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MODELS OF SKETCHES

by Michael BARR

RÉSUMÉ. Le but de cet aticle est d'étudier les catégories qui sont des catégories de modèles d'une esquisse. Les résultats principaux répondent à la question de la caractérisation du type de l'esquisse par rapport aux propriétés de sa catégorie de modèles. Les esquisses qu'on emploie sont un peu différentes de celles de A. et C. Ehresmann, mais la notion découle de la leur.

INTRODUCTION.

Ever since the dissertation of Lawvere [1963], categorists have been interested in the question of presenting categories as categories of functors and natural transformations between them. One of the earliest attempts to go beyond product preserving functors (and thus equational theories) was Ehresmann's categories of sketched structures. The concept of sketch was laid out at least as early as [Ehresmann, 1966], but the most complete exposition seems to be [Bastiani & Ehresmann, 1972]. Meantime, Isbell [1972] was exploring a similar idea, again motivated by the idea of describing interesting categories as categories of functors and natural transformations.

In [Barr & Wells, 1985], a variation of Ehresmann's sketches was the main conceptual tool used to present abstract theories. The sketches used there were not even categories. This has been criticized as a retrograde step and it seems appropriate to say a few words about our reasons. The principal reason is that our sketches are the closest thing we could find to a naive presentation and thus seemed to reflect actual mathematical usage better than any substitute. In retrospect, there is an even more important reason for sketches being useful ; in many of the most interesting and commonplace cases they are finite, realized on a computer. In view of the growing use of theories in theoretical computer science (see [Ehrig & Mahr, 1985] and the many references found there), this can be a very great advantage.

Some open questions were raised in [Barr & Wells, 1985] as to identifying types of sketches from properties of their categories of models. An example of the type of question is this : what property of the category of commutative von Neumann regular rings would allow one to predict *from the categorical structure,* that it has a presentation as the category of models of an equational theory, even though it is normally presented with an existential quantifier and thus one would expect it to be presented using a regular theory. The fact is that there is both a regular presentation and a purely equational one means that we must also consider the question of knowing if two sketches have the same models. The question we are really addressing is when a sketch is equivalent in that sense to one with such and such property.

The equivalence relation used is the least equivalence relation such that $S_1 \to S_2$ is an equivalence if it induces an equivalence between the categories of set-valued models. This relation is too crude in general. However, we will mostly be considering coherent sketches here and if they are at most countable, the conceptual completeness theorem of [Barr & Makkai, to appear] can be readily shown to imply that if a morphism induces an equivalence between the categories of set-valued models, then it induces an equivalence between their classifying toposes and hence between their categories of models valued in any topos. More generally, sketches need have no models at all, and this equivalence relation is clearly much too crude.

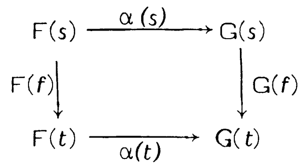
In the past few years, the study of initial objects in categories of models has entered theoretical computer science in a substantial way. The arguments in this paper are partly inspired by the study of initial objects, even though the end results are independent of that. A forthcoming paper of Wells and this author will explore the relevance of these results in computer science.

I would like to thank Charles Wells with whom I have had many fruitful discussions on this subject. I am grateful for the support I have received from the National Science and Engineering Research Council and from the Ministère de l'Education de la Province du Québec through grants from the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche, especially to the Groupe Interuniversitaire en Etudes Catégoriques.

SKETCHES.

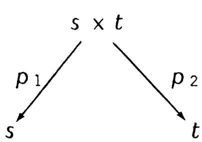
By a sketch, we mean something a little more general that that considered in [Barr & Wells, 1985], but also a little different in that we do not suppose identity arrows chosen. See the discussion following the definition of commutative diagram below for an explanation of how we deal with identity arrows.

A sketch $S = (G, D, L, C)$ consists of a graph G, a set D of diagrams in G, a set L of cones in G and a set C of cocones in G. A morphism of sketches is a morphism of the graphs which preserves the diagrams, cones and cocones. The sketch underlying a category has as graph the one gotten by forgetting the composition, and has for diagrams all the commutative diagrams, for cones all limit cones and for cocones all colimit cocones. Thus a morphism of a sketch in a category (also called a **model** of the sketch in that category) must take all the diagrams to commutative diagrams, all the cones to limits and all the cocones to colimits. A morphism of models is, as usual, a natural tranformation. Note that the notion of natural transformation between two morphisms from graph G to a category E makes perfectly good sense. If $F, G : G \to E$ are two morphisms, a natural transformation $\alpha : F \to G$ consists of a family $\alpha(s)$ of morphisms indexed by the nodes of G such that $\alpha(s) : F(s) \to G(s)$ and if $f: s \to t$ is an arrow of G, then the diagram

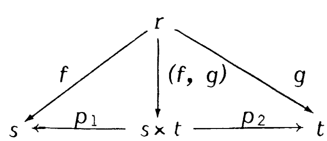


commutes.

There is a certain notation, shown to me originally by John Gray, that makes the describing of sketches much easier. If S is a sketch, a node in the graph G is called a **sort** of S. If s and t are sorts, it makes no sense to speak of a sort that is a product of s and t. Just for that reason, we are free to use the notation $s \times t$ to mean what we want. We may and do use such a notation as a shorthand for a node, having, in the first instance, no connection with s or t, but also as implying, by its very name, the existence of a cone



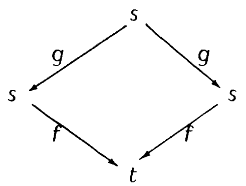
in L. Let us say that such a cone is defined *implicitly* by this notation. In a similar vein, a node named 1 is understood to be the vertex of an empty cone. There is a similar convention for arrows. If $f : r \to s$ and $g : r \to t$ are arrows in the graph, then an arrow named $(f, g) : r \to s \times t$ implies that there is a diagram



in D. Analogous remarks apply to

$$f : q \to s, \quad g : r \to t \text{ and } \quad f \times g : q \times r \to s \times t.$$

An arrow written $f : s \rightarrowtail t$ is understood to imply the existence of a cone



Of course the duals of these conventions give implicit cocones. There is one more convention giving diagrams. If we have two paths in G between the same pair of objects,

$$s \xrightarrow{f_1} \xrightarrow{f_2} \cdots \xrightarrow{f_n} t \quad \text{and} \quad s \xrightarrow{g_1} \xrightarrow{g_2} \cdots \xrightarrow{g_m} t$$

then we will indicate that the diagram

$$s \; \substack{\xrightarrow{f_1} \xrightarrow{f_2} \cdots \xrightarrow{f_n} \\ \xrightarrow[g_1]{} \xrightarrow[g_2]{} \cdots \xrightarrow[g_m]{}} \; t$$

is in D by writing

$$f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1 .$$

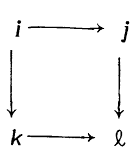
In the definition of free monoid, we always have an empty string. When monoids are generalized to categories, we also allow empty strings, but only between a node and itself. Thus in the diagram above, we will allow the case that $m = 0$, say, but only when $s = t$. The statement that such a diagram commutes in a category means simply that the composite

$$f_n \circ f_{n-1} \circ \cdots \circ f_1 = id.$$

Some care must be exercised in translating naive diagrams into formal diagrams. For example, if one wants to say of the diagram

$$r \; \substack{\xrightarrow{d^0} \\ \xrightarrow[d^1]{}} \; s \xrightarrow{d} t$$

that $d \circ d^0 = d \circ d^1$, it is not sufficient to put in a diagram based on a graph of that shape, for our definition would force $d^0 = d^1$, which is not intended. Instead, we must use a diagram based on the graph Q with the shape



and a morphism

$$D : Q \to S \quad \text{with} \quad D(j) = D(k) = s.$$

Similar remarks apply in the case that one wants to force a morphism to be idempotent without forcing it to be the identity.

Using this notation, it is often not necessary to write down many diagrams, cones or cocones at all.

If S is a sketch and s is a sort of S, then we define a new sketch S_S as follows. If S does not contain an object 1, we add it. In addition, we add an arrow $1 \to s$. We think of this sketch as having a new constant of type s.

Many of our results below are phrased in terms of initial objects in the category of models. An initial object in a category is, as usual, an object that has exactly one map to every other object. Clearly a category that has an initial object is connected. Now if a category is not connected, it is still possible that each component has an initial object. If that is so, the initial objects of the components will be called an **initial family.** Such a family has the property that each object in the category has a unique morphism from a unique one of the objects. An example of an initial family is the set of prime fields in the category of fields. If there is no danger of confusion, we will continue to call an object of an initial family an initial object.

A generalization in another direction is given by an object that has a morphism to every other object, without it necessarily being unique. The category whose objects are monoids, but with morphisms that are multiplicative maps, does not have an initial object, but the one element monoid has a morphism to every other monoid. The identity element can go to any idempotent element. Let us call such an object **quasi-initial.**

A sketch is called an **FP sketch** (for finite product) if there are no cocones and all the cones are discrete and finite. The category of models of such a sketch is evidently the algebras for a multi-sorted equational theory.

A sketch is called an **LE sketch** (for left exact) if there are no cocones and all the cones are finite. In both of these cases, we will generally omit mention of the cocones entirely, writing, "Let (G, D, L) be an FP (or LE) sketch".

A sketch is called a **regular sketch** if its cones are finite and the cocones are of the form implicit in the notation $s \twoheadrightarrow t$.

A sketch is called an **FS sketch** (for finite sum) if the cones are finite and all the cocones are finite and discrete.

A sketch is called a **coherent sketch** if the cones are finite and the cocones are either finite discrete or of the form $s \twoheadrightarrow t$.

A sketch is called a **geometric sketch** if the cones are finite and the cocones are either discrete or of the form $s \twoheadrightarrow t$.

More general sketches are indeed possible. We could, for instance, add operators that are to be interpreted in models as universal quantifiers or types that are to be instantiated as power objects. These constructions are certainly possible but we will not explore them here. Our sketches represent the most that can be done with generalized exactness conditions.

THE MAIN RESULTS.

Theorem 1. *Let S be a coherent sketch and let* Mod(S) *be the category of models of S. Then of the following, we have* (i) => (ii) => (iii).

(i) We can find morphisms of sketches $S \to S^{\#} \leftarrow S'$ that are equivalences with S' an FS sketch and such that the nodes in $S^{\#}$ can be built inductively out of nodes of S using cones and S' has the same nodes as $S^{\#}$;

(ii) Every connected diagram in Mod(S) *has a pointwise limit ;*

(iii) For any sort s of S, the sketch S_s has an initial family of models.

Remark. It is an open question whether (iii) => (i). Later results suggest it might, but I have not been able to find a proof. I note that iii is really the statement that for each sort s, the functor

$$\mathrm{ev}(s) : \mathrm{Mod}(S) \to \mathbf{Set}$$

given by $\mathrm{ev}(s)(M) = M(s)$ is multi-representable which means that the functor is a disjoint union of representable functors. Although he doesn't appear to use the word, this notion is essentially due to Diers. See, for example, [Diers, 1980].

Proof. (i) => (ii). Since the nodes of $S^{\#}$ and hence of S' are built in the way described and limits commute with limits, we can prove this under the assumption that S is an FS sketch.

Let $D : \mathcal{J} \to A = \mathrm{Mod}(S)$ be a diagram with $\mathcal{J}$ connected. A pointwise limit of models is a left exact functor since limits commute with limits.

As for commuting with finite sums, it is sufficient to show that if $\mathcal{J}$ is connected and

$$E_1 : \mathcal{J} \to \mathbf{Set} \qquad \text{and} \qquad E_2 : \mathcal{J} \to \mathbf{Set}$$

are functors with limit sets S_1 and S_2, respectively, then $S_1 + S_2$ is the limit of $E_1 + E_2$. This is applied as follows. Whenever there is a cocone $s = s_1+s_2$ in the sketch, let E_k denote the diagram $D(s_k)$ in **Set** and S_k denote the set $\lim E_k$, for $k = 1, 2$. Then it follows that $S_1 + S_2$ is the limit of

$$D(s_1) + D(s_2) \simeq D(s_1 + s_2)$$

as it should be. To verify the assertion, let (x_i) be an element of the limit. Then for each i, either $x_i \in E_1(i)$ or $x_i \in E_2(i)$. This determines a partition of $\mathcal{I}$ into two pieces, the indices i for which $x_i \in E_1(i)$ and those for which $x_i \in E_2(i)$. It is clear that if there is an arrow $i \to j$ in $\mathcal{I}$, then i and j are in the same half of the partition. Thus if $\mathcal{I}$ is connected, then either $x_i \in E_1(i)$ for all i or $x_i \in E_2(i)$ for all i. Thus either $(x_i) \in S_1$ or $(x_i) \in S_2$.

(ii) $\Rightarrow$ (iii). The category $\mathrm{Mod}(S_s)$ has all limits if $\mathrm{Mod}(S)$ does. In fact, the evident underlying functor $\mathrm{Mod}(S_s) \to \mathrm{Mod}(S)$ creates limits. Since the category $\mathrm{Mod}(S_s)$ has pullbacks, it is easy to see that if two algebras are in the same component, there is an algebra that maps to each of them. By a straightforward cardinality argument, we see that every algebra contains a subalgebra bounded in cardinality by the larger of the size of the sketch and $\aleph_0$. Thus there is a *set* of algebras with the property that each algebra admits a morphism from at least one algebra in the set. Since intersections are connected limits, we may suppose the algebras in this set contain no proper subalgebras. (It is quite common in model theory to suppose that the models are non-empty ; we do not make such an assumption here. See Appendix A for a discussion.) There may well be morphisms between these minimal algebras, but we can deal with that as follows. Let $A = A_0$ be a minimal algebra. Suppose there is a non-isomorphism $A_1 \to A_0$ where A_1 is also minimal. Continue this way until we either find an algebra A_n with no non-isomorphic morphisms from a minimal algebra or we have constructed an infinite sequence. In the latter case, let A_ω be a minimal subalgebra of the inverse limit (which is over a connected diagram). Continuing in this way, we build a sequence indexed by all ordinals. In that case, there being only a set of non-isomorphic minimal ordinals $\alpha > \beta$ with $A_\alpha \simeq A_\beta$. In that case, we have both the arrow $A_\alpha \to A_\beta$ that comes from the diagram and the isomorphism. But the equalizer of those two arrows is a subalgebra of A_α which must be all of A_α. This implies that the arrow in the diagram $A_\alpha \to A_\beta$ is an isomorphism. From the sequence

$$A_\beta \simeq A_\alpha \longrightarrow \cdots \longrightarrow A_{\beta+1} \longrightarrow A_\beta$$

we see that there is a map $A_\beta \to A_{\beta+1}$ that splits $A_{\beta+1} \to A_\beta$. This means that either the latter map is an isomorphism or $A_{\beta+1}$ has a proper subalgebra, both of which contradict our construction. ◊

Theorem 2. *Let S be a coherent sketch and let* $\mathrm{Mod}(S)$ *be the category* *of models of S. Then the following are equivalent :*

(i) There is a sketch morphism $S' \subset S$ which is an equivalence with S' a regular sketch and S' has the same objects as S ;

(ii) The category Mod(S) *has pointwise products ;*

(iii) The category Mod(S) *has pointwise finite products ;*

(iv) For every sort s, the category Mod(S_s) *is connected ;*

(v) For any sort s of S, the sketch S_s has a quasi-initial model.

Proof. We will show that (i) => (ii) => (iii) => (iv) => (i) and that (ii) => (v) => (iv).

(i) => (ii). We may suppose that S is a regular sketch. Products commute with arbitrary limits and with regular epis. Thus the pointwise product of models is a model and is, of course, the product in the category of models.

(ii) => (iii). Trivial.

(iii) => (iv). As above, the fact that the products are computed pointwise allows us to show that the category of models of S_s still has finite products and hence is connected.

(iv) => (i). it is clear that if we have a discrete cocone

$$s = s_1 + s_2 + \dots + s_n$$

and equations that force all but one s_i to be empty in every model, then we can replace the discrete cocone by the requirement that $s_i \to s$ be an isomorphism. This can be done either by making the inclusion of $s_i \to s$ a cone or by adding an inverse arrow. Thus it is sufficient to show that if there is a decomposition $s = r + t$, then either M(r) is empty in every model M or M(t) is empty in every model M or the category of models of S_s is not connected.

So suppose we have a model M_r with $M_r(r) \neq \emptyset$ and another model M_t with $M_t(t) \neq \emptyset$. Choose elements $\rho \in M_r(r)$ and $\tau \in M_t(t)$. A model of S_s is a pair (M, σ) where $\sigma \in M(s)$. Since $M(s) = M(r) + M(t)$, the models come in two varieties according as $\sigma \in M(r)$ or $\sigma \in M(t)$. Let Mod_r and Mod_t denote the full subcategories consisting of the two varieties of models. Then (M_r, ρ) is an object of Mod_r and (M_t, τ) is an object of Mod_t ; thus neither subcategory is empty. If we show that there are no morphisms between objects in the subcategories, it will follow that the category of models of S_s is not connected. But if (M, σ) is an object of Mod_r and (M', σ') an object of Mod_t and $f : (M, \sigma) \to (M', \sigma')$ a morphism between them, we see from the commutative diagram

$$\begin{array}{ccc} M(r) & \longrightarrow & M(s) \\ {\scriptstyle f(r)}\big\downarrow & & \big\downarrow{\scriptstyle f(s)} \\ M'(r) & \longrightarrow & M'(t) \end{array}$$

that

$$\sigma' = f(s)(\sigma) = f(r)(\sigma) \in M'(r),$$

a contradiction.

(ii) => (v). Since the forgetful functor $\mathrm{Mod}(S_s) \to \mathrm{Mod}(S)$ creates products (the pointwise nature of the products is crucial here), the former category continues to have products. Now let S' be the sketch derived by splitting each regular epi in S. That is, whenever $g : t \twoheadrightarrow s$ is a cocone in S, add a new arrow $g : s \to t$ and an equation $f \circ g = \mathrm{id}$. The arrow $S \to S'$ induces a functor $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$ and this functor is readily seen to be surjective on objects, since any model M of S can be made into a model of S', generally in many different ways by splitting arbitrarily each $M(t) \twoheadrightarrow M(s)$. Of course, morphisms will not remain morphisms of the new theory, so the induced functor is not full. It is clear that the epis can now be dropped from the sketch S' without changing the category of models, so that that category has, by Theorem 1, an initial family $\{M_i\}$. Since every object of $\mathrm{Mod}(S')$ allows just one arrow from just one M_i, it is clear that each model of S allows at least one arrow from at least one M_i. Since now the category $\mathrm{Mod}(S)$ has products, the product of the M_i is obviously a quasi-initial model.

(v) => (iv). Trivial. ◊

The category of fields of a fixed finite characteristic p has an initial object Z_p (and is thus connected) but if one adjoins a new constant of type non-zero the resultant category of models is no longer connected. There is one component for each prime ideal in $Z_p[x]$ Thus it is necessary that in parts (iii) and (iv) above the conditions be satisfied not merely in S, but also in S_s.

Theorem 3. *Let S be a coherent sketch and let* Mod *(S) be the category of models of S. Then the following are equivalent :*

(i) We can find morphisms of sketches $S \to S^{\#} \leftarrow S'$ *that are equivalences with* S' *an LE sketch and such that the nodes in* $S^{\#}$ *can be built inductively out of the nodes of S using cones and* S' *has the same nodes as* $S^{\#}$ *;*

(ii) Every diagram in Mod *(S) has a pointwise limit ;*

(iii) For any sort s of S, the sketch S_s *has an initial model.*

Proof. (i) => (ii). Because of the fact that the nodes of S' are built with limit cones from the nodes of S and limits commute with limits, it is sufficient to show that models of an LE sketch have limits, which is well (in fact, an application of the fact that limits commute with limits).

(ii) => (iii). From Theorem 1, each component has an initial model and if there are products, there is just one component.

(iii) => (i). Condition (iii) is equivalent to the supposition that for each sort $s \in S$, the functor $\mathrm{ev}(s) : \mathrm{Mod}(S) \to$ **Set** is represented by a model we will denote M_S. By Theorem 2, we can suppose that S is a regular sketch. let $t \twoheadrightarrow s$ be an epi cocone in S. Then for each model M, $M(t) \twoheadrightarrow M(s)$. In particular, $M_S(t) \twoheadrightarrow M_S(s)$. We have,

$$\begin{aligned} M_S(t) &\to M_S(s) \\ \mathrm{ev}(t)(M_S) &\to \mathrm{ev}(s)(M_S) \\ \mathrm{Nat}(\mathrm{Hom}(M_S, -), \mathrm{ev}(t)) &\to \mathrm{Nat}(\mathrm{Hom}(M_S, -), \mathrm{ev}(s)) \\ \mathrm{Nat}(\mathrm{ev}(s), \mathrm{ev}(t)) &\to \mathrm{Nat}(\mathrm{ev}(s), \mathrm{ev}(s)). \end{aligned} \qquad (*)$$

The equivalences above are either the Yoneda Lemma or come from the fact that M_S represents $\mathrm{ev}(s)$.

There is an obvious representation of S into the functor category $\mathbf{Set}^{\mathrm{Mod}(S)}$, that sends the sort s to $\mathrm{ev}(s)$. There is a least regular subcategory $S^{\#}$ of $\mathbf{Set}^{\mathrm{Mod}(S)}$ that contains the image of S. Although $S^{\#}$ can easily be obtained as an intersection, I prefer to build it up from below. So let $S_0 = S$ and construct S_1 by adding to S_0 all the following (to simplify notation, we will refer to the object $\mathrm{ev}(s)$ in $\mathbf{Set}^{\mathrm{Mod}(S)}$ simply as s and similarly for arrows) :

(a) When $f : r \to s$ and $g : s \to t$ are composable arrows in S_0, put $g \circ f$ into S_1.

(b) When s and t are sorts in S_0, put $s \times t$ and the projection arrows into S_1.

(c) If $f : r \to s$ and $g : r \to t$ are in S_0, put $\langle f, g \rangle : r \to s \times t$ into S_1.

(d) When $f, g : s \rightrightarrows t$ are a parallel pair in S_0, put their equalizer into S_1.

(e) When $f : r \to s$ in S_0 equalizes f and g, put the induced arrow from r to the equalizer into S_1.

(f) When $f : s \twoheadrightarrow t$ is a cocone in S_0 and $g : s \to r$ coequalizes the kernel pair of f, then put into S_1 the unique arrow $h : t \to r$ for which $h \circ f = g$.

Make S_1 into a sketch by taking for diagrams all diagrams that commute in $\mathbf{Set}^{\mathrm{Mod}(S)}$, for cones all the finite cones that are limits in $\mathbf{Set}^{\mathrm{Mod}(S)}$ and for cocones all the arrows that are pointwise surjective in $\mathbf{Set}^{\mathrm{Mod}(S)}$. I should note that the nature of $\mathbf{Set}^{\mathrm{Mod}(S)}$ is such that a diagram commutes or a cone or cocone is a limit or colimit if and only if it is in *every* model. Thus every model $M : S \to$ **Set** has a unique extension to a model of S_1 and every morphism of models has a similar extension. The morphism $S_0 \to S_1$ induces an equivalence

$$\mathbf{Set}^{\mathrm{Mod}(S_1)} \to \mathbf{Set}^{\mathrm{Mod}(S_0)}.$$

This process can be carried out with S_1 to produce a sketch S_2 that also has the same category of models as S_0. We can thus build a chain of sketches

$$S_0 \subset S_1 \subset S_2 \subset \ldots$$

all with the same property. Their union $S^{\#}$ evidently has the same property and is a regular category. Moreover, the evaluation

$$S^{\#} \to \mathbf{Set}^{\mathrm{Mod}(S)}$$

is full and faithful. Although this fact can be ferreted out from the proof of the main theorem of [Barr, 1971], it is made explicit in [Makkai, 1980] and [Barr, to appear]. From (*), it follows that

$$\mathrm{Hom}_{S^{\#}}(s, t) \twoheadrightarrow \mathrm{Hom}_{S^{\#}}(s, s)$$

which means that there is in $S^{\#}$ a splitting for the given epi cocone. This means that, if S' is the LE sketch with the same graph, diagrams and cones of $S^{\#}$, it will have the same category of models as $S^{\#}$, hence as S. ◊

The reader will note that the full embedding theorem for regular categories was crucial in making this argument work. I am indebted to M. Makkai for suggesting that I try this. It is notable in being the first example I have come on in which the full power of the full embedding theorem has been used. In previous applications, mainly, if not exclusively, limited to diagram chasing, the existence of a regular embedding into a functor category that reflects isomorphisms would have readily sufficed. The fact that there is not a similar full embedding theorem for coherent categories explains why I cannot prove that the three statements of Theorem 1 are not equivalent. In fact, as is observed in [Barr, to appear], the existence of a full embedding into a functor category would force the lattices of complemented subobjects to be atomic. It is easy to find an example of a coherent Grothendieck topos in which this fails. For example, the category of sheaves on a non-discrete Stone space will do.

In order to explain our final result, we have to explain what we mean by saying that a category is closed under ultraproducts. Unlike limits and colimits, an ultraproduct is not defined by any universal mapping property. Of course, if the category has products and (filtered) colimits, then it has ultraproducts constructed as colimits of products (see the Appendix for more details). But usually the category of models of a coherent theory (such as the theory of fields) lacks products and hence does not have categorical ultraproducts.

Let S be a geometric sketch. Let Reg(S) denote the sketch derived from S by omitting all the discrete cocones. Then Reg(S) is a regular sketch and hence its category of models has products. The category of models of any geometric sketch has filtered colimits (they commute with finite limits and all colimits), hence the category of models of Reg(S) has categorical ultraproducts. It is clear that there is a full inclusion Mod(S) $\to$ Mod(Reg(S)). We will say that Mod(S) is **closed under ultraproducts** if any ultraproduct in Mod(Reg(S)) of objects of Mod(S) is an object of Mod(S).

This definition defines a property of a particular presentation of a theory. It is not true that a theory that is closed under ultrafilters in one presentation will be closed in all presentations. Robert Paré has suggested the following example. Consider the sketch S with countably many sorts s_1, s_2, ... and no other structure. This has the same models in **Set** (and in any topos with countable sums) as the theory to which a sort s and a cocone $s = s_1 + s_2 + \ldots$ has been added. Yet the first theory is closed under ultraproducts and the second one isn't. Thus the following theorem is a theorem about a particular presentation of a theory, rather than the theory itself. It is not known whether it is necessary to suppose the hypotheses for extensions by a single constant, but that is what this proof requires.

Theorem 4. *Let S be a geometric sketch. Then S is equivalent to a coherent sketch if and only of for all sorts s of S, the models of S_s are closed under ultraproducts.*

The sense of "equivalent" is simply this : that if there are any infinite discrete cocones in the sketch, it should be the case that all but finitely many of them are empty in every model and they can simply be deleted from the sketch. The resultant sketch has the same category of models as the original. There is no a priori guarantee that this will be the case for every possible codomain topos.

Proof. Suppose there is an infinite discrete cocone $s = s_1 + s_2 + \ldots$ in S. We can remove from the sketch and from the above sum any sort which is actually empty in every model. If the resultant sum is still infinite, then we must have for each i a model M_i for which $M_i(i) \neq \emptyset$. Choose an element $\sigma \in M_i(i)$. Then (M_i, σ_i) is a model of S_s. Let **u** be a non-principal ultrafilter on the index set and (M, σ) be the ultraproduct over **u.** If M is a model of S_s, then $\sigma \in M(s) = \Sigma M(s_i)$ so that $\sigma \in M(s_i)$ for a unique i. Now a non-principal ultraproduct does not depend on any one coordinate. Thus (M, σ) is also the ultraproduct of all the (M_j, σ_j), $j \neq i$. For any $J \in$ **u** with $i \notin J$, we have a commutative diagram in the category of Reg(S_s) models

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \prod_{j \in J}(\sigma_j)} \downarrow & & \downarrow \\ \prod_{j \in J} M_j(s) & \longrightarrow & M(s) \end{array} .$$

But the map on the left actually lands in $\Pi_j \Sigma_{k \neq i} M_j(s_k)$ which is taken by the lower map to $\Sigma_{k \neq i} M(s_k)$, a contradiction. ◊

Of course all these theorems are proved under the hypothesis that the category in question is the category of all models of some kind of theory. They are relative results giving conditions on the category of models of some theory being actually the category of models of a less rich theory. There are absolute results at one end of the scale. For example, the Gabriel-Ulmer Theorem [1971] can easily be seen to imply that a category is the category of models of a left exact theory if and only if it is complete, has filtered colimits and the latter commute with finite limits. (See also Kelly, [1982].) And of course, we have the well-known characterization of a category of models of a finite product sketch as regular with effective equivalence relations and sufficiently many projectives. Beyond that, we have no answers to the question of absolute characterizations. This appears to be an interesting and difficult question. The only property that I know of that is possessed by the category of models of any geometric theory is filtered colimits.

APPENDIX.

The purpose of this Appendix is a short discussion of the reasons for allowing the empty set to be a model (or, in the case of a multi-sorted theory, one of the sets in a model) of a theory.

It is an arbitrary requirement that results in the category of models being less complete that it naturally is. For example, there is a theorem in universal algebra that says, in effect, that every collection of submodels of a left exact theory is either empty or is a submodel. In certain parts of abstract computer science, a data type is defined to be an initial algebra for a theory. But a theorey will have a non-empty initial algebra only if there are constants in the theory. It may be unrealistic to imagine empty data types, but the point is that the theory takes a very unnatural aspect if that is built in to the core. Some parametrizing types have no constant until they are introduced via the parameters.

The only argument for banning the empty model that has any force comes from the observation that if (M_i) is a collection of models and M is a non-principal ultraproduct of the M_i, then one wants and expects that $M(s)$ will be empty if and only if the set of i for which $M_i(s)$ is null belongs to the ultrafilter. if one takes the traditional definition of an ultraproduct as a quotient of the product, the ultraproduct will be empty as soon as one factor is. There is a simple way around this difficulty which is to take a slightly different definition of ultraproducts.

This definition of the ultraproduct is as the colimit of the products taken over the subsets in the ultrafilter. If we let **u** denote an ultrafilter of subsets of the index set I, then for any $J \in \mathbf{u}$ we can form the product $\Pi\{M_i(s) \mid i \in J\}$. For $K \in \mathbf{u}$, $K \subset J$, there is an obvious map

$$\Pi\{M_i(s) \mid i \in J\} \to \Pi\{M_i(s) \mid i \in K\}$$

gotten by restricting coordinates. This gives a directed system of these products and the colimit can be defined to be the ultraproduct. Not only does this give the correct definition in case there is a small set of indices i for which $M_i(s) = \emptyset$, but it also gives an easy proof of the left exactness of the ultraproduct construction as well as showing that it exists if (as often happens) products and filtered colimits do. Of course, this definition is easily seen to agree with the older one in case all the factors are non-empty.

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