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NOTE ON HOMOTOPY PULLBACKS IN ABELIAN CATEGORIES

by Thomas MÜLLER

In this Note we inform about conditions in (abelian) categories with homotopy system under which a homotopy pullback is a homotopy push-out and vice versa. In particular, the category of chain complexes over an abelian category together with the usual homotopy system fulfills these conditions. As an easy consequence of this result we have Mather's cube Theorems and their duals (cf. [8], Section 3).

Let $\underline{C}$ always be a category provided with a homotopy system (I, j_0, j_1, q) (cf. [5], 0.5) which fulfills the Kan-conditions E(2), E(3) (cf. [5], 0.6) and so induces in a canonical way the structure of a category enriched over $\underline{Gd}$, the category of groupoids (cf. [3], 2.4). The 2-morphisms in $\underline{C}$ are equivalence classes of homotopies ($\{H\}$) but besides these we calculate with the homotopies (H) themselves as well.

DEFINITION 1. a) A homotopy commutative square in $\underline{C}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\swarrow H"] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

is called a *homotopy pullback* (HPB for short) if:

(i) to every triple (u, v, K) where $u \in \underline{C}(E, B)$, $v \in \underline{C}(E, C)$, and $K: ku \simeq lv$, there exist a $h \in \underline{C}(E, A)$ and homotopies

$F: u \simeq ih$, $G: jh \simeq v$ such that $\{lG\}+\{HIh\}+\{kF\} = \{K\}$,

(ii) given two triples (h, F, G), (h', F', G') where $h, h' \in \underline{C}(E, A)$, $F: u \simeq ih$, $F': u \simeq ih'$, $G: jh \simeq v$, $G': jh' \simeq v$ such that

$$\{lG\}+\{HIh\}+\{kF\} = \{K\} = \{lG'\}+\{HIh'\}+\{kF'\},$$

there exists a homotopy $\phi: h \simeq h'$ such that

$$\{F'\} = \{i\phi\} + \{F\}, \quad \{G\} = \{G'\} + \{j\phi\}.$$

b) A *homotopy pushout* (HPO for short) is defined dually.

DEFINITION 2. Two homotopy commutative squares in $\underline{C}$

$$(1) \quad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\swarrow H"] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd} \qquad (2) \quad \begin{tikzcd} A' \arrow[r, "i'"] \arrow[d, "j'"'] \arrow[dr, phantom, "\swarrow H'"] & B' \arrow[d, "k'"] \\ C' \arrow[r, "l'"] & D' \end{tikzcd}$$

are called *equivalent* (we write $(1) \diamond (2)$ for short) if a homotopy commutative cube in $\underline{C}$

$$(C) \qquad \begin{tikzcd} & B \arrow[rr, "h_2"] \arrow[dd, "k" near start] \arrow[drr, phantom, "F^3"] & & B' \arrow[dd, "k'"] \\ A \arrow[ur, "i"] \arrow[rr, "h_1" near start] \arrow[dd, "j"'] \arrow[dr, phantom, "H"] \arrow[urr, phantom, "F^1"] \arrow[drr, phantom, "F^2"] & & A' \arrow[ur, "i'"] \arrow[dd, "j'" near start] \arrow[dr, phantom, "H'"] & \\ & D \arrow[rr, "h_4" near start] & & D' \\ C \arrow[ur, "l"] \arrow[rr, "h_3"'] \arrow[urr, phantom, "F^4"] & & C' \arrow[ur, "l'"'] & \end{tikzcd}$$

(coherence condition :

$$\{H' I h_1\} + \{k' F^1\} + \{F^3 I i\} = \{l' F^2\} + \{F^4 I j\} + \{h_4 H\})$$

exists where h_i, $1 \leq i \leq 4$, are homotopy equivalences.

REMARK. (Cf. [7], (1.1.15).) This relation is an equivalence relation.

LEMMA 3 *(cf. [7], (1.2.4)). If a homotopy commutative square is equivalent to a homotopy pullback (pushout), then it is a HPB (HPO).*

LEMMA 4 *(cf. [7], (1.2.5)). a) If, in the homotopy commutative cube (C) above, the left and right faces are homotopy pullbacks and h_2, h_3, h_4 are homotopy equivalences, then so is h_1.*

b) If, in the homotopy commutative cube (C) above, the left and right faces are homotopy pushouts and h_1, h_2, h_3 are homotopy equivalences, then so is h_4.

From now on we assume

1. that I has a right adjoint,

2. that $\underline{C}$ has pullbacks of diagrams $B \xrightarrow{k} D \xleftarrow{l} C$ where k is a fibration,

3. that $\underline{C}$ has pushouts of diagrams $B \xleftarrow{i} A \xrightarrow{j} C$ where i is a cofibration.

LEMMA 5 *(cf.* [2], *(4.2)). a) Every morphism f in $\underline{C}$ factors as $f = hi$ where i is a cofibration and h is a homotopy equivalence.*

b) Every morphism f in $\underline{C}$ factors as $f = ph$ where h is a homotopy equivalence and p is a fibration.

LEMMA 6 *(cf.* [7], *(1.4.8)). Let*

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{l} & D \end{array}$$

be a commutative square in $\underline{C}$.

a) If $(*)$ is a pullback and k is a fibration, then $(*)$ is a HPB.

b) If $(*)$ is a pushout and i is a cofibration, then $(*)$ is a HPO.

LEMMA 7 *(cf.* [7], *(2.5.1)). a) Every HPO in $\underline{C}$ is equivalent to a pushout in $\underline{C}$*

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{l} & D \end{array}$$

where i is a cofibration and j is a fibration.

b) Every HPB in $\underline{C}$ is equivalent to a pullback in $\underline{C}$ of the form $(*)$, where k is a fibration and l is a cofibration.

PROOF. a) Let

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "H"] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

be a HPO in $\underline{C}$. We first replace β by a fibration: By Lemma 5, we have $\beta = jh_1$ where j is a fibration and h_1 is a homotopy equivalence. If h'_1

is a homotopy inverse for h_1, there is a homotopy commutative square

$$\begin{tikzcd} A' \arrow[r, "a h_1'"] \arrow[d, "j"'] & B \arrow[d, "\gamma"] \arrow[dl, phantom, "K"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

where $K \in \{\delta j G\} + \{H l h_1'\}$ and $G : h_1 h_1' \simeq 1_{A'}$. We now replace $a h_1'$ by a cofibration: By Lemma 5, we have $a h_1' = h i$ where i is a cofibration and h is a homotopy equivalence. Then we obtain a homotopy commutative square

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "j"'] & B' \arrow[d, "\gamma h"] \arrow[dl, phantom, "K"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

Finally, we form a pushout of the diagram $B' \xleftarrow{i} A' \xrightarrow{j} C$. Then we get a commutative square

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "j"'] & B' \arrow[d] \\ C \arrow[r] & D' \end{tikzcd}$$

which is of the required form. Since there exists a homotopy

$$G' : h_1' h_1 \simeq 1_A \quad \text{such that} \quad \{h_1 G'\} = \{G l h_1\}$$

(cf. [7], (1.1.10)), it is clear, by the Lemmas 3, 4, 6 above, that the last square is equivalent to the HPO at the beginning of the proof.

b) This is the dual of a. □

DEFINITION 8. Let $\mathfrak{A}$, $\mathfrak{B}$ be two classes of morphisms in $\underline{C}$.

a) $\mathfrak{A} \times \mathfrak{B}$ is called *pullback-stable* (pb-stable for short) in $\underline{C}$ if, in every pullback in $\underline{C}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

we have $i \in \mathfrak{A}$, $j \in \mathfrak{B}$ whenever $l \in \mathfrak{A}$, $k \in \mathfrak{B}$.

b) $\mathfrak{A} \times \mathfrak{B}$ is called *pushout-stable* (po-stable for short) in $\underline{C}$ if, in every pushout in $\underline{C}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

we have $l \in \mathfrak{A}$, $k \in \mathfrak{B}$ whenever $i \in \mathfrak{A}$, $j \in \mathfrak{B}$.

LEMMA 9 *(cf. [1], (8.1.1)). Let $\underline{C}$ have zero-morphisms.*

a) If, in a pullback

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

l is conormal and i is an epimorphism, then this pullback is also a pushout.

b) If, in a pushout

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

i is normal and l is a monomorphism, then this pushout is also a pullback.

NOTATION. Let $M(\underline{C})$ be the class of monomorphisms in $\underline{C}$, $E(\underline{C})$ the class of epimorphisms, $N(\underline{C})$ the class of normal morphisms, $Con(\underline{C})$ the class of conormal morphisms, $F(\underline{C})$ the class of fibrations, $C(\underline{C})$ the class of cofibrations in $\underline{C}$.

THEOREM 10. *Let $\underline{C}$ have zero-morphisms.*

a) If $C(\underline{C}) \subset N(\underline{C})$ and $C(\underline{C}) \times F(\underline{C})$ is po-stable in $\underline{C}$, then every HPO in $\underline{C}$ is a HPB in $\underline{C}$.

b) If $F(\underline{C}) \subset Con(\underline{C})$ and $C(\underline{C}) \times F(\underline{C})$ is pb-stable in $\underline{C}$, then every HPB in $\underline{C}$ is a HPO in $\underline{C}$.

PROOF. a) By the Lemmas 7 and 3, it is sufficient to prove the theorem in the case where the given HPO is a pushout

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

where $\alpha \in C(\underline{C})$, $\beta \in F(\underline{C})$. By assumption, we have

$$\delta \in C(\underline{C}) \subset N(\underline{C}) \subset M(\underline{C}) \quad \text{and} \quad \gamma \in F(\underline{C}).$$

By Lemma 9, this square is a pullback and hence, by Lemma 6, a HPB.

b) This is the dual of a. □

REMARK. If, in Theorem 10, $\underline{C}$ is an abelian category, we can replace $C(\underline{C}) \subset N(\underline{C})$ by $C(\underline{C}) \subset M(\underline{C})$ and $F(\underline{C}) \subset Con(\underline{C})$ by $F(\underline{C}) \subset E(\underline{C})$.

LEMMA 11 *(cf. [7], (2.5.6)). Let $\underline{C}$ be an abelian category and $S(\underline{C})$ the class of sections, $R(\underline{C})$ the class of retractions in $\underline{C}$. Then $S(\underline{C}) \times R(\underline{C})$ is po-stable and pb-stable as well.*

PROOF. a) Let

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, dash, "\beta"'] & B \arrow[d, dash, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

be a pushout in $\underline{C}$ where $\alpha \in S(\underline{C})$, $\beta \in R(\underline{C})$. It follows $\delta \in S(\underline{C}) \subset M(\underline{C})$, and therefore, by Lemma 9, this pushout is also a pullback. We now obtain an exact sequence

$$(\mathrm{S}) \qquad 0 \longrightarrow A \xrightarrow{[\alpha, \beta]} B \oplus C \xrightarrow{<\gamma, -\delta>} D \longrightarrow 0.$$

Since $\alpha, \delta \in S(\underline{C})$, $\beta \in R(\underline{C})$, there exist $r_\alpha \in \underline{C}(B, A)$, $r_\delta \in \underline{C}(D, C)$ and $s_\beta \in \underline{C}(C, A)$ where

$$r_\alpha \alpha = 1_A, \quad r_\delta \delta = 1_C \quad \text{and} \quad \beta s_\beta = 1_C.$$

We define

$$\sigma \in \underline{C}(B, A) \quad \text{by} \quad \sigma := r_\alpha + s_\beta r_\delta \gamma.$$

It is easy to check that $<\sigma, -s_\beta>[\alpha, \beta] = 1_A$, i.e. the sequence (S) splits. Hence there exists a section $s \in \underline{C}(D, B \oplus C)$ for $<\gamma, -\delta>$. We define

$$s_\gamma \in \underline{C}(D, B) \quad \text{by} \quad s_\gamma := <1_B, -\alpha s_\beta> s,$$

and one verifies that $\gamma s_\gamma = 1_D$, i.e. $\gamma \in R(\underline{C})$.

b) The dual statement has a dual proof. □

Now, let $\underline{A}$ be an abelian category and $\partial \underline{A}$ be the category of chain complexes over $\underline{A}$ provided with the homotopy system defined in [4], 2; then it is well-known that $\partial \underline{A}$ is an abelian category, that its homotopy system fulfills the Kan-conditions E(2), E(3) and that the cylinder functor I has a right adjoint.

Further, by [4], Proposition 1 and its dual, a morphism f in $\partial \underline{A}$ is a cofibration (fibration) in $\partial \underline{A}$ iff f_q is a section (retraction) in $\underline{A}$ for each $q \in Z$.

Finally, one verifies that a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

is a pushout (pullback) in $\partial \underline{A}$ iff

$$\begin{tikzcd} A_q \arrow[r, "\alpha_q"] \arrow[d, "\beta_q"'] & B_q \arrow[d, "\gamma_q"] \\ C_q \arrow[r, "\delta_q"] & D_q \end{tikzcd}$$

is a pushout (pullback) in $\underline{A}$ for each $q \in Z$.

Thus, in view of Lemma 11 above, we obtain immediately:

COROLLARY 12 *(cf.* [7], *(2.5.8)). $C(\partial \underline{A}) \times F(\partial \underline{A})$ is po-stable and pb-stable as well.*

Since

$$C(\partial \underline{A}) \subset M(\partial \underline{A}) = N(\partial \underline{A}), \quad F(\partial \underline{A}) \subset E(\partial \underline{A}) = Con(\partial \underline{A}),$$

we conclude from Theorem 10:

COROLLARY 13 *(cf.* [7], *(2.5.9)). Every HPO in $\partial \underline{A}$ is a HPB in $\partial \underline{A}$, and vice versa.*

Let

(R) $$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

be the composition of two homotopy commutative squares in $\underline{C}$.

LEMMA 14 *(cf.* [7], *(1.2.8), (1.2.10), (1.4.7)). a) Let* (1) *be a HPO. Then* (2) *is a HPO iff* (R) *is a HPO.*

b) Let (2) *be a HPB. Then* (1) *is a HPB iff* (R) *is a HPB.*

COROLLARY 15. *Let* $\underline{C}$ *have zero-morphisms,*

$$C(\underline{C}) \subset N(\underline{C}), \quad F(\underline{C}) \subset Con(\underline{C}),$$

and let $C(\underline{C}) \times F(\underline{C})$ *be po-stable and pb-stable in* $\underline{C}$. *Then*

a) If two of the diagrams (1), (2), (R) *are HPOs, then so is the third.*

b) If two of the diagrams (1), (2), (R) *are HPBs, then so is the third.*

PROOF. a) By Lemma 14, we have only to prove the case where (2) and (R) are HPOs. By Theorem 10 a and Lemma 14, the square (1) is a HPB, and, by Theorem 10 b, we conclude that (1) is a HPO.

b) This is the dual of a. □

COROLLARY 16 *(Cube Theorems). Under the circumstances of Corollary 15 we consider the homotopy commutative cube* (C) *of Definition 2.*

a) If the front and left faces are HPBs and if the top and bottom faces are HPOs, then the right and rear faces are HPBs.

b) If the right and rear faces are HPOs, and if the top and bottom faces are HPBs, then the front and left faces are HPOs.

c) If all vertical faces are HPBs, and if the bottom face is a HPO, then the top face is a HPO.

d) If all vertical faces are HPOs, and if the top face is a HPB, then the bottom face is a HPB.

PROOF. a to d are easily proved by a «diagram chasing». We prove a for example. By Theorem 10, we consider the left face of the cube to be a HPO. Since, by Lemma 14, the composition of the left and bottom faces is a HPO, it follows that the composition of the top and right faces is a HPO (using Lemma 3 and the fact that the cube is homotopy commutative). Again by Lemma 14, the right face is a HPO and therefore, by Theorem 10, is a HPB too. Similarly, we prove that the rear face is a HPB. □

REMARK. In particular, by Corollary 13, Mather's cube theorems and their duals hold in the category of chain complexes (cf. [8], Section 3).

By Corollary 16, imitating and dualizing the proof of [6], Theorem 1, we get

COROLLARY 17 *(Commuting homotopy limits and colimits). Given a homotopy commutative diagram in* $\underline{C}$,

$$\begin{tikzcd}
A_1 \arrow[r] & C_1 & B_1 \arrow[l] \\
A_0 \arrow[u] \arrow[r] \arrow[d] & C_0 \arrow[u] \arrow[d] & B_0 \arrow[l] \arrow[u] \arrow[d] \\
A_2 \arrow[r] & C_2 & B_2 \arrow[l]
\end{tikzcd}$$

under the circumstances of Corollary 15 (for example, in the category of chain complexes) homotopy pullbacks and pushouts commute (in the sense of [6]*) if either*

a) the two left-hand or two right-hand squares are HPBs,

or b) the two top or bottom squares are HPOs.

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