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THE FREE ADJUNCTION

by Stephen SCHANUEL and Ross STREET

RÉSUMÉ. Les adjonctions dans une 2-catégorie K correspondent aux 2-foncteurs de la 2-catégorie Adj "adjonction libre" vers K . On donne ici une description explicite de cette 2-catégorie Adj .

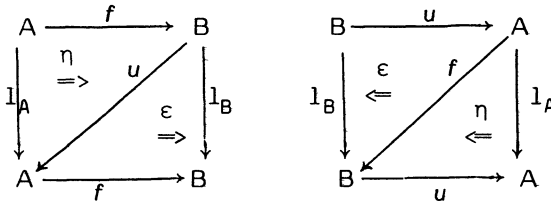
In obtaining the unit-counit expression of adjunction, Kan [2] made it clear that adjunctions can be defined in an arbitrary 2-category K . An adjunction in K consists of arrows

$$f : A \rightarrow B, \quad u : B \rightarrow A$$

and 2-cells

$$\eta : 1_A \Rightarrow uf, \quad \epsilon : fu \Rightarrow 1_B$$

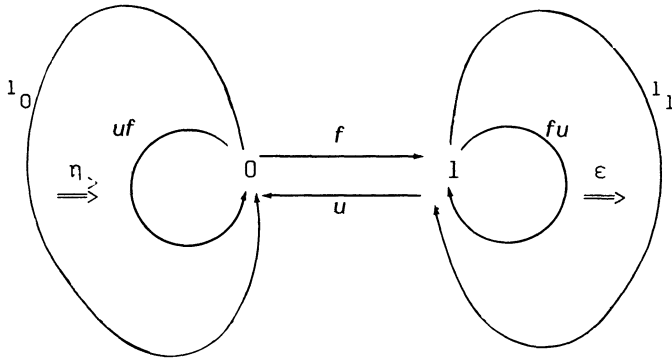
such that the 2-cells, obtained by pasting [3] the following diagrams, are the identity 2-cells of f, u respectively.



Using presentations by computads [4], one can see that there is a 2-category Adj such that a 2-functor $Adj \rightarrow K$ precisely amounts to an adjunction in K . The purpose of this Note is to provide a concrete description of the 2-category Adj : *the free adjunction*.

In generating monads (= standard constructions) from adjoint functors, Huber [1] made it clear that describing Adj is related to describing the *free monad* Mnd . A concrete description of Mnd is easy: it has one object 0 , the hom-category $Mnd(0, 0)$ is the category Δ of finite ordinals and order-preserving functions, and composition $\Delta \times \Delta \rightarrow \Delta$ is ordinal sum.

The 2-category Adj has two objects $0, 1$, say, since an adjunction in K involves two objects A, B . The full sub-2-category with object 0 should be the free monad and the full sub-2-category with object 1 should be the free comonad; hence,



$$\text{Adj}(0, 0) \simeq \Delta, \quad \text{Adj}(1, 1) \simeq \Delta^{\text{OP}}.$$

(Auderset [0] has clarified the above aspects of Adj.)

It has been observed by many people (although [5], p. 131, is the only reference which comes to mind) that Δ^{OP} is isomorphic to the category of non-empty ordinals and first-and-last-element-preserving order-preserving functions. It turns out that $\text{Adj}(0, 1)$, $\text{Adj}(1, 0)$ can also be taken as categories whose objects are the non-empty ordinals and whose arrows are respectively last-element-preserving, first-element-preserving arrows in Δ .

This leads to the following 2-category Badj . The objects are finite-ordinals $p = \{0, 1, \dots, p-1\}$. Objects of the category $\text{Badj}(p, q)$ amount to $m : p \rightarrow q$ where m is a finite ordinal with both $p \leq m$ and $q \leq m$. An arrow $\vartheta : m \Rightarrow m'$ in $\text{Badj}(p, q)$ is an order-preserving function $\vartheta : m \rightarrow m'$ satisfying

$$\vartheta(i) = \begin{cases} i & \text{for } 0 \leq i < p, \\ m' - m + i & \text{for } m - q \leq i < m; \end{cases}$$

composition of such arrows is composition of functions. The composition functor

$$\text{Badj}(q, r) \times \text{Badj}(p, q) \rightarrow \text{Badj}(p, r)$$

takes

$$(n : q \rightarrow r, m : p \rightarrow q) \text{ to } m - q + n : p \rightarrow r$$

and

$$(\varphi : n \Rightarrow n', \vartheta : m \Rightarrow m') \text{ to } \psi : m - q + n \Rightarrow m' - q + n'$$

given by the formula

$$\psi(i) = \begin{cases} \vartheta(i) & \text{for } i < m, \\ \varphi(i - m + q) + m' - q & \text{for } i \geq m - q. \end{cases}$$

This composition is functorial, associative, and the identity of p is $p : p \rightarrow p$.

The full sub-2-category of Badj consisting of the objects $0, 1$ is Adj . There is a distinguished adjunction $f : A \rightarrow B, u : B \rightarrow A, \eta, \epsilon$ in Adj given as follows :

$$A = 0, B = 1, f = 1 : 0 \rightarrow 1, u = 1 : 1 \rightarrow 0, \\ \eta : 0 \Rightarrow 1 = uf, \quad \epsilon : fu = 2 \Rightarrow 1$$

are the unique such functions.

It is now possible to prove the universal property.

Proposition. *For each adjunction in a 2-category K , there exists a unique 2-functor $\text{Adj} \rightarrow K$ whose value at the distinguished adjunction is the given adjunction.* \diamond

Kan's adjoint functors gave us the setting for "free" constructions in Mathematics; Adj gives the "free expression of freeness".

The inclusion $\text{Mnd} \rightarrow \text{Adj}$ induces a 2-functor

$$[\text{Adj}, K] \rightarrow [\text{Mnd}, K]$$

from the 2-category of adjunctions in K to the 2-category of monads in K . The technique for obtaining adjoints to such a 2-functor is also due to Kan [2] (*Kan extensions*). In this case (as pointed out in [0]) the adjoints yield the Kleisli and Eilenberg-Moore constructions on monads in K [3].

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