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JENÖ SZIGETI

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ON LIMITS AND COLIMITS IN THE KLEISLI CATEGORY

by Jenö SZIGETI

1. INTRODUCTION.

Given a triple (monad) $R = (R, \alpha, \beta)$ on the category \mathcal{A} the standard constructions of the Eilenberg-Moore category \mathcal{A}^R and the Kleisli category \mathcal{K}_R are well known. The computation of \mathcal{A}^R -limits easily can be derived from the computation of the corresponding \mathcal{A} -limits. On the other hand the case of \mathcal{A}^R -colimits proved to be far more complicated. The various \mathcal{A}^R -colimits (and $\mathcal{A}(R)$ -colimits of R -algebras) were thoroughly investigated in a lot of papers (e. g. in [1-3, 5, 9, 10]). In this way the raising of the similar questions for the Kleisli category \mathcal{K}_R is quite natural. This paper makes an attempt to get closer to the problem of the completeness and the cocompleteness of \mathcal{K}_R . The canonical functor $S: \mathcal{K}_R \rightarrow \mathcal{A}^R$ will play a central role in our development. It will be proved in 2 that $\dashv S$ is equivalent to the cocompleteness of \mathcal{K}_R under certain circumstances. In 3 we shall use the assumption $S \dashv$ in order to prove completeness for \mathcal{K}_R . Mention must be made that these results are powerless in concrete instances. One can regard them as an alternative approach to the problem.

Part 4 deals with the existence of a left adjoint to R , where R is the base functor of some R . The main result of 4 states that if \mathcal{A} has coequalizers of all pairs then the relative adjointness $\dashv_R R$ is equivalent to $\dashv R$.

2. COCOMPLETENESS.

Given a triple $R = (R, \alpha, \beta)$ on the category \mathcal{A} an adjoint pair $F \dashv U$ consisting of functors $F: \mathcal{A} \rightarrow \mathcal{B}$, $U: \mathcal{B} \rightarrow \mathcal{A}$ with unit $\epsilon: 1_{\mathcal{A}} \rightarrow U \circ F$ and counit $\delta: F \circ U \rightarrow 1_{\mathcal{B}}$ is said to be an R -adjunction on \mathcal{A} if

$$R = U \circ F, \quad \alpha = \epsilon \quad \text{and} \quad \beta = U \delta F.$$

The Kleisli category \mathcal{K}_R of R is defined in the following manner (cf. [8, 11, 12]). Objects are the same as in \mathcal{A} , namely $|\mathcal{K}_R| = |\mathcal{A}|$. A morphism $r: a \rightarrow a'$ in \mathcal{K}_R (the notation \rightarrow for \mathcal{K}_R -arrows will be used throughout this paper) is given by an \mathcal{A} -morphism $r: a \rightarrow R a'$. The $a \rightarrow a$ unit in \mathcal{K}_R is $\alpha_a: a \rightarrow R a$ and the \mathcal{K}_R -composition ∇ for

$$a \xrightarrow{r} a' \xrightarrow{r'} a''$$

is given by $r' \nabla r = \beta_{a''} \circ (R r') \circ r$. The functors of the Kleisli R -adjunction

$$F_R: \mathcal{A} \rightarrow \mathcal{K}_R \quad \text{and} \quad U_R: \mathcal{K}_R \rightarrow \mathcal{A}$$

are defined by

$$a \xrightarrow{f} a' \quad \Big| \xrightarrow{F_R} \quad a \xrightarrow{\alpha_a \circ f} a' \quad \text{and} \quad a \xrightarrow{r} a' \quad \Big| \xrightarrow{U_R} \quad R a \xrightarrow{\beta_{a'} \circ R r} R a'$$

U_R is faithful since $U_R r$ uniquely determines r by

$$(U_R r) \circ \alpha_a = \beta_{a'} \circ (R r) \circ \alpha_a = \beta_{a'} \circ \alpha_{R a'} \circ r = l_{R a'} \circ r = r.$$

The well known initiality of the Kleisli R -adjunction means that any R -adjunction $F \dashv U$ involves a unique $\mathcal{K}_R \rightarrow \mathcal{B}$ functor making (2:1) commute.

$$\begin{array}{ccc}
 & \mathcal{K}_R & \\
 F_R \nearrow & & \searrow U_R \\
 \mathcal{A} & & \mathcal{A} \\
 F \searrow & & \nearrow U \\
 & \mathcal{B} &
 \end{array} \tag{2:1}$$

Take $F \dashv U$ to be the Eilenberg-Moore R -adjunction with $F^R: \mathcal{A} \rightarrow \mathcal{A}^R$, and $U^R: \mathcal{A}^R \rightarrow \mathcal{A}$, then there is a unique $S: \mathcal{K}_R \rightarrow \mathcal{A}^R$ such that (2:2) commutes:

$$\begin{array}{ccc}
 & \mathcal{K}_R & \\
 F_R \nearrow & & \searrow U_R \\
 \mathcal{A} & & \mathcal{A} \\
 F^R \searrow & & \nearrow U^R \\
 & \mathcal{A}^R &
 \end{array} \tag{2:2}$$

The existence and the uniqueness of such a functor S also follow from the terminal property of the Eilenberg-Moore R -adjunction. A more explicit

definition for S is the following

$$a \xrightarrow{r} a' \quad \vdash^S \langle Ra, \beta_a \rangle \xrightarrow{\beta_{a'} \circ Rr} \langle Ra', \beta_{a'} \rangle .$$

$U^R \circ S = U_R$ implies that S is faithful just as U_R .

2.1. PROPOSITION. Let \mathcal{A}_R have coequalizers of all pairs (with a common coretraction) then S has a left adjoint.

PROOF. Since $F_R \dashv U_R = U^R \circ S$ an application of Johnstone's adjoint lifting theorem can give $\dashv S$ (\mathcal{A}_R can be regarded as \mathcal{A}_R^1 , where 1 is the trivial triple on \mathcal{A}_R). //

The question of \mathcal{A}_R -coproducts doesn't cause difficulties.

2.2. PROPOSITION. Let \mathcal{A} have coproducts then \mathcal{A}_R has coproducts.

PROOF. For the objects $a_i \in |\mathcal{A}_R|$ ($i \in I$) let $p_i : a_i \rightarrow x$ ($i \in I$) be an \mathcal{A} -coproduct, then $a_x \circ p_i : a_i \rightarrow x$ ($i \in I$) will be the required \mathcal{A}_R -coproduct. //

In (2.3) a certain converse of Proposition (2.1) will be established.

2.3. THEOREM. Let \mathcal{A} be cocomplete (with an initial object), \mathcal{A}^R have coequalizers of all pairs (with a common coretraction) and suppose that each partial R -algebra admits a free completion in $\mathcal{A}(R)$ (consequently $\mathcal{A}(R)$ has free algebras). If S has a left adjoint then \mathcal{A}_R is cocomplete.

To prove the above theorem we need the following lemma.

2.4. LEMMA. Let \mathcal{A} have finite coproducts and suppose that each partial T -algebra admits a free completion in $\mathcal{A}(T)$ (apart from this $T : \mathcal{A} \rightarrow \mathcal{A}$ is arbitrary). Then the canonical embedding functor $B^T : \mathcal{A}(T) \rightarrow (T \downarrow \mathcal{A})$ has a left adjoint.

PROOF. For an object $u : Ta \rightarrow b$ in $(T \downarrow \mathcal{A})$ consider the following partial T -algebra

$$T(a \amalg b) \xleftarrow{Tj_a} Ta \xrightarrow{u} b \xrightarrow{j_b} a \amalg b,$$

where j_a and j_b are coproduct injections. The free completion (2.3) of this partial T -algebra in $\mathcal{A}(T)$ immediately yields an initial object in the

comma category $(T a \xrightarrow{u} b \downarrow B^T)$. //

$$\begin{array}{ccc}
 & T a & \\
 T j_a \swarrow & & \searrow u \\
 T(a \amalg b) & & a \amalg b \\
 T s^* \downarrow & & \downarrow s^* \\
 T z^* & \xrightarrow{v^*} & z^*
 \end{array} \quad (2.3)$$

PROOF OF (2.3). Since \mathcal{U}_R has coproducts by (2.2) we have to deal only with coequalizers. Consider a set $r_i: a \rightarrow a'$ ($i \in I$) of parallel \mathcal{U}_R -morphisms. Form the coequalizer

$$\begin{array}{ccc}
 a & \xrightarrow{r_i} & R a' \xrightarrow{e} x \\
 & \vdots & \\
 & &
 \end{array}$$

in \mathcal{U} and let (2.4) represent an initial object in

$$(R a' \xrightarrow{e} x \downarrow B^R \circ E^R \circ S),$$

where

$$\mathcal{U}^R \xrightarrow{E^R} \mathcal{U}(R) \quad \text{and} \quad \mathcal{U}(R) \xrightarrow{B^R} (R \downarrow I \downarrow \mathcal{U})$$

are canonical embeddings

$$\begin{array}{ccc}
 R a' & \xrightarrow{R e^*} & R^2 a^* \\
 e \downarrow & & \downarrow \beta_{a^*} \\
 x & \xrightarrow{t^*} & R a^*
 \end{array} \quad (2.4)$$

The existence of this initial object is clear since $\downarrow B^R$ by (2.4) and $\downarrow E^R$ by the assumptions that \mathcal{U}^R has coequalizers of all pairs (with a common coretraction) and that $\mathcal{U}(R)$ has free algebras (Johnstone's adjoint lifting Theorem works again because of the free R -algebra adjunction is always monadic). We claim that $e^*: a' \rightarrow a^*$ is the \mathcal{U}_R -coequalizer of the morphisms r_i ($i \in I$). (2.4) proves that

$$e^* \nabla r_i = t^* \circ e \circ r_i = t^* \circ e \circ r_m = e^* \nabla r_m$$

for all $i, m \in I$. Suppose that $\hat{e} \nabla r_i = \hat{e} \nabla r_m$ ($i, m \in I$) for an \mathcal{U}_R -morphism $\hat{e}: a' \rightarrow \hat{a}$. The \mathcal{U} -coequalizer property of e gives a unique $\hat{t}: x \rightarrow R \hat{a}$

making (2:5) commute

$$\begin{array}{ccc}
 R a' & \xrightarrow{R \hat{e}} & R^2 \hat{a} \\
 e \downarrow & & \downarrow \beta_{\hat{a}} \\
 x & \xrightarrow{\hat{t}} & R \hat{a}
 \end{array} \quad (2:5)$$

Now

$$\langle \hat{e}, \hat{t} \rangle : \langle a', e, x \rangle \rightarrow B^R E^R S \hat{a}$$

is in $(R \downarrow 1_{\mathcal{Q}})$, so there exists a unique $k : a^* \dashrightarrow \hat{a}$ in \mathcal{Q}_R such that (2:6) commutes.

$$\begin{array}{ccc}
 & & B^R E^R S a^* \\
 \langle e^*, t^* \rangle & \nearrow & \downarrow B^R E^R S k \\
 \langle a', e, x \rangle & & B^R E^R S \hat{a} \\
 \langle \hat{e}, \hat{t} \rangle & \searrow &
 \end{array} \quad (2:6)$$

Using the facts that S is faithful and e is epimorphic in \mathcal{Q} one can easily obtain that (2:6) and (2:7) are equivalent for a $k : a^* \dashrightarrow \hat{a}$. //

$$\begin{array}{ccc}
 & & a^* \\
 & \nearrow e^* & \downarrow k \\
 a' & & \hat{a} \\
 & \searrow \hat{e} &
 \end{array} \quad (2:7)$$

3. COMPLETENESS.

3.1. THEOREM. Let \mathcal{Q} be complete and suppose that S has a right adjoint, then \mathcal{Q}_R is complete.

PROOF. Given a functor $D : \mathcal{D} \rightarrow \mathcal{Q}_R$ observe that an \mathcal{Q}_R -cone $\xi : a \dashrightarrow D$ is the same thing as an \mathcal{Q} -cone $\xi : a \rightarrow U_R \circ D$ and vice versa.

If $q : x \rightarrow U_R \circ D$ is a limit cone in \mathcal{Q} then there is a unique $b : R x \rightarrow x$, making the diagram (3:1) commute.

$$\begin{array}{ccc}
 x & \xrightarrow{q} & U_R \circ D \\
 b \uparrow & & \uparrow \beta D \\
 R x & \xrightarrow{R q} & R \circ U_R \circ D
 \end{array} \quad (3:1)$$

The adjointness $S \dashv$ yields a terminal object $\nu^*: Sa^* \rightarrow \langle x, b \rangle$, in $(S \downarrow \langle x, b \rangle)$. We claim that

$$\xi^*: a^* \xrightarrow{\alpha_{a^*}} Ra^* \xrightarrow{\nu^*} x \xrightarrow{q} U_R \circ D$$

is an \mathcal{U}_R -limit cone for D . Let $\xi: a \rightarrow D$ be a cone, then there is a unique $t: a \rightarrow x$ with $q \circ t = \xi$. Easily it can be seen that $b \circ Rt: Sa \rightarrow \langle x, b \rangle$ is a morphism in \mathcal{U}^R , so there exists a unique $k: a \rightarrow a^*$ making (3:2) commute.

$$\begin{array}{ccc} Sa & \xrightarrow{b \circ Rt} & \langle x, b \rangle \\ Sk \downarrow & & \nearrow \nu^* \\ Sa^* & & \end{array} \quad (3:2)$$

(3:1) makes q to be a $\langle x, b \rangle \rightarrow S \circ D$ limit cone in \mathcal{U}^R , consequently for a morphism $k: a \rightarrow a^*$ (3:2) is equivalent to (3:3).

$$\begin{array}{ccc} Sa & \xrightarrow{b \circ Rt} & \langle x, b \rangle \\ Sk \downarrow & & \nearrow \nu^* \\ Sa^* & & \end{array} \xrightarrow{q} S \circ D, \quad (3:3)$$

$$a \xrightarrow{\xi_d} Dd \xrightarrow{S} Sa \xrightarrow{q_d \circ b \circ Rt} SDd \text{ and } a^* \xrightarrow{\xi_d^*} Dd \xrightarrow{S} Sa^* \xrightarrow{q_d \circ \nu^*} SDd$$

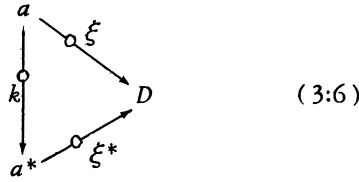
are proved by (3:4) and (3:5) respectively.

$$\begin{array}{ccccc} Ra & \xrightarrow{R\xi_d} & R^2Dd & \xrightarrow{\beta Dd} & RDd \\ & \searrow Rt & \uparrow Rq_d & & \uparrow q_d \\ & & Rx & \xrightarrow{b} & x \end{array} \quad (3:4)$$

$$\begin{array}{ccccccc} Ra^* & \xrightarrow{R\alpha_{a^*}} & R^2a^* & \xrightarrow{R\nu^*} & Rx & \xrightarrow{Rq_d} & R^2Dd \xrightarrow{\beta Dd} RDd \\ & \searrow I_{Ra^*} & \downarrow \beta_{\alpha^*} & & \downarrow b & & \nearrow q_d \\ & & Ra^* & \xrightarrow{\nu^*} & x & & \end{array} \quad (3:5)$$

Since S is faithful we obtained that for a $k: a \rightarrow a^*$:

$$(3:2) \iff (3:3) \iff (3:6).$$



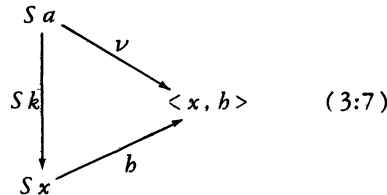
//

3.2. THEOREM. Let \mathcal{Q}_R have coequalizers of all sets of parallel morphisms and suppose that $U_R: \mathcal{Q}_R \rightarrow \mathcal{Q}$ preserves these coequalizers, then S has a right adjoint.

PROOF. At first we prove that $b: Sx \rightarrow \langle x, b \rangle$ is a weak terminal object in $(S \downarrow \langle x, b \rangle)$. For an object $\nu: Sa \rightarrow \langle x, b \rangle$ in $(S \downarrow \langle x, b \rangle)$ define the morphism $k: a \rightarrow x$ in \mathcal{Q}_R as

$$a \xrightarrow{\alpha_a} Ra \xrightarrow{\nu} x \xrightarrow{\alpha_x} Rx.$$

Clearly (3:7) commutes since $b \circ R\nu = \nu \circ \beta_a$.



Let $e^*: x \rightarrow a^*$ be the \mathcal{Q}_R -coequalizer of those morphisms $r: x \rightarrow x$ for which $b \circ (Sr) = b$. Now

$$\begin{array}{ccccc}
 U_R x & \xrightarrow{U_R r} & U_R x & \xrightarrow{U_R e^*} & U_R a^* \\
 & \vdots & & &
 \end{array}$$

is a coequalizer diagram in \mathcal{Q} by the preservation property of U_R . $RU_R e^*$ is also a coequalizer since $R = U_R \circ F_R$ and F_R preserves coequalizers by $F_R \dashv U_R$. Thus $RU_R e^*$ is proved to be an epimorphism. Accordingly, (3:8) is a coequalizer diagram in \mathcal{Q}^R , i.e. S preserves the above mentioned (and any other) coequalizer.

Since $b \circ (Sr) = b$ for all the considered « r »'s there exists a unique $\nu^*: Sa^* \rightarrow \langle x, b \rangle$ in \mathcal{Q}^R with $\nu^* \circ (Se^*) = b$. A standard terminal object

$$\begin{array}{ccccc}
 U_{\mathbb{R}} x & \xrightarrow{U_{\mathbb{R}} r} & U_{\mathbb{R}} x & \xrightarrow{U_{\mathbb{R}} e^*} & U_{\mathbb{R}} a^* \\
 \beta_x \uparrow & & \vdots & & \vdots \\
 & & & & \beta_x \uparrow \\
 & & & & \vdots \\
 & & & & \beta_{a^*} \uparrow \\
 R U_{\mathbb{R}} x & \xrightarrow{R U_{\mathbb{R}} r} & R U_{\mathbb{R}} x & \xrightarrow{R U_{\mathbb{R}} e^*} & R U_{\mathbb{R}} a^*
 \end{array} \quad (3:8)$$

argument can prove that $\nu^*: S a^* \rightarrow \langle x, b \rangle$ is terminal in $(S \downarrow \langle x, b \rangle)$. //

3.3. COROLLARY. Let \mathcal{A} be complete, $\mathcal{A}_{\mathbb{R}}$ have coequalizers of all sets of parallel morphisms and suppose that $U_{\mathbb{R}}$ preserves these coequalizers, then $\mathcal{A}_{\mathbb{R}}$ is complete. //

4. LEFT ADJOINTS BY DENSITY.

We start this part with two simple but extremely powerful lemmas.

4.1. LEMMA. Let $H: \mathcal{H} \rightarrow \mathcal{B}$ be a functor and $F: \mathcal{A} \rightarrow \mathcal{B}$, $U: \mathcal{B} \rightarrow \mathcal{A}$ be the functors of an adjoint pair $F \dashv U$ with unit $\epsilon: 1_{\mathcal{A}} \rightarrow U \circ F$. Suppose that $(a \downarrow U \circ H)$ has an initial object, then there exists an initial object in $(F a \downarrow H)$.

PROOF. If $f^*: a \rightarrow U H \omega^*$ is initial in $(a \downarrow U \circ H)$, then $\langle H \omega^*, f^* \rangle$ is in $(F a \downarrow H)$. The initial property of $\epsilon_a: a \rightarrow U F a$ gives a unique morphism $g^*: F a \rightarrow H \omega^*$ in \mathcal{B} making (4:1) commute.

$$\begin{array}{ccc}
 & & U F a \\
 & \nearrow \epsilon_a & \downarrow U g^* \\
 a & & \\
 & \searrow f^* & \\
 & & U H \omega^*
 \end{array} \quad (4:1)$$

Clearly $\langle \omega^*, g^* \rangle$ is initial in $(F a \downarrow H)$. //

4.2. LEMMA. Let H be as in (4.1), $D: \mathcal{D} \rightarrow \mathcal{B}$ be a functor with a colimit cocone $\gamma: D \rightarrow b$ and suppose that $(D d \downarrow H)$ has an initial object for all $d \in |\mathcal{D}|$. If \mathcal{H} is \mathcal{D} -cocomplete, then there exists an initial object in $(b \downarrow H)$.

PROOF. For $d \in |\mathcal{D}|$ let $\langle \omega(d), \eta_d \rangle$ be an initial object in $(D d \downarrow H)$ and define the functor $D^*: \mathcal{D} \rightarrow \mathcal{H}$ by

$$d \rightarrow d' \quad \downarrow D^* \quad \omega(d) \rightarrow \omega(d'),$$

where $\omega(d) \rightarrow \omega(d')$ is the unique morphism making (4:2) commute.

$$\begin{array}{ccc} D d & \xrightarrow{\eta d} & H \omega(d) \\ \downarrow & & \downarrow \\ D d' & \xrightarrow{\eta d'} & H \omega(d') \end{array} \quad (4:2)$$

Thus η becomes a $D \rightarrow H \circ D^*$ natural transformation. If $\gamma^*: D^* \rightarrow \omega^*$ is a colimit cocone then there is a unique $g^*: b \rightarrow H \omega^*$ making (4:3) commute.

$$\begin{array}{ccccc} D & \xrightarrow{\eta} & H \circ D^* & \xrightarrow{H \gamma^*} & H \omega^* \\ & \searrow \gamma & & \nearrow g^* & \\ & & b & & \end{array} \quad (4:3)$$

The purely technical details of the verification that $\langle \omega^*, g^* \rangle$ is initial in $(b \downarrow H)$ are omitted. //

Now we can prove the main result of Section 4.

4.3. THEOREM. Let \mathcal{A} have coequalizers of all pairs and $R: \mathcal{A} \rightarrow \mathcal{A}$ be the base functor of some triple $R = (R, \alpha, \beta)$. Suppose that R has a left adjoint relative to R , i. e. $(R a \downarrow R)$ has an initial object for all $a \in |\mathcal{A}|$. Then there is a left adjoint to R , namely $\dashv R$.

PROOF. Since $R = U_R \circ F_R$ with the functors of the Kleisli R -adjunction $F_R \dashv U_R$ one can apply (4.1) in order to obtain an initial object in $(F_R R a \downarrow F_R)$. $F_R R a = R a$ and

$$\begin{array}{ccccc} & \xrightarrow{\alpha_{Ra}} & & & \\ R a & \xrightarrow{\quad \circ \quad} & R a & \xrightarrow{I_{Ra}} & a \\ & \xleftarrow{R \alpha_a} & & & \end{array}$$

is a coequalizer diagram in \mathcal{A}_R , so (4.2) gives an initial object in $(a \downarrow F_R)$. Thus $\dashv F_R$ and $\dashv U_R$ implies $\dashv R$. //

4.4. COROLLARY. Let \mathcal{A} have coequalizers of all pairs and $R: \mathcal{A} \rightarrow \mathcal{A}$ be the base functor of some triple $R = (R, \alpha, \beta)$. Suppose that for each object $a \in |\mathcal{A}|$ there is an integer $n \geq 1$ such that $(R^n a \downarrow R)$ has an initial object. Then there is a left adjoint to R , namely $\dashv R$. //

To illustrate the force of Lemmas 4.1 and 4.2 we give a very simple proof for an important theorem due to W. Tholen (see in [15]).

4.5. THEOREM. Let H, F, U be as in (4.1) with counit $\delta : F \circ U \rightarrow 1_{\mathfrak{B}}$. If (i) or (j) holds, then $\dashv H$ is equivalent to $\dashv U \circ H$.

(i) \mathfrak{K} has coequalizers of all pairs and each $\delta_b (b \in |\mathfrak{B}|)$ is a regular epimorphism;

(j) \mathfrak{K} has coequalizers of all pairs with a common coretraction and each $\delta_b (b \in |\mathfrak{B}|)$ is a regular epimorphism with a kernel pair.

PROOF. $\dashv H \Rightarrow \dashv U \circ H$ is trivial. Set $\dashv U \circ H$. If

$$\bar{b} \begin{array}{c} \xrightarrow{\pi_0} \\ \xrightarrow{\pi_1} \end{array} FUb \xrightarrow{\delta_b} b$$

is a coequalizer in \mathfrak{B} then so is (4:4) since $\delta_{\bar{b}}$ is epimorphic.

$$FU\bar{b} \begin{array}{c} \xrightarrow{\pi_0 \circ \delta_{\bar{b}}} \\ \xrightarrow{\pi_1 \circ \delta_{\bar{b}}} \end{array} FUb \xrightarrow{\delta_b} b \quad (4:4)$$

By (4.1) there is an initial object in $(FU\bar{b} \dashv H)$ and in $(FUb \dashv H)$. Let $\gamma : D \rightarrow b$ of (4.2) represent the coequalizer (4:4), then (4.2) gives an initial object in $(b \dashv H)$. If in addition

$$\bar{b} \begin{array}{c} \xrightarrow{\pi_0} \\ \xrightarrow{\pi_1} \end{array} FUb \xrightarrow{\delta_b} b$$

is a kernel pair, then $(FU\bar{b}) \circ F\epsilon_{Ub}$ is a common coretraction of $\pi_0 \circ \delta_{\bar{b}}$ and $\pi_1 \circ \delta_{\bar{b}}$, where $t : FUb \rightarrow \bar{b}$ is the unique morphism making (4:5) commute.

$$\begin{array}{ccccc} & & FUb & \xrightarrow{\delta_b} & b \\ & \nearrow^{1_{FU\bar{b}}} & \uparrow \pi_0 & & \uparrow \delta_b \\ & & \bar{b} & \xrightarrow{\pi_1} & FUb \\ & \nearrow^t & & & \\ FUb & & & & \\ & \searrow_{1_{FU\bar{b}}} & & & \end{array} \quad (4:5)$$

Now $\gamma : D \rightarrow b$ (and consequently $\gamma^* : D^* \rightarrow \omega^*$) represents a coequalizer of a pair with a common coretraction. Essentially we proved that the full

subcategory of \mathcal{B} consisting of all objects of the form Fa ($a \in |\mathcal{A}|$) is a dense subcategory. //

REFERENCES.

1. ADAMEK, J., Colimits of algebras revisited, *Bull. Austr. Math. Soc.* 15 (1976).
2. ADAMEK, J. & TRNKOVA, V., Varietors and machines, *COINS Technical Report 78-6*, Univ. of Mass. at Amherst (1978).
3. ADAMEK, J. & KOUBEK, V., Simple construction of colimits of algebras, to appear.
4. ANDREKA, H., NEMETI, I. & SAIN, I., Cone injectivity and some Birkhoff-type theorems in categories, *Proc. Coll. Universal Algebra (Esztergom 1977)*, Coll. Soc. J. Bolyai, North Holland, to appear.
5. BARR, M., Coequalizers and free triples, *Math. Z.* 116 (1970).
6. EILENBERG, S. & MOORE, J., Adjoint functors and triples, *Ill. J. Math.* 9 (1965).
7. JOHNSTONE, P. T., Adjoint lifting theorems for categories of algebras, *Bull. London Math. Soc.* 7 (1975).
8. KLEISLI, H., Every standard construction is induced by a pair of adjoint functors, *Proc. A. M. S.* 16 (1965).
9. KOUBEK, V. & REITERMAN, J., Categorical constructions of free algebras, colimits and completions of partial algebras, *J. Pure Appl. Algebra* 14 (1979).
10. LINTON, F. E. J., Coequalizers in categories of algebras, *Lecture Notes in Math.* 80, Springer (1969).
11. MACLANE, S., *Categories for the working mathematician*, Springer GTM 6 1971.
12. MANES, E. G., *Algebraic Theories*, Springer GTM 26, 1976.
13. THOLEN, W., Adjungierte Dreiecke, Colimites und Kan-Erweiterungen, *Math. Ann.* 217 (1975).
14. THOLEN, W., On Wyler's taut lift theorem, *Gen. Top. Appl.* 8 (1978).
15. THOLEN, W., WISCHNEWSKY, M. B. & WOLFF, H., Semi-topological functors III: Lifting of monads and adjoint functors, *Seminarberichte* 4, Fachbereich für Math., Fernuniversität Hagen (1978).
16. ULMER, F., Properties of dense and relative adjoint functors, *J. Algebra* 8 (1968).

Mathematical Institute
Hungarian Academy of Sciences
Reáltanoda u. 13-15
H-1053 BUDAPEST. HONGRIE