure with respect to any n -group of invariance, if $n \leq 5$.

We turn now to the orthogonal group G_6 . Let H_6 denote the corresponding associated group. On the ground of Remark 2, we can obtain the Deltheil's system of H_6 by suppressing in (11) the first equation.

It is not difficult to check that this system has only one solution (up to a constant)

$$\begin{aligned} \phi(a_1, a_2, a_3, a_4, a_5) &= \\ &= a_1^2 ((a_2 a_4 + a_3 a_5)^2 + a_1^2 (a_4^2 + a_5^2)) (a_1^2 + a_2^2 + a_3^2)^{-3/2}. \end{aligned}$$

Hence we have

PROPOSITION 2. With respect to the orthogonal group \mathcal{F}_5 admits the elementary measure

$$\begin{aligned} a_1^2 ((a_2 a_4 + a_3 a_5)^2 + a_1^2 (a_4^2 + a_5^2)) (a_1^2 + a_2^2 + a_3^2)^{-3/2} \\ da_1 \wedge da_2 \wedge da_3 \wedge da_4 \wedge da_5. \end{aligned}$$

As a consequence we have the main result (see also Theorems 2, 3 and Proposition 1):

COROLLARY. The family of varieties \mathcal{F}_5 is measurable in spite of the non-measurability of the group associated to its maximal group of invariance.

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