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MINIMAL ATLASES OF MANIFOLDS *

by Alberto CAVICCHIOLI and Luigi GRASSELLI

RÉSUMÉ. On montre que chaque "ball-intersection atlas" minimal d'une n-variété M connexe et linéaire par morceaux a exactement n boules si la frontière de M est non vide. Ceci améliore divers résultats connus relatifs aux recouvrements par boules minimaux des variétés.

1. INTRODUCTION.

Given a connected compact n-manifold M, a natural invariant of M is the minimal number of balls which are needed to cover M.

Following [SN] the *Ljusternik-Schnirelmann category* (resp. the *strong Ljusternik-Schnirelmann category*), written cat M (resp. C(M)), is the minimal number of open contractible subsets (resp. of balls) of M which suffice to cover M. Obviously

$$C(M) \geqq \text{cat } M .$$

W. Singhof proved that C(M) = cat M if cat M is not too small compared with the dimension of M.

If M is a closed connected combinatorial n-manifold ($n > 0$) which is geometrically $[n/r]$-connected, $r \geqq 2$, then M can be covered by r combinatorial balls [Z2]. If M is r-connected and $r \leqq n-3$, then $[n/(r+1)]+1$ balls suffice to cover M as was later proved by E.C. Zeeman for PL-manifolds [Z1] and by E. Luft in the topological case [L].

Classical results for particular classes of spaces are :

1° A closed piecewise-linear 3-manifold covered by 3 open 3-balls is a 3-sphere-with-handles [HM].

2° If M is a locally trivial n-dimensional sphere bundle over a sphere, having a cross-section, then M admits coverings by 3 open n-balls [M1].

Theorems which improve some quoted statements are obtained in [M2, PD, S1, S2] by making use of *residual sets,* a concept introduced in [DH].

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Relations between the Poincaré conjecture and ball coverings arguments are studied in [OS, Z2].

In order to cover a manifold with balls whose intersections are nice, R. Osborne and J. Stern proved this theorem : If M is a closed k-connected topological n-manifold and $q = \min\{k, n-3\}$, then M can be covered by p open balls if $p(q+1) > n$. Further, these balls may be chosen so that the intersection of any collection of them is $(q-1)$-connected.

The boundary case is also considered in [OS, KT].

In the present paper, we prove that each minimal "ball-intersection atlas" of a connected piecewise-linear n-manifold M has exactly n balls if ∂M is non-void. This improves some results of [OS] and [KT] in the piecewise-linear category.

2. NOTATIONS.

Let Δ_n be the set $\{0, 1, ..., n\}$ and $N_n = \Delta_n - \{0\}$. The symbol $\#A$ means the cardinality of the set A.

All (compact) spaces and maps considered belong to the piecewise-linear (PL) category in the sense of [H] or [Z1]. The prefix PL will always be omitted.

The ball-complexes B_1, B_2 are said to be *abstractly isomorphic* if there exists a bijection $f : B_1 \to B_2$ preserving the face-incidence relation.

An *n-pseudocomplex* K is an n-dimensional principal ball-complex in which every r-ball, considered with all their faces, is abstractly isomorphic with the complex underlying an r-simplex ([HW], p. 49). K is said to be a *pseudodissection* of the polyhedron $|K|$. By $S_r(K)$ and K^s, we respectively denote the set of all the r-balls of K and the s-skeleton of K. We shall also call *r-simplex* (resp. *vertex*) each r-ball (resp. 0-ball) of K.

Given a simplex s in an n-pseudocomplex K, the *disjoined star* std(s, K) is defined to be the disjoint union of the n-simplexes of K containing s, with re-identification of the $(n-1)$-faces containing s and of their faces. The subcomplex

$$\mathrm{lkd}(s, K) = \{\tau \in \mathrm{std}(s, K) \mid \tau \cap s = \emptyset\}$$

is called the *disjoined link* of s in K. If K is a pseudodissection of a manifold, the star st(s, K) and the link lk(s, K) of a simplex s in K are not necessarily balls or spheres ; however, std(s, K) and lkd(s, K) are the balls or spheres obtained by a minimal set of severings on st(s, K) and lk(s, K) respectively. A vertex v of an n-pseudocomplex K will be called a *cone-vertex* if it belongs to all n-simplexes of K (or, equivalently, if st(v, K) = K).

An r-simplex s of a closed n-pseudomanifold K (cf. [SP]) is said to be *regular* (resp. *singular*) if lkd(s, K) is (resp. is not) a combinatorial $(n-r-1)$-sphere.

An *identification system* of a principal n-pseudocomplex K is defined to be a set G of simplicial isomorphisms such that, for any pair

$$s_\alpha^{n-1},\ s_\beta^{n-1} \in S_{n-1}(K),$$

there exists at most one map

$$\varphi_{\alpha\beta} : \bar{s}_\alpha^{n-1} \to \bar{s}_\beta^{n-1}$$

belonging to G. Let $\sim_G$ be the equivalence relation on

$$S(K) = \bigcup_{r \in \Delta_n} S_r(K)$$

defined as follows :

$s_\alpha^h \sim_G s_\beta^k$ iff $s_\alpha^h = s_\beta^k$ or there exists a sequence of isomorphisms in G (or their inverses) taking one to the other.

The symbol $\tilde{K}_G$ will denote the *quotient complex* $S(K)/\sim_G$.

3. MINIMAL BALL COVERINGS.

Let M be a closed connected n-manifold and $B = \{B_i \mid i \in I\}$ be a finite set of closed n-balls such that $M = \bigcup_{i \in I} B_i$.

Definition 1. B is said to be a P_0-*ball covering* if it satisfies the following property :

(P_0) For every $i, j \in I$ ($i \neq j$),

$$B_i \cap B_j = \partial B_i \cap \partial B_j$$

has $(n-1)$-manifolds as connected components.

B is said to be a P_1-*ball covering* if it satisfies the following property

(P_1) For every $i, j \in I$ ($i \neq j$),

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has $(n-1)$-balls as connected components.

B is said to be a P_2-*ball covering* if it satisfies the following property

(P_2) For every $J \subset I$, $\# J = k$, $k \leqq n+1$,

$$\bigcap_{j \in J} B_j = \bigcap_{j \in J} (\partial B_j)$$

has $(n-k+1)$-balls as connected components.

Obviously $P_2 \Rightarrow P_1 \Rightarrow P_0$.

Definition 2. Let M be an n-manifold with h ($h > 0$) boundary components M_j ($j \in N_h$) and $B = \{ B_i \mid i \in I \}$ be a finite set of closed n-balls such that $M = \bigcup_{i \in I} B_i$. B is said to be a P_α-*ball covering* ($\alpha \in \Delta_2$) of M if B satisfies the property P_α and

$$B_j = \{ B_i \cap M_j \mid i \in I \}$$

is a P_α-ball covering of the closed $(n-1)$-manifold M_j, for every $j \in N_h$.

Note that a P_0-ball (resp. P_1-ball) covering is a *ball covering* (resp. *strong ball covering)* in the sense of [IY, KT] (resp. [FG2]).

Let M be a connected n-manifold. For $\alpha \in \Delta_2$, define :

$$b_\alpha(M) = \min \{ \# B \mid B \text{ is a } P_\alpha\text{-ball covering of } M \} .$$

Obviously,

$$b_0(M) \leqq b_1(M) \leqq b_2(M).$$

The following results are known.

Proposition 1. *1° If* M *is a closed n-manifold,* $b_2(M) = n+1$ [P1, FG1].
2° If M *has non-empty connected boundary,* $b_2(M) \leqq n$ [FG2].
3° If M *has non-empty boundary,* $b_0(M) \leqq n$ [KT]. ◊

The statements 2 and 3 of the above proposition can be obtained as easy consequences of the following :

Proposition 2. *If* M *is a connected n-manifold with non-empty boundary, then* $b_2(M) = n$.

Proof. We first prove that $b_2(M) \leqq n$ by exhibiting a P_2-ball covering B^* of M with n balls. Let M_i ($i \in N$) be the boundary components of M, M'_i a copy of M_i and $\varphi_i : M_i \to M'_i$ the identification map. Let w_i ($i \in N_h$) be a point such that the adjunction space

$$Q = M \cup_{\varphi_1} (w_1 * M'_1) \cup_{\varphi_2} \dots \cup_{\varphi_h} (w_h * M'_h)$$

is a closed n-pseudomanifold.

Moreover, if K is a simplicial triangulation of Q, the set of the singular simplexes of K is $\{w_i \mid i \in N_h\}$ and the disjoined star of each simplex of K is strongly-connected.
We give an inductive algorithm for constructing a pseudodissection $\tilde{K}_p$ ($0 \leqq p \leqq n$) of Q such that $S_0(\tilde{K}_p)$ has p regular cone-vertices. Set $\tilde{K}_0 = K$. Let now A_j ($j \in N_p$) be a regular cone-vertex of $\tilde{K}_p$. There exist a finite sequence $\xi_1 = \{\sigma_\alpha^{n-p}\}_{\alpha=0}^{s}$ of all the $(n-p)$-simplexes of $\tilde{K}_p$ not containing $A_1, \ldots, A_p$ and a finite sequence $\varepsilon_1 = \{\tau_\beta^{n-1}\}_{\beta=1}^{s}$ of $(n-1)$-simplexes of $\tilde{K}_p$ such that, for every $\beta \in N_s$,

$$\tau_\beta^{n-1} \in \mathrm{st}(\sigma_\beta^{n-p}, \tilde{K}_p) \cap \mathrm{st}(\sigma_\gamma^{n-p}, \tilde{K}_p)$$

for some $\gamma < \beta$. For each $\sigma_\alpha^{n-p} \in \xi_1$, consider the disjoined star $\mathrm{std}(\sigma_\alpha^{n-p}, \tilde{K}_p)$ and glue them pairwise together by identifying the two copies of every $(n-1)$-simplex of ε_1. The pseudocomplex B so obtained is a pseudodissection of an n-ball. Moreover, there exists an identification system G on B such that the quotient $\tilde{B}_G$ is isomorphic with $\tilde{K}_p$. Define A_{p+1} as an interior point of B and set $\Sigma = A_{p+1} * \partial B$. If G' is the identification system induced by G on Σ, set $\tilde{K}_{p+1} = \tilde{\Sigma}_{G'}$.
There exist a finite sequence $\xi_2 = \{v_\delta\}_{\delta=0}^{u}$ of all the vertices of $\tilde{K}_n$ different from the regular cone-vertex A_j ($j \in N_n$) and a finite sequence $\varepsilon_2 = \{\rho_\delta^{n-1}\}_{\delta=1}^{u}$ of $(n-1)$-simplexes of $\tilde{K}_n$ such that, for every $\delta \in N_u$,

$$\rho_\delta^{n-1} \in \mathrm{st}(v_\delta, \tilde{K}_n) \cap \mathrm{st}(v_\mu, \tilde{K}_n),$$

for some $\mu < \delta$. Note that

$$\{w_i\}_{i=1}^{h} \subset \{v_\delta\}_{\delta=0}^{u}.$$

By the strong connectedness of $\mathrm{std}(w_i, \tilde{K}_n)$, it is possible to obtain a triangulated n-ball B_i ($i \in N_h$) such that :

1° all the vertices of B_i belong to ∂B_i,
2° w_i is a cone-vertex of B_i,
3° there exists an identification system G_i on B_i such that $\tilde{B}_{i\,G_i}$ is isomorphic with $\mathrm{std}(w_i, \tilde{K}_n)$.

Let ξ_3 be the finite sequence obtained from ξ_2 by considering the disjoined stars of all the regular vertices of ξ_2 and all the n-balls B_i 's. By identifying the elements of ξ_3 along suitable $(n-1)$-simplexes of ξ_2, we can obtain exactly h triangulated n-balls $D_1, \ldots, D_h$ such that

$$\{w_k \mid k \in N_h\} \cap D_i = \{w_i\}.$$

If

$$\Sigma_i = \overline{\partial D_i - \mathrm{st}(w_i, \partial D_i)},$$

set $C_i = w_i * \Sigma_i$.

There exist an identification system G^* induced by ξ_3 and a triangulated n-ball E obtained from $C_1, \ldots, C_h$ such that $|\tilde{E}_{G^*}| = Q$, $A_1, \ldots, A_n$ are cone-vertices of $\tilde{E}_{G^*}$ and

$$S_0(\tilde{E}_{G^*}) = \{A_j \mid j \in N_n\} \cup \{w_i \mid i \in N_h\} .$$

Set $T = \tilde{E}_{G^*}$. If T' is the first barycentric subdivision of T, define

$$B = \{B_i \mid i \in N_{n+h}\},$$

where

$$B_j = st(A_j, T') \quad \text{if} \quad 1 \leqq j \leqq n,$$
$$B_j = st(w_j, T') \quad \text{if} \quad n+1 \leqq j \leqq n+h.$$

Note that, by construction,

$$B_i \cap B_j = \emptyset \quad \text{if } i \neq j, \quad \text{and} \quad i, j \in N_{n+h} - N_n .$$

$B^* = \{B_i \mid i \in N_n\}$ is a P_2-ball covering of M.
Now we show that no such covering of smaller cardinality exists. Let

$$B = \{B_i \mid i \in N_k\} \qquad (k < n)$$

be a P_2-ball covering of M. For each $i \in N_k$,

$$H_j(B_i \cap M_s) = \begin{cases} 0 & \text{if } j > 0 \\ Z & \text{if } j = 0 \end{cases}$$

$H_j(.)$ being the j-th homology group. The Mayer-Vietoris sequence gives :

$$\ldots \to H_j(B_1 \cap M_s) \oplus H_j(B_2 \cap M_s) \to H_j((B_1 \cup B_2) \cap M_s) \to H_{j-1}(B_1 \cap B_2 \cap M_s) \to \ldots$$

Then

$$H_j((B_1 \cup B_2) \cap M_s) = 0 \qquad \text{if} \quad j \geqq 2,$$

while, for $j = 1$, it is a free abelian group (possibly zero). By induction on $m \leqq k$, the Mayer-Vietoris sequence gives :

$$0 = H_j((\bigcup_{i=1}^{m-1} B_i) \cap M_s) \oplus H_j(B_m \cap M_s) \to H_j((\bigcup_{i=1}^{m} B_i) \cap M_s) \to$$
$$\to H_{j-1}((\bigcup_{i=1}^{m-1} B_i) \cap B_m \cap M_s) = 0.$$

Then

$$H_j((\bigcup_{i=1}^{m} B_i) \cap M_s) = 0 \qquad \text{if} \quad j \geqq m ,$$

while, for $j = m-1$, it is a free abelian group. If $k < n$, setting $m = k$, we have that

$$H_j(M \cap M_s) = H_j(M_s)$$

vanishes for $j \geqq k$ and is a free abelian group for $j = k-1$. In particular $H_{n-1}(M_s) = 0$ and $H_{n-2}(M_s)$ is a free abelian group.

This is a contradiction because either $H_{n-1}(M_s) = Z$ or $H_{n-1}(M_s) = 0$ and $H_{n-2}(M_s)$ has torsion, M_s being a closed $(n-1)$-manifold. ◊

Remark. For the proof of $b_2(M) \geqq n$ it is sufficient that each B_j is a P_2-ball covering of M_j without assuming the property P_2 for B in the interior of M.

Note that Proposition 2 improves the statement of the Theorem 4.1 in [OS] in the case $q = 0$.

4. MINIMAL ATLASES.

A *BI-atlas* (ball-intersection atlas) of a closed connected n-manifold M in the sense of [P2] is a finite covering

$$\mathcal{A} = \{V_\alpha \mid \alpha \in A\}$$

of M such that :

a) each V_α is an open n-ball,

b) the intersection of any number of V_α's has open balls as connected components.

In order to define a concept of BI-atlas for manifolds with boundary, we need the following

Definition 3. Let M be a connected n-manifold. An open subset **P** of M is said to be an *open n-quasi-ball* if **P** is homeomorphic with the union of an open n-ball B with a finite number (possibly null) of open disjoint $(n-1)$-balls on ∂ B.

Definition 4. A finite covering $\mathcal{A} = \{V_\alpha \mid \alpha \in A\}$ of a connected n-manifold M with h ($h > 0$) boundary components M_i ($i \in N_h$) is said to be a *BI-atlas* if the following conditions hold :

a') each V_α is an open n-quasi-ball,

b') the intersection of any number of V_α's has open quasi-balls as connected components,

c') $$\mathcal{A}_i = \{V_\alpha \cap M_i \mid \alpha \in A\}$$

is a BI-atlas of the closed $(n-1)$-manifold M_i ($i \in N_h$).

Let us define

$$a(M) = \min\{\# \mathcal{A} \mid \mathcal{A} \text{ is a BI-atlas of } M\}.$$

A BI-atlas $\mathcal{A}$ of M such that $\# \mathcal{A} = a(M)$ is said to be a *minimal atlas* of M.

In ([P2], Proposition 5.1), M. Pezzana proved that $a(M) = n+1$ for every closed connected n-manifold M.

Proposition 3. *If* M *is a connected n-manifold with h* $(h > 0)$ *boundary components* M_i $(i \in N_h)$, $a(M) = n$.

Proof. Let Q be the closed n-pseudomanifold constructed as in Proposition 2 startingfrom M. If $T = E_{G^*}$ is the pseudodissection of Q obtained in Proposition 2, the interior of the space $|std(A_i, T)|$, underlying the disjoined star of each cone-vertex $A_i \in S_0(T)$ $(i \in N_n)$, is an open n-ball of Q. If T' is the first barycentric subdivision of T, set

$$B_i = |st(A_i, T')|.$$

The polyhedron $M' = \bigcup_{i=1}^{n} B$ is homeomorphic with M.
Since $M' \subset Q$, the collection

$$A = \{ |std(\overset{o}{A}_i, T)| \cap M' \mid i \in N_n \}$$

is a BI-atlas of M' such that $\# A = n$. In fact, each connected component of $|Std(\overset{o}{A}_i, T)| \cap \partial M'$ is an open collar of the $(n-1)$-ball

$$|Std(b_{ir}, T')| \cap \partial M',$$

b_{ir} being the barycenter of the edge $< A_i, w_r >$ for some singular vertex $w_r \in S_0(T)$. This proves that $a(M) \leqq n$.
Conditions b' and c' of Definition 4 give $a(M) \geqq n$, according to a Mayer-Vietoris argument as in Proposition 2. ◊

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