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A GENERALIZED GLOBAL DIFFERENTIAL CALCULUS : APPLICATION TO INVARIANCE UNDER A LIE GROUP. I

by Joseph JOHNSON

RÉSUMÉ. La théorie du Calcul Différentiel sur les variétés différentiables est généralisée (sans l'affaiblir) en utilisant des fonctions localement définies C^∞ (ou analytiques réelles, ou analytiques complexes) de 1, 2, 3, ... variables comme opérateurs, et en introduisant des relations de commutativité appropriées. La théorie est de plus précisée par des axiomes qui permettent de recoller les informations locales en une information globale. Ce qu'on obtient possède les principales propriétés de l'algèbre commutative, en particulier la possibilité d'ajouter des indéterminées et de résoudre des systèmes d'équations. De plus, on peut prendre des limites et colimites aussi générales (mais petites) que l'on veut. La théorie est appliquée pour généraliser les théorèmes de Lie aux espaces de dimension arbitraire (même infinie) et sans restriction sur la nature des singularités qui peuvent intervenir.

INTRODUCTION.

In synthetic differential geometry one can discern at least two distinct trends. One of these is the "global approach" as represented by the theory of C^∞-algebras [9]. At the other extreme is the "local approach", where something concrete is hypothesized about the local nature of what is being studied [10]. This paper, while hewing to the global point of view, obtains very detailed information about local structure (cf. discussion preceding (6.7)). What we seem to learn from the approach used here is that one can carry differential calculus very far without knowing anything whatever about the underlying topological structure of the spaces involved. This approach has therefore an obvious advantage for studying geometric structures so highly "fractal" that one cannot easily lay down a minimal set of axioms for them. The theory pursued here somewhat resembles the theory of C^∞-algebras, but our operators are certain functions of 1, 2, ... variables whose domains are arbitrary open subsets of euclidean space (rather than the entire euclidean space itself). In this way we are able to handle the real-analytic and complex-analytic cases together with the C^∞-case. Also we can look at functions like $1/z$ as operators. Of course, requiring more operators than for C^∞-algebras also gives our objects more structure.

The paper is in two parts. Part I constructs a tower of categories $U \supset C \supset K$ such that each inclusion has a left adjoint. We finally arrive at a category K in which a lot of the usual business of differential geometry can be done. Each of these categories is both complete and cocomplete. The potential usefulness of U and C, except as approximations to K needed to derive the properties of K, is only hinted at here (cf. (3.2), (8.2), and the discussion that precedes (3.2)).

Even though the theory of this paper makes no specific hypotheses about the local nature of the spaces studied, it does nonetheless provide a framework for ideas such as are found in [6]. Also it is possible that the dual of the category K is cartesian closed, so that, since K^{op} contains manifolds as a full subcategory, we would get an embedding of the manifold category into a cartesian closed one. However, obtaining an embedding of the manifold category with this property is in no wise the goal of this paper.

Rather the justification for the approach followed here is that the categories we generate allow us to practice differential or analytic geometry by imitating commutative algebra. This application of the theory is shown in Part II of the paper, where the theory of Part I is used to derive a treatment of invariant theory that is more flexible and also much more general than the traditional one. The proofs on occasion are hard to find, but once found (sometimes by using geometric intuition and dualizing, sometimes from algebraic intuition) are elegant and simple. The extra generality one gets and the possibility of using algebraic as well as geometric intuition justify the added measure of difficulty. The fact that the present approach rides roughsod over singularities (like a large horse galloping over little bits of cactus) makes it a very convenient tool. In the case of complex-analytic geometry, the approach followed here is more thoroughly algebraic (and more general) than for instance that of [7].

The notion that seems to most fundamentally underlie the mathematics of this paper is that of commutative inverse semigroup. It was my understanding of the related concept of prering (cf. [8]) that caused me to realize the approach taken here could be viable.

The contents of this paper have been circulated in two (larger) preprints since 1984, and I have received many constructive comments about them. Conversations with Paul Cherenack, Anders Kock, Fred Linton and David Yetter in particular have been helpful in crafting the present version.

For more detail, the reader can consider the following outline and the paper itself.

PART I. GENERAL THEORY OF UNIVERSES.

§1. *Definitions and elementary properties of universes.* Defines the operators for the particular type of universal algebra (weak universe) studied here and makes clear the type of commutativity relations that are imposed on these to define the notion of a universe (object of $\mathcal{U}$).

§2. *Structure theory of universes.* Studies the additive inverse semigroup associated to a universe and gives information on how universes are put together. It is shown how equalities and inequalities can be solved within the category of universes and how one can adjoin indeterminates to a universe.

§3. *Piecing together global information from local information.* Takes up the question of how local data can be harvested into global information. This is done through introduction of the category C of cohesive universes.

§4. *Phantom decomposition of universes.* A very spiritual section that speaks for itself.

§5. *Representability of certain functors* $C \to$ **sets**. Shows us how to construct the kinds of cohesive universes needed in the sequel.

§6. *Local theory of cohesive universes.* Introduces points for a cohesive universe and defines the local universe at a point. It is shown that a local universe is essentially a special type of local ring, and that dividing this local ring by an arbitrary proper ideal produces a new local universe (cf. (6.5)). In (6.6) it is shown that solving systems of equations in a local universe amounts to modding out by ideals.

§7. *Topological universes.* Defines the category K of topological universes. These are the cohesive universes that have enough points to allow one to distinguish between open sets.

§8. *Sober spaces and the spaces* **S**C. Shows that the category of sober spaces is exactly the category we get when we consider the set of points for various cohesive universes. The spaces **S**C, where C is a limit or colimit in *K*, are shown to be easily computable in principle

§9. *Derivations and tangent spaces.* This essentially introduces the tangent bundle and the notion of admissible derivation.

§10. *Infinitesimals and Taylor polynomials.*

§11. *Integration of one-parameter families and Taylor's Theorem.* Shows that every one-parameter family of elements of a topological universe has an integral which is also a one-parameter family of elements of that universe. This is used to show that, for arbitrary topological universes, Taylor's Theorem with the integral form of the remainder is valid.

PART II. INVARIANCE UNDER A LIE GROUP.*

This part of the paper only uses universes that are in the category *K*. In Part II we show how the theory of Part I can be used to provide a vast generalization of a part of differential geometry which, in its classical form, requires the use of a number of different techniques.

§12. *Actions of a group universe on a universe.* It is shown that an element that is locally invariant and has an invariant domain is invariant. Right-invariant vector fields are introduced. A group action is shown to have an orbit space which is itself the set of points of a universe. There is no apparent general need for "slice theorems" in this theory (cf. [11]).

§13. *Action of a local group universe on a universe.* It is shown that local invariance of c is equivalent to all Lie derivatives of c being equal to zero.

§14. *Low-order terms in the power series expansion of* ψ. Taylor's Theorem is used to show that the set of right-invariant vector fields and the Lie alebra of the group are isomorphic as Lie algebras.

*) This Part will be published in Volume XXVII-4 (1986).

0. NOTATION.

This paper has some very special conventions.

Categories. If A, B $\in \mathcal{Z}$, $\mathcal{Z}$ a category, $(A,B) = (A,B)_{\mathcal{Z}}$ is the set of morphisms A $\to$ B. A$\uparrow$B is the canonical morphism from A to B when the context makes clear what A$\uparrow$B would be (e.g., if A is an initial object of $\mathcal{Z}$). The conventions of [1] are generally followed here. Z-$\mathcal{Z}$ for Z in the category $\mathcal{Z}$ is the category of morphisms in $\mathcal{Z}$ with domain Z,

$$(i: Z \to Z',\ j: Z \to Z'')_{Z-\mathcal{Z}} = \{\ f \in (Z',Z'') \mid fi = j\ \}.$$

We define $\mathcal{Z}$-Z analogously. S $\dashv$ T means S is a left adjoint of T. If I is a diagram scheme (cf. [1]), C_I is the constant diagram of scheme I associated to C.

Sets. $|A|$ denotes the underlying set of A when the context makes its meaning clear. C$|$B means the image of C in B if C $\subset$ A or C $\in$ A, when the context specifies a map A $\to$ B. If S, T $\in$ **sets**,

$$f : S \dashrightarrow T \quad \text{means} \quad f \in (U,T)_{\mathbf{sets}} \text{ for some set } U \subset S.$$

Write U = dom f. The category of *weak sets* denoted **wsets** has the same objects as **sets**, but

$$(S,T)_{\mathbf{wsets}} = \{\ f : S \dashrightarrow T\ \}.$$

If S, T, V $\in$ **wsets**, $f \in (S,T)$, $g \in (T,V)$, then $g \circ f \in (S,T)$ is defined by

$$(g\circ f)(s) = g(f(s)),$$
$$\text{dom } (g\circ f) = \{\ s \in S \mid s \in \text{dom } f,\ f(s) \in \text{dom } g\ \}.$$

Our abbreviation for closure, say in a topological space, is cl.

In what follows, K = $\mathbf{R}$ or K = $\mathbf{C}$. If P is any finite set,

$$M_P = (P,|K|)_{\mathbf{sets}}$$

and is called a *euclidean manifold*. If X is any set, define

$$K_X = (X,|K|)_{\mathbf{wsets}}.$$

Define $B_P = K_{M_P}$ if $\#P < \infty$, $B_p = B_{[1,p]}$ if $p \in \mathbf{N}$, where

$$[1,p] = \{ 1, \ldots, p \} \quad (= \emptyset \text{ if } p = 0).$$

We note that if $a \in K$, we have $a_p \in B_p$ defined by

$$a_p : M_p \to K \quad \text{with} \quad M_p = M_{[1,p]}, \quad a_p(x) = a \quad \text{for every } x.$$

If $f : P \to Q$, P, Q finite, define M_f: $M_Q \to M_P$ by $M_f(b) = b \circ f$. The euclidean manifold M_P has *coordinate functions* $z_p^P \in B_{M_P}$ defined by

$$(z_p^P)(a) = a(p), \quad p \in P.$$

If $1 \leq q \leq p$, we write z_q^p for $z_p^{[1,p]}$. As usual, $K \simeq M_1$, and $x \in M_p$ can be identified with

$$(x_1, \ldots, x_p), \quad \text{where} \quad x_q = x_q^p(x).$$

We note that $M_{[1,0]} = M_\emptyset$ has exactly one element $\emptyset$ (considered as equal to the empty graph). We can identify B_0 with $K \amalg \{\emptyset_0\}$, where $\emptyset_0$ is a symbol, by

$$f \mapsto \emptyset_0 \text{ if } \operatorname{dom} f = \emptyset, \; f \mapsto f(\emptyset) \text{ if } \operatorname{dom} f \neq \emptyset.$$

If $F \in B_p$ and $F_1, \ldots, F_p \in K_X$, $p > 0$, we have an element denoted $F(F_1, \ldots, F_p)$ of K_X defined by

$$(F(F_1, \ldots, F_p))(x) = F(F_1(x), \ldots, F_p(x))$$

with dom $F(F_1, \ldots, F_p)$ defined as all $x \in X$ such that the right-hand side of this equation makes sense. The paper that follows is basically a particular way of generalizing this observation about K_X. We note that K_X has a unique element $\emptyset_X$ such that dom $\emptyset_X = \emptyset$.

The reader must now pick one of the following as a synonym for *admissible:* C^∞, real-analytic, complex-analytic. One must then let $K = \mathbf{R}$ if "admissible" means C^∞ or real-analytic, $K = \mathbf{C}$ if it means complex-analytic. Let $A_r \subset B_r$ consist of all those functions with open domain which are admissible functions of the points of that domain. If the reader wishes to axiomatize the theory that follows more fundamentally than is done here, he will note that initially we use very few special properties of K and the sequence $A_1, A_2, \ldots$, but that by the end of the paper, the list of properties used grows quite long.

PART I: GENERAL THEORY OF UNIVERSES

1. DEFINITION AND ELEMENTARY PROPERTIES OF UNIVERSES.

A set U is called a *weak universe* if for each $h = 1, 2, \dots$ (but not for $h = 0$), and each element (F, u) of $A_h \times U^h$, an element

$$F(u) = F(u_1, \dots, u_h)$$

of U is given. Put another way, U is a universal algebra with set of operators $A_1 \amalg A_2 \amalg A_3 \amalg \dots$, the elements of A_h acting with arity h. If U, V are weak universes, $f \in (|U|, |V|)$ is called a ***morphism*** if whenever $h > 0$, $F \in A_h$, $u \in U^h$, we have

$$f(F(u)) = F(fu), \quad \text{where} \quad fu = (fu_1, \dots, fu_h).$$

(It is tiresome to endlessly repeat the caveat $h > 0$, so we usually will not do that, and shall consider that $h > 0$ is understood.)

We note that since the elements of A_h are not necessarily globally defined functions, the definitions given here do not fit within the framework of "Lawvere theories" (cf. [1], p. 220). We let $\mathcal{W}$ be the category of weak universes, and note that $\mathcal{W}$ is a category of sets with algebraic structure as defined in 11.1.9 of [1]. In time we shall, in the spirit of [1], introduce commutativity relations to define the subcategory of $\mathcal{W}$ that we shall be studying.

For the case where admissible means C^∞, we get examples of weak universes by considering a C^r-manifold M and letting $U \subset K_{|M|}$ consist of all C^s functions (for a fixed $s \le r$) whose domains are open subsets of M. Such examples, however, do not even begin to suggest the range of possibilities.

We fix a countable sequence of symbols $z_1, z_2, \dots$. The set of *elementary expressions* is defined inductively by arbitrary use and reuse of the following two rules:

1) z_i is an elementary expression, $i = 1, 2, \dots$;

2) If $E_1, \dots, E_h$ $(h > 0)$ are elementary expressions and $F \in A_h$, then $F(E_1, \dots, E_h)$ is an elementary expression.

For instance

$$F(G(z_1, H(z_2, z_2)), z_5, H(z_1, z_3)) \quad \text{where } F \in A_3,\ G, H \in A_2$$

is an elementary expression. If we use a different sequence of symbols, say w_1, w_2, ..., we shall speak instead of elementary expressions in w_1, w_2, ... We can speak of an *elementary expression in a finite sequence of symbols* w_1, ...w_p. If E is an elementary expression in w_1, ..., w_p, but not in w_1, ..., w_{i-1}, w_{i+1}, ..., w_p, we shall say that E *effectively involves* w_i, or that E is not *free of* w_i. We write $E = E(w_1, \ldots, w_p)$ if E is an elementary expression in w_1, ..., w_p. The example that was given is an elementary expression in z_1, z_2, z_3, z_4, z_5 free of z_4, hence also is an elementary expression in z_1, z_2, z_3, z_5. If an elementary expression has a nest of brackets ((... ( ) ...)) with n pairs of brackets, but none with more than n pairs, we call $c(E) = n$ the *complexity* of E. In the example, $c(E(z_1, \ldots, z_5)) = 3$. We have

$$c(z_i) = 0, \qquad c(F(z_1, \ldots, z_p)) = 1 \quad \text{if} \quad F \in A_p.$$

An equation $E_1 = E_2$ where E_1, E_2 are elementary expressions is called an *elementary equation*. If $u_1, \ldots, u_p \in U \in \mathcal{V}$, and if $E(z_1, \ldots, z_p)$ is an elementary expression, then $E(u_1, \ldots, u_p) \in U$ can be defined by replacing z_i by u_i for every i and evaluating by using the maps $A_n \times U^n \to U$ that define the weak universe structure on $|U|$. Thus, in our example,

$$\begin{aligned} & E(u_1, \ldots, u_5) = F(v, u_5, w) \\ \text{where} \quad & v = G(u_1, t),\ t = H(u_2, u_2),\ w = H(u_1, u_3). \end{aligned}$$

Given any elementary equation

$$E_1(z_1, \ldots, z_p) = E_2(z_1, \ldots, z_p)$$

and a weak universe U, we say $u \in U^n$ *is a solution of* $E_1 = E_2$ if $E_1(u) = E_2(u)$. An elementary equation $E_1(z_1, \ldots, z_p) = E_2(z_1, \ldots, z_p)$ which has every element of U^p as a solution is called an *elementary identity for* U. We shall then say that U *satisfies* $E_1 = E_2$. We observe that every K_X satisfies the identity $\emptyset_1(z_1) = \emptyset_1(z_2)$, where $\emptyset_r \in A_r$ is defined by dom $\emptyset_r = \emptyset$.

An elementary equation $E_1(z_1, \ldots, z_p) = E_2(z_1, \ldots, z_p)$ is called *balanced* if z_i is effectively involved in E_1 and E_2 for every $i = 1, \ldots, p$. If an elementary equation is balanced and is an elementary identity for each of the weak universes A_1, A_2, A_3, ..., we shall say it is a *fundamental identity*. A weak universe for which every fundamental identity is an identity will be called a *universe*. Let $\mathcal{U}$ be the full subcategory of $\mathcal{V}$ supported by the universes. We shall see that there are interesting examples of universes for which the identity $\emptyset(z_1) = \emptyset(z_2)$ fails.

If we would show that the weak universe U is a universe, we would need, according to our definition, to show that if $E_1 = E_2$ is any fundamental identity, then U satisfies $E_1 = E_2$. The list of fundamental identities is uncountably infinite, however, and though for example any K_X is a universe, that is not apparent at this point. We shall proceed now to develop a "constructible" list of fundamental identities such that any weak universe which satisfies all of them will be a universe.

We have first that

$$(1.1.1) \qquad z_1{}^1(z_1) = z_1$$

is a fundamental identity, since to say that $U \in \mathcal{V}$ satisfies (1.1.1) just means

$$z_1{}^1(u) = \mathrm{Id}_K(u) = u \quad \text{if} \quad u \in U,$$

where
$$\mathrm{Id}_K(x) = x \quad \text{for all} \quad x \in K.$$

A weak partition of $1, \ldots, p$ $(p > 0)$ is given by writing

$$\{1, \ldots, p\} = \cup_{i=1}{}^h(e_{i_1}, \ldots, e_{i p_i})$$

where $p_i > 0$ for every i. Given such a weak partition and any family $u_1, \ldots, u_p$, we set

$$u_{ij} = u_{e_{ij}}, \qquad 1 \leq i \leq h, \quad 1 \leq j \leq p_i$$

and write u_{i*} for the sequence $u_{i1}, \ldots, u_{ip_i}$. If $F \in A_h$ and $F_i \in A_{p_i}$ for $1 \leq i \leq h$, then

$$(1.1.2) \quad (E(z_1{}^p, \ldots, z_p{}^p))(z_1, \ldots, z_p) = E(z_1, \ldots, z_p)$$

where $E = F(F_1(z_{1*}), \ldots, F_h(z_{h*}))$ is a fundamental identity. To prove this, note that if $U \in \mathcal{V}$ and $u \in U^p$, then (1.1.2) means that

$$(F(\ldots, F_i(z_{i*}{}^p), \ldots))(u_1, \ldots, u_p) = F(\ldots, F_i(u_{i*}), \ldots).$$

This equation holds if $U = K_X$ for any set X, hence for any A_h since $A_h \subset B_h = K_{M_h}$ as a weak subuniverse. Any fundamental identity of type (1.1.1) or (1.1.2) will be called a *seminal identity*. We note that any fundamental identity, in particular any seminal identity, is a "commutativity relation" in the sense of 11.1.9 of [1].

PROPOSITION (1.2). Any weak universe that satisfies every seminal identity is a universe.

For (1.2) we use a lemma that generalizes (1.1.2).

LEMMA (1.2.1). Let $U \in W$ *satisfy every seminal identity, and let* $E(z_1, \ldots, z_p)$ *be an elementary expression that is not free of any* z_i. *Then* U *satisfies*

$$(E(z_1{}^p, \ldots, z_p{}^p))(z_1, \ldots, z_p) = E(z_1, \ldots, z_p).$$

We use induction on $r = c(E)$. If $r = 0$, then $p = 1$ and $E(z_1) = z_1$. Then

$$(z_1(z_1{}'))(u) = z_1{}'(u) = u$$

by (1.1.1). Now let $r > 0$. We can write

$$E(z_1, \ldots, z_p) = F(E_1(z_{1*}), \ldots, E_h(z_{h*})),$$

using an appropriate weak partition of 1, ..., p, where each $E_i(z_{i*})$ is elementary with complexity $< r$. Using induction on r and (1.1.2) with $F_i = E_i(z_1{}^{pi}, \ldots, z_{pi}{}^{pi})$ we get, if $u \in U^p$ that

$$(E(z_1{}^p, \ldots, z_p{}^p))(u) = (F(\ldots, E_i(z_{i*}{}^p), \ldots))(u) =$$
$$(F(\ldots, E_i)(z_{i*}{}^{pi})(z_{i*}{}^p), \ldots))(u) = F(\ldots, F_i(u_{i*}), \ldots) =$$
$$F(\ldots, E_i(u_{i*}), \ldots) = E(u),$$

proving (1.2.1).

To prove (1.2), let U be as in (1.2.1) and let

$$E_1(z_1, \ldots, z_p) = E_2(z_1, \ldots, z_p)$$

be any fundamental identity. If $u \in U^p$, then, since A_p satisfies $E_1 = E_2$, we have

$$E_1(u) = (E_1(z_1{}^p, \ldots, z_p{}^p))(u) = (E_2(z_1{}^p, \ldots, z_p{}^p))(u) = E_2(u),$$

proving (1.2).

It is evident that if X is any set, K_X will satisfy every seminal identity. If U is any universe and $V \to U$ a one-one morphism in $\mathcal{V}$, then V is also a universe. Also, any surjective morphism $U \to V$ in $\mathcal{V}$ where $U \in \mathcal{U}$ implies $V \in \mathcal{U}$. Thus from K_X we get a whole host of universes, and we see that every weak universe we have considered so far is in fact a universe.

It follows from 11.5.1 of [1] that $\mathcal{U}$ is complete and from 11.5.3 that $|\ | : \mathcal{U} \to$ **sets** preserves limits. In particular, if $(U_i)_{i \in I}$ is a family of universes, then $\prod_{i \in I} |U_i|$ has the universe structure defined by

$$(F(u_1, \ldots, u_p))_i = F(u_{1i}, \ldots, u_{pi})$$

if $\quad u_j = (u_{ji})_{i \in I} \in \prod_{i \in I} |U_i|, \quad 1 \leq j \leq p \quad$ and $\quad F \in A_p,$

and this is $\prod_{i \in I} U_i$. In the next pages we shall also see that $\mathcal{U}$ is cocomplete.

To establish that $\mathcal{U}$ is cocomplete, we need to look at a construction which can be done in $\mathcal{U}$ that seems not to be expressible in terms of standard notions of category theory. We shall consider any ordered set I as a category where $(i,j) : i \to j$ is unique if $i \leq j$ and doesn't exist otherwise (so

$$(i,j) = \{ (i,j) \} \quad \text{if} \quad i \leq j).$$

Let I be an upper semi-lattice (the sup $i \vee j$ of i and j always exists), $T: I \to \mathcal{U}$ any functor. We define a universe U with $|U| = \coprod_{i \in I} |Ti|$. If $t \in Ti \subset U$, let $\gamma_t = i$. Assume that $F \in A_h$, $t_1, \ldots, t_h \in U$, and set

$$i = \gamma_1 \vee \ldots \vee \gamma_h \quad \text{where} \quad \gamma_i = \gamma_{t_i}.$$

Let $t_j|Ti$ denote the image of t_j under $T(\gamma_j) \to Ti$, and define

$$F(t_1, \ldots, t_h) = F(t_1|Ti, \ldots, t_h|Ti) \in Ti \subset U,$$

using the fact that Ti is a universe to form the right-hand side of the equation. This makes U into a weak universe $\coprod T$ that we shall call the *concatenation* of T. It is easily seen that $\coprod T$ satisfies any given balanced elementary equation iff every Ti satisfies this equation. Thus $\coprod T$ is a universe.

It should be observed that any one-element set is a universe in a unique way, a so-called *one-element universe*. Concatenations of one-element universes can be used to disprove many naïve conjectures. We note that if $i \neq j$, $t_i \in Ti$, $t_j \in Tj$, then $\emptyset_1(t_i) \in Ti$, so $\emptyset_1(t_i) \neq \emptyset_1(t_j)$, $\emptyset_1(z_1) = \emptyset_1(z_2)$ is not an identity for $\amalg T$ if $\#I > 1$.

Let S be any set, and let I be the upper semi-lattice of all finite non-empty subsets of S where if P, Q $\in$ I, P $\leq$ Q means P $\subset$ Q. We have a functor $T\colon I \to \mathcal{U}$ defined by $T(P) = A_P$ where $A_P \subset K_{M_P}$ consists of all admissible functions $M_P \dashrightarrow K$ with open domain. Define $\mathbf{U}_I \in \mathcal{U}$ to be $\amalg T$. We let $Tu = A_u : A_P \to A_Q$ if $u : P \to Q$ is any map of finite sets.

LEMMA (1.3). $I \mapsto \mathbf{U}_I$ *is a left adjoint for* $|\ |\colon\ \mathcal{U} \to$ sets.

We need to show that for any given I, $\mathbf{U}_I$ represents the functor $\mathcal{U} \to$ sets defined by $U \mapsto (I, |U|)$ (cf. [1], 16.4.5). Define

$$I^*\colon I \to |\mathbf{U}_I| \quad \text{by} \quad i \mapsto t_i = z_1^{\{i\}} \in A_{\{i\}} \subset \mathbf{U}_I.$$

We need to show that if $f : I \to |U|$, there exists a unique morphism

$$\phi_f\colon \mathbf{U}_I \to U \quad \text{with} \quad \phi_f I^* = f.$$

LEMMA (1.3.1). *Suppose* $u : [1,q] \to [1,p]$ *is surjective and* $F \in A_q$. *Then*

$$(A_u F)(z_1, \ldots, z_p) = F(z_{u1}, \ldots, z_{uq})$$

is a fundamental identity.

Indeed, $A_u F = F(z_{u1}, \ldots, z_{uq})$.

To show that ϕ_f is unique, note that if $P = \{x_1, \ldots, x_p\} \subset I$, then $F(z_{x_1}{}^P, \ldots, z_{x_p}{}^P)$ for $F \in A_P$ is a typical element of A_P. Necessarily

$$\phi_f(F(z_{x_1}{}^P, \ldots, z_{x_p}{}^P)) = \phi_f(F(I^* x_1, \ldots, I^* x_p)) = F(fx_1, \ldots, fx_p),$$

so ϕ_f is unique.

To show ϕ_f exists, let $x_1, \ldots, x_p \in I$ be all distinct, and let $P = \{x_1, \ldots, x_p\}$. Let $w_j = z_{x_j}{}^P$. Then $F(w_1, \ldots, w_p) \in \mathbf{U}_I$ can be written thus for only one $F \in A_P$. We define

$$\phi_f(F(w_1, \ldots, w_p)) = F(fx_1, \ldots, fx_p).$$

When ϕ_f is so defined, we note that the formula just given will continue to hold even if the x_i are not distinct or are chosen in some other order. Indeed, let

$$\{y_1, \ldots, y_q\} = \{x_1, \ldots, x_p\}$$

have cardinality p and let $G \in A_q$. Let us show that

$$\phi_f(G(z_{y_1}{}^P, \ldots, z_{y_q}{}^P)) = G(fy_1, \ldots, fy_q).$$

Let $y_j = x_{uj}$, $1 \leq j \leq q$ where $u : [1,q] \twoheadrightarrow [1,p]$. Then

$$\phi_f(G(z_{y_1}{}^P, \ldots, z_{y_q}{}^P)) = \phi_f((A_u G)((z_{x_1}{}^P, \ldots, z_{x_p}{}^P)) =$$
$$(A_u G)(f(x_1), \ldots, f(x_p)) = G(f(y_1), \ldots, f(y_q))$$

using (1.3.1).

***LEMMA** (1.3.2). $\phi_f \in U$.*

Let $u_1, \ldots, u_h \in U_I$, $F \in A_h$. Write $u_i = F_i(w_{i1}{}^{P_i}, \ldots, w_{ip_i}{}^{P_i})$ where

$$w_{ij}{}^{P_i} = z_{x_{ij}}{}^{P_i}, \qquad P_i = (x_{i1}, \ldots, x_{ip_i}).$$

Set

$$P = P_1 \cup \ldots \cup P_h, \qquad x_1, \ldots, x_p = x_{11}, \ldots, x_{1p_1}; \ldots ; x_{h1}, \ldots, x_{hp_h};$$

i.e., we have here a weak decomposition of $1, \ldots, p$, and this is the indexing associated to it. Then

$$\phi_f(F(u_1, \ldots, u_h)) = \phi_f(F(\ldots, F_i(w_{i*}{}^P), \ldots)) =$$
$$(F(\ldots, F_i(w_{i*}{}^P), \ldots))(fx_1, \ldots, fx_p) = F(\ldots, F_i(fx_{i*}), \ldots)) =$$
$$F(\ldots, \phi_f(F_i(w_{i*}{}^{P_i})), \ldots) = F(\phi_f u_1, \ldots, \phi_f u_h).$$

***THEOREM** (1.4). U is cocomplete.*

Let $D : I \to U$ be a diagram in U, and let $S = \coprod_{i \in I} |D(i)|$. Let $U \in U$ and $T: D \to I_U$ a morphism of D into the constant diagram associated to U. Then define $f_T: U_S \to U$ by $(f_T S^*)|_{D_i} = Ti$ for each $i \in I$. *Define* $u \sim v$ for $u, v \in U_S$ if $f_T u = f_T v$ for every $T: D \to I_U$.

Let $V = U_S/\sim$. The natural maps $Di \to V$ constitute an element of (D, I_V) that exhibits V as a colimit of D.

2. STRUCTURE THEORY OF UNIVERSES.

From now on, fundamental identities will be, in most cases, treated as obvious and be used without any special explanation. We use the addition function

$$z_1{}^2 + z_2{}^2 : K^2 \to K$$

to define an addition on a weak universe U as follows:

$$u + v = (z_1{}^2 + z_2{}^2)(u,v) \quad \text{if} \quad u, v \in U.$$

Similarly we let

$$uv = (z_1{}^2 z_2{}^2)(u,v).$$

If U is a universe, fundamental identities give us

$$u + v = v + u, \quad uv = vu, \quad u(v+w) = uv + uw$$

and associativity laws for addition and multiplication. Our study of an arbitrary universe U will first focus on the additive structure of U.

The function neg: $K \to K$ defined by $\text{neg}(a) = -a$ is in A_1, so we can define $-u = \text{neg}(u)$, $u \in U$. As usual we write $u - v$ for $u + (-v)$. Then $x = -u$ solves $u+x+u = u$ (because of a formal identity), so $(U,+)$ is a "regular" semigroup ([2], p. 10). By v.4.5 of [2], p. 159, U is an "inverse semigroup", a notion which has an abundant literature (cf. [3]). It is this discovery about U that led me to realize that the present approach to differential calculus might be a viable one. It lies at the heart of the entire theory.

We can use the fact that U is a universe to give quick proofs of facts that hold as well for arbitrary commutative semigroups. If $u \in U$, define $0u = 0_1(u)$ (where $0_1 : K \to K$ is defined by $0_1(a) = 0$). If $U = K_X$, X any set, $0u$ is just the zero function on the domain of u. Thus for an arbitrary universe U, we shall think of $0u$ as being somehow the domain of u, an analogy that will be endlessly exploited here.

Define $u \preceq v$ $(u, v \in U)$ if $u + 0v = v$. (In our analogy,

$$u \preceq v \Leftrightarrow v = u\,|_{\mathrm{dom}\, v}\,.)$$

We note that :

$u \preceq u$ since $u + 0u = u$, $u \preceq v \preceq u \Rightarrow u = v$, $u \preceq v \preceq w \Rightarrow u \preceq w$

(since $w = v + 0w = u + 0v + 0w = u + 0(v + 0w) = u + 0w$),

$u \preceq v, w \Rightarrow v + 0w = w + 0v$, $u \preceq v$ and $0u = 0v \Rightarrow u = v$.

Also

$$u \preceq u', v \preceq v' \Rightarrow u{+}v \preceq u'{+}v', uv \preceq u'v'.$$

The set $0U = \{ 0u \mid u \in U \}$ plays a very special role in the theory and acts like our family of "open sets". The elements n in $0U$ are characterized by the equation $n + n = n$, i.e.,

$$n + n = n \Leftrightarrow n \in 0U.$$

We note that

$$0U + 0U \subset 0U, \quad (0U)(0U) \subset 0U.$$

In fact, if $n, p \in 0U$, then $n{+}p = np$. Note $n \in 0U \Rightarrow n = 0n$.

If $n \in 0U$, let $U_n = \{ u \in U \mid 0u = n \}$. Then $|U| = \coprod_{n \in 0U} U_n$. We shall see that this decomposition of U into the sets U_n has very agreeable properties.

LEMMA (2.1). *Let $n, p \in 0U$, $*$ = plus or times ($+$ or $\times$). Then* $U_n * U_p \subset U_{n+p}$.

Let $u \in U_n$, $v \in U_p$. Then

$$0(u*v) = (0u)*(0v) = (0n)*(0p) = 0(n*p) = n*p = n + p.$$

In particular, $U_n * U_n \subset U_n$. If $u \in U_n$, then

$$u + n = u = 1(0u) = 1(n)u,$$

so U_n is an associative ring with $n = 0_{U_n}$, $1(n)$ = identity of U_n.

LEMMA (2.2). $0U$, *with the order induced from* U, *is an upper semilattice.*

If $n, p \in 0\mathrm{U}$, then $n, p \leq n + p$. Suppose $n, p \leq u \in \mathrm{U}$. Then $u = n + 0u \in 0\mathrm{U}$, so $n + p \leq u + u = u$. Thus $n + p$ is the sup $n \vee p$ of n and p in $0\mathrm{U}$ (and in U as well).

If $a, b \in \mathrm{U}$, $n, p \in 0\mathrm{U}$, $* = +$ or $\times$, then

$$a*b + n*p = (a+n)*(b+p) \tag{2.3}$$

since $$a*b + (0n)*(0p) = (a+0n)*(b+0p).$$

If $n, n' \in 0\mathrm{U}$, $n \leq n'$, then $\mathrm{U}_n + n' \subset \mathrm{U}_{n'}$. By (2.3),

$$r_n{}^{n'} : U_n \to \mathrm{U}_{n'} \quad \text{defined by} \quad r_n{}^{n'}(u) = u + n'$$

respects + and ×. Also

$$r_n{}^{n'}(1(n)) = 1(0n) + n' = 1(0n + 0n') = 1(n'),$$

so $r_n{}^{n'}$ is a unitary ring morphism. Thus we have $|\mathrm{U}| = \coprod_{n \in 0\mathrm{U}} \mathrm{U}_n$, where $\{ \mathrm{U}_n \mid n \in 0\mathrm{U} \}$ is a family of rings directed by an upper semilattice.

We pause to observe that we can work in $\mathcal{U}$ in very much the same way that we can work in the category of rings. Let $\mathrm{U} \in \mathcal{U}$, u, v in U. We shall write (u,v) as "$u = v$" when we wish to think of (u,v) as representing an equation. Let $\mathrm{E} = \{ u_i = v_i \mid i \in \mathrm{I} \}$ be a family of equations on U ($u_i, v_i \in \mathrm{U}$ all i) where I is any indexing set. If $f \in (\mathrm{U},\mathrm{W})_{\mathcal{U}}$, f is a *solution* of E if $fu_i = fv_i$ for every $i \in \mathrm{I}$. Let

$$\mathrm{TW} = \{ f \in (\mathrm{U},\mathrm{W}) \mid f \text{ is a solution of E} \} = (\mathrm{Sol\ E})(\mathrm{W}).$$

If Sol E is represented by $g : \mathrm{U} \to \mathrm{Z} \in (\mathrm{Sol\ E})(\mathrm{Z})$ (so that $\mathrm{TW} = (\mathrm{Z},\mathrm{W})$) we shall say that g is a *generic solution* of E.

PROPOSITION (2.4). *Let* E *be a family of equations on* $\mathrm{U} \in \mathcal{U}$. *Then* E *has a generic solution* $\mathrm{U} \to \mathrm{U/E}$.

If $u, u' \in \mathrm{U}$, define $u \sim u'$ if for any solution f of E we have $f(u) = f(u')$. If $u_i \sim u'_i$, $1 \leq i \leq p$ and $\mathrm{F} \in \mathrm{A}_p$, then

$$\mathrm{F}(u_1, \ldots, u_p) \sim \mathrm{F}(u'_1, \ldots, u'_p), \quad \text{so} \quad \mathrm{U}/\!\sim\; \in \mathcal{U}.$$

Obviously $\mathrm{U} \to \mathrm{U}/\!\sim$ is a generic solution of E. Notation for this map will be $u \mapsto u/\mathrm{E}$, $u \in \mathrm{U}$.

The reader able to imagine many equations one might impose on the elements of a universe might not observe that using equations, we can state what it means for elements of a universe to satisfy what is usually referred to as an "inequality". For instance, let $a \in K = R$, and let $g_a(x) = x$, $x > a$, $g_a \in A_1$. Then the inequality $u > 0$ (in the numerical sense, not in the sense we have been using) can be expressed as $g_0(u) = u$. Also $u \geq 0$ (numerically) can be expressed by the system $\{ g_a(u) = u \mid a < 0 \}$.

Other ideas from algebra also have counterparts for $\mathcal{U}$. Let $U \in \mathcal{U}$, $Y \in$ **sets**, and let $S = |U| \amalg Y$. Let $i : |U| \to |\mathbf{U_S}|$ and $j : Y \to |\mathbf{U_S}|$ be the compositions of $S^*: S \to \mathbf{U_S}$ with $|U| \to S$ and $Y \to S$ respectively. Now let $\mathbf{U_S} \to V$ be the generic solution of the following set of equations of E :

$$S^*(F(u_1, \ldots, u_p)) = F(S^*u_1, \ldots S^*u_p), \qquad p \in \mathbf{N},\ u \in U_p,$$

where we consider that $|U| \subset \mathbf{U_S}$ using i. We shall define $U{\bullet}Y = \mathbf{U_S}/E$ and let $Y \mapsto |U{\bullet}Y|$ be induced by j. The following is immediate.

PROPOSITION (2.5). $Y \mapsto U{\bullet}Y$ *is a left adjoint for* $|\ |: U\text{-}\mathcal{U} \to$ **sets**.

If $0 \in U \in \mathcal{U}$ and $0 + u = u$ for every $u \in U$, we call 0 an *identity element* of U, and write $0 = 0_U$ since it must be unique. We shall denote by $\mathcal{U}_0$ the subcategory of $\mathcal{U}$ supported by those universes that have an identity element, defining

$$(U,V)_{\mathcal{U}} = \{ f \in (U,V)_{\mathcal{U}} \mid f(0_U) = 0_V \}.$$

If $U \in \mathcal{U}_0$, $a \mapsto a_1(0_U)$ defines an element of $(A_0,U)_{\mathcal{U}_0}$ giving us, in a sense, a 0-ary operator of A_0 on U. In fact, A_0 is an initial object of $\mathcal{U}_0$. We define $U_0 = U_{0U}$ for $U \in \mathcal{U}_0$, and we call U_0 the set of *global elements* of U. We have a functor $(\ )_0 : U\text{-}\mathcal{U}_0 \to$ **sets** defined by $V \mapsto V_0$.

PROPOSITION (2.6). *Let* $U \in \mathcal{U}_0$. *Then* $(\ )_0$ *has a left adjoint* $U(\ )$: **sets** $\to U\text{-}\mathcal{U}_0$.

Evidently $U \mapsto U{\bullet}Y$ for any set Y is an injection as is $Y \mapsto |U{\bullet}Y|$ (since $U\text{-}\mathcal{U}$ possesses objects with more than one element), so we can consider that $U, Y \subset U{\bullet}Y$. Let $U(Y) = (U{\bullet}Y)/E$, where E is the set of all equations $v + 0_U$ for $v \in U{\bullet}Y$ together with all equations $0y = 0_U$,

$y \in Y$. Since $U{\bullet}Y \to (U{\bullet}Y)/E$ is surjective, $0_U/E$ is an identity for $U(Y)$. Then

$$(U(Y),V)_{U-UO} = \{\ f \in (U{\bullet}Y,V)_{U-UO} \mid f(0Y) = \{0_V\}\ \} = (Y,V_0).$$

We can use (2.6) for a new look at the universes A_h. We have $A_h = A_0([1,h])$.

3. PIECING TOGETHER GLOBAL INFORMATION FROM LOCAL INFORMATION.

If u, $v \in U \in \mathcal{U}$, we shall say that u and v *match* if $u + 0v = v + 0u$. A subset M of U is *matching* if every two elements of M match. We write $\mathcal{M}(U)$ for the set of all matching subsets of U, and note that any subset of U which has a lower bound is necessarily in $\mathcal{M}(U)$. Also $0U \in \mathcal{M}(U)$. We note the following axiom that is satisfied by every universe K_X.

(3.1.1) Every matching subset of U *has a greatest lower bound (glb).*

The reader will note that (3.1.1) resembles the axiom that distinguishes a presheaf from a sheaf. This statement can be made exact using the theory of inverse semigroups.

Let U satisfy (3.1.1). Let $F, G \in A_h$, $u, v \in U^h$. Define $u + 0v \in U^h$ by $(u + 0v)_i = u_i + 0v_i$, and assume $u + 0v = v + 0u$ and also $F + 0G = G + 0F$. Then , using some fundamental identities, we get

$$F(u) + 0G(v) = F(u + 0v) + 0G(v + 0u) =$$
$$(F + 0G)(u + 0v) = G(v) + 0F(u).$$

It follows that if $\mathcal{F} \in \mathcal{M}(A_h)$, $M_1, \ldots, M_h \in \mathcal{M}(U)$, then

$$\mathcal{F}(M_1, \ldots, M_h) = \{\ F(u_1, \ldots, u_h) \mid F \in \mathcal{F},\ u \in U^h\ \} \in \mathcal{M}(U).$$

We consider the following axiom:

$$\textbf{(3.1.2)}\quad \mathrm{glb}(\mathcal{F}(M_1, \ldots, M_h)) = (\mathrm{glb}\ \mathcal{F})(\mathrm{glb}\ M_1, \ldots, \mathrm{glb}\ M_h).$$

We note that if F, G $\in A_n$, u, $v \in U^n$, F $\leq$ G, $u_i \leq v_i$ for all i, then $F(u) \leq G(v)$, so $\geq$ holds automatically in (3.1.2). Thus in fact (3.1.2) abbreviated X = Y is equivalent to X $\leq$ Y and also to 0X = 0Y. If U $\in$ *U* satisfies (3.1.1) and (3.1.2), we shall call U *cohesive*. If U is cohesive, glb 0U is an identity element for U since

$$u \leq u + \text{glb } 0U \leq u + 0u = u.$$

If U and V are cohesive, any $f \in (U,V)_{U_0}$ is called a *morphism* provided f(glb M) = glb (fM) for every M $\in$ *M*(U). We let *C* denote the category of cohesive universes.

Let U $\in$ *C*, M, N $\in$ *M*(U), $*$ = + or $\times$. Then, (glb M)$*$(glb N) = glb (M$*$N) by (3.1.2). In particular, if $n \in$ 0U and M $\subset$ 0U, then glb (n+M) = n + glb M. The elements of 0U under + and $\times$ behave like the family of all subsets of a given set under $\cap$ and $\cup$. Comparison with a type of algebra commonly considered in quantum mechanics suggests that the algebra of 0U under these operations for the case when U is an entirely arbitrary universe is analogous to the "logic" of a quantum mechanical system. Since 0U is "Boolean" when U $\in$ *C*, i.e. glb (n + M) = n + glb M, and the logic of quantum mechanical systems is non-Boolean, this suggests that the universes which are not cohesive, or satisfy (3.1.1) but not (3.1.2), can possibly be of interest. Nonetheless the main thrust of what follows will be the study of universes which are cohesive. The following is an example of a non-cohesive universe where the theory which follows gives no information whatever.

EXAMPLE (3.2). Take K = **R**. If f, $g \in A_1$, define

$$f \sim g \quad \text{if} \quad f + 0_{(a,\infty)} = g + 0_{(a,\infty)} \text{ for some } a \in K.$$

Then U = $A_1/\sim$ has a unique structure of universe such that $A_1 \to U$ (written $f \mapsto f/\sim$) is a morphism of universes. Let g : U $\to$ C be any morphism of U into a cohesive universe C, and let $w = g(z/\sim)$, where $z = \mathrm{Id}_K$. Since any element of A_1 can be written $f(z)$ with $f \in A_1$, any element of g(U) can similarly be written $f(w)$. If $a \in$ K, then

$$\mathrm{Id}_{(a-1,a+1)}(z/\sim) = \emptyset(z/\sim)$$

since
$$\mathrm{Id}_{(a-1,a+1)} + 0_{(a+1,\infty)} = \emptyset + 0_{(a+1,\infty)}.$$

Therefore

$$w = \text{glb } \{ \mathrm{Id}_{(a-1,a+1)}(w) \mid a \in K \} = \emptyset(w).$$

If $t = f(w) \in g$(U), then $t = f(\emptyset(w)) = \emptyset(w)$, i.e., g(U) = $\{w\}$.

If M is an admissible manifold, M with all its structure can be recovered from A_M. The reader will be able, after reading some more of this paper, to fashion a proof of this as well as of the fact that

$$(N, M)_{\text{admissible manifolds}} \to (A_M, A_N)_C$$

given by $$f \mapsto (u \to u \circ f)$$

is a bijection.

***THEOREM* (3.3).** *C is complete.*

We use repeatedly the following (cf. [4], p. 11).

***LEMMA* (3.3.1).** *Let $(S_i)_{i \in I}$ be a family of ordered sets, and let $S = \Pi_{i \in I} S_i$ have the "product order"*

$$s \leq t \quad \text{if} \quad s_i \leq t_i \quad \text{for every } i \in I.$$

Let π_i: $S \to S_i$ be the i-th projection. If $T \subset S$, we have

$$\text{glb}_S\, T = (\text{glb}_{S_i}(\pi_i T))_{i \in I} \ ,$$

and either side of this equation is defined whenever the other side is.

Let $C_i \in C$, $i \in I$, and let U be the universe $\Pi_{i \in I} C_i$. The order

$$u \leq v \ (\Leftrightarrow v = u + 0v)$$

on U is the product order derived from the order structures that exist on the C_i. If $M \in M(U)$, $\pi_i M \in M(C_i)$ for every i, so glb M exists and is $(\text{glb}_{C_i}(\pi_i M))_{i \in I}$ by (3.3.1). Let $F \in M(A_h)$, M_1, ..., $M_h \in M(U)$. Then with $\underline{M} = \text{glb } M$, and j a typical element of $\{1,\dots,h\}$,

$$(F(\underline{M}_1, \dots, \underline{M}_h))_i = F((\underline{M}_1)_i, \dots, (\underline{M}_h)_i) = F(\dots, \underline{\pi_i M_j}, \dots) =$$
$$\text{glb}(F(\dots, \pi_i M_j, \dots)) = \text{glb}[(F(M_1, \dots, M_h))_i] = [\text{glb}(F(M_1, \dots, M_h))_i],$$

so $U \in C$. Calculations like the above (but easier) show that if $f \in (A,U)_U$, $A \in C$, then $f \in (A,U)_C$ iff $\pi_i f \in ((A,C)_i)_C$ for every $i \in I$. Thus U is a product in C of the C_i. It is easy to show that if f, g : $C \to C'$ in C, $\{ c \in C \mid fc = gc \}$ is the difference kernel in C of f and g, so (3.3) follows.

4. PHANTOM DECOMPOSITION OF UNIVERSES.

The most general cohesive universe C can be strung together in a very complicated way. However, there is a crude decomposition of C into simpler pieces.

LEMMA *(4.1). Let $p \in \mathrm{U} \in \mathcal{U}$. The following conditions on p are equivalent:*

1) $p \in 0\mathrm{U}$ and U_p is the zero ring;
2) $p \in 0\mathrm{U}$ and $1(p) = p$;
3) $p = \emptyset_1(p)$.

Assume 1. Then

$$1(p) = 1_{\mathrm{U}_p} = 0_{\mathrm{U}_p} = p,$$

so $1 \Rightarrow 2$.

Assume 2, and let $\mathrm{W} = \mathrm{K}\backslash\{1\}$. Let $0_\omega\colon \mathrm{W} \to \mathrm{K}$ be $0_\omega(z) = 0$, $z \neq 1$. Then

$$p = 0(p) = 0_\omega(0p) = 0_\omega(1(p)) = \emptyset(p),$$

so $2 \Rightarrow 3$.

Assume 3. Then

$$0(p) = 0(\emptyset(p)) = \emptyset(p) = p = 1(\emptyset(p)) = 1(p).$$

Therefore $\mathrm{U}_p = \{p\}$ is the zero ring and $3 \Rightarrow 1$.

Lemma (4.1) characterizes in several ways the elements of

$$\emptyset(\mathrm{U}) = \{\, p \in \mathrm{U} \mid p = \emptyset(p) \,\},$$

elements which will be called *phantoms* of U. If $h \in \mathbf{N}$, $\emptyset(\mathrm{A}_h) = \{\emptyset_h\}$, and there exist cohesive universes C with $\#\emptyset\mathrm{C} > 1$ (e.g., concatenate $\{0\} \to \{1\}$ to get $\mathrm{C} = \{0,1\}$). In any case, $\#\emptyset\mathrm{C} \geq 1$ since $\emptyset(0_\mathrm{C}) \in \emptyset\mathrm{C}$. If $p \in \emptyset(\mathrm{U})$, $\mathrm{U} \in \mathcal{U}$, let $R(p) = \{\, u \in \mathrm{U} \mid \emptyset(u) = p \,\}$ define the *realm* of the phantom p. Then $\mathrm{U} = \bigcup_{p \in \emptyset(u)} R(p)$ and this union is a disjoint one. This decomposition is especially nice when $\mathrm{U} \in \mathcal{C}$.

LEMMA *(4.2). Let $\mathrm{U} \in \mathcal{U}$. Then:*

1) $\emptyset(\mathrm{U})$ is a subuniverse of U;
2) If $p \in \emptyset(\mathrm{U})$, then $R(p)$ is a subuniverse of U.

PROPOSITION (4.3). Let $C \in \mathcal{C}$. *Then* $\emptyset(C)$ *and all the* $R(p)$ *are cohesive. Moreover* $\{ R(p) \mid p \in \emptyset(C) \}$ *is directed and* C *is its concatenation.*

The first two assertions hold since

$$\text{glb } \emptyset(M) = \emptyset(\text{glb } M) \Rightarrow \text{glb } \emptyset M \in \emptyset C, \quad \emptyset(\text{glb } M) = p \quad \text{if} \quad M \subset R(p).$$

Since $\emptyset(C) = 0\emptyset(C)$, $\emptyset(C)$ is an upper semilattice. Let $u_1, \ldots, u_n \in C$, $F \in A_n$. Then if $p_i = \emptyset(u_i)$,

$$\emptyset(F(u_1, \ldots, u_n)) = F(\emptyset(u_1), \ldots, \emptyset(u_n)) = p_1 + \ldots + p_n.$$

We need to show C is the concatenation of the $R(p)$. First we make $p \mapsto R(p)$ into a functor $\emptyset C \to \mathcal{C}$. Let

$$n_p = 0_{R(p)} = \text{glb}(0R(p)) \quad \text{if} \quad p \in \emptyset(C).$$

Then $\emptyset(n_p) = p$. Also

$$p \leq q \Rightarrow \emptyset(\text{glb } \{ n_p, n_q \}) = p$$

so (by the definition of n_p),

$$n_p \leq \text{glb } \{ n_p, n_q \} \leq n_q, \quad \text{i.e.,} \quad p \leq q \Rightarrow n_p \leq n_q.$$

If $u \in R(p)$, $\emptyset(u + n_q) = n_q$, so $R(p) + n_q \subset R(q)$ (if $n_p \leq n_q$). For $p \leq q$, define $r_p{}^q : R(p) \to R(q)$ by $r_p{}^q(u) = u + n_q$. One sees that $r_p{}^q \in (R(p), R(q))_{\mathcal{C}}$ so that $R(\ )$ is now a functor $\emptyset C \to \mathcal{C}$. We have already seen that $C = \amalg_{p \in \emptyset C} R(p)$, so (4.3) will be proved once we show that when $u \in C^n$, $F \in A_n$ and $p = \emptyset(u_1) + \ldots + \emptyset((u_n)$, then $F(u) = F(u_1 + n_p, \ldots, u_n + n_p)$. Now $0u_1 + \ldots + 0u_n \geq n_p$, since p is its phantom. It follows that

$$0F(u) = 0F(u) + 0u_1 + \ldots + 0u_n \geq n_p.$$

Then as

$$0u \geq n = n+n \Rightarrow u+n = u,$$

$$F(u) = F(u) + n_p = F(u_1 + n_p, \ldots, u_n + n_p).$$

5. REPRESENTABILITY OF CERTAIN FUNCTORS $\mathcal{C} \to$ sets.

We shall write $C \subset_{\mathcal{C}} C'$ to mean that $C \subset C'$ and that the inclusion map is a morphism in $\mathcal{C}$. We shall use this same convention for $\mathcal{U}$ and other categories of universes as appropriate.

THEOREM (5.1). *Let* T *be a subfunctor of* $(\mathrm{U},\ )_{\mathcal{U}}: \mathcal{C} \to$ sets *where* $\mathrm{U} \in \mathcal{U}$. *Assume* T *satisfies the following conditions:*

1) $\mathrm{C} \subset_{\mathcal{C}} \mathrm{C}' \Rightarrow \mathrm{TC} = (\mathrm{TC}') \cap (\mathrm{U},\mathrm{C})_{\mathcal{U}}$;

2) If $\mathrm{C} = \prod_{i \in I} \mathrm{C}_i$, *the natural bijection* $(\mathrm{U},\mathrm{C})_{\mathcal{U}} = \prod_{i \in I}(\mathrm{U},\mathrm{C}_i)_{\mathcal{U}}$ *identifies* TC *with* $\prod_{i \in I}\mathrm{TC}_i$.

Then T *is representable.*

The conditions 1 and 2 of (5.1) mean simply that T preserves limits. By 10.3.9 of [1], to show that T is representable, we need to show there exists a set $\mathcal{D}$ of objects of $\mathcal{C}$ such that if $\mathrm{C} \in \mathcal{C}$ and $f \in \mathrm{TC}$, it is possible to write $f = hg$ where $g: \mathrm{U} \to \mathrm{D} \in \mathcal{D}$ and $h : \mathrm{D} \to \mathrm{C}$. If $\mathrm{C} \in \mathcal{C}$ and S is a subset of C, there exists a smallest C' such that $\mathrm{S} \subset \mathrm{C}' \subset_{\mathcal{C}} \mathrm{C}$. This smallest C' will be written $\mathrm{cl}_{\mathcal{C}}\mathrm{S}$, or $\mathrm{cl}_{\mathcal{C},\mathrm{C}}\ \mathrm{S}$. Let $f : \mathrm{U} \to \mathrm{D}$ be a morphism in $\mathcal{U}$. We shall call f ***minimal*** if $\mathrm{cl}_{\mathcal{C}}(f(\mathrm{U})) = \mathrm{D}$. If also $f' : \mathrm{U} \to \mathrm{D}'$ is minimal, we shall call f, f' *equivalent* if there exists an isomorphism $g: \mathrm{D} \to \mathrm{D}'$ in $\mathcal{C}$ (or equivalently, in $\mathcal{U}$) such that $gf = f'$. The proof of (5.1) will be complete once we show the following

LEMMA (5.1.1). *Let* $\mathrm{U} \in \mathcal{U}$ *be fixed,* $\mathrm{C} \in \mathcal{C}$ *variable. The equivalence classes of minimal elements of* $(\mathrm{U},\mathrm{C})_{\mathcal{U}}$ *form a set.*

Let $f : \mathrm{U} \to \mathrm{C}$ be minimal, and let

$$\mathcal{Q} = \{\ \mathrm{Q} \subset \mathrm{U} \mid f(\mathrm{Q}) \in \mathcal{M}(\mathrm{C})\ \}.$$

If $\mathrm{Q}_1, \ldots, \mathrm{Q}_p \in \mathcal{Q}$ and $\mathrm{F} \in \mathrm{A}_p$, then

$$f(\mathrm{F}(\mathrm{Q}_1, \ldots, \mathrm{Q}_p)) = \mathrm{F}(f\mathrm{Q}_1, \ldots, f\mathrm{Q}_p) \in \mathcal{M}(\mathrm{C}),$$

so $\mathcal{Q}$ is a weak universe. Define $g: \mathcal{Q} \to \mathrm{C}$ by $g(\mathrm{Q}) = \mathrm{glb}(f(\mathrm{Q}))$. Since f is minimal, (5.1.2) below implies that g is surjective. Also

$$g(\mathrm{F}(\mathrm{Q}_1, \ldots, \mathrm{Q}_p)) = \mathrm{glb}(\mathrm{F}(f\mathrm{Q}_1, \ldots, f\mathrm{Q}_p)) = \mathrm{F}(g\mathrm{Q}_1, \ldots, g\mathrm{Q}_p),$$

so g is a morphism of weak universes.

If $\mathrm{Q}, \mathrm{Q}' \in \mathcal{Q}$, define $\mathrm{Q} \sim \mathrm{Q}'$ to mean $g\mathrm{Q} = g\mathrm{Q}'$. Clearly we have an induced isomorphism of universes $\mathcal{Q}/\!\sim\ \to \mathrm{C}$, and it defines an equivalence of f with the map $\mathrm{U} \to \mathcal{Q}/\!\sim$ given by $u \mapsto \langle u\rangle/\!\sim$. Now, $\mathcal{Q} \in \mathcal{P}(\mathcal{P}(\mathrm{U}))$ and $\sim\ \in \mathcal{P}(\mathcal{P}(\mathrm{U})\times\mathcal{P}(\mathrm{U}))$. The structure of $\mathcal{Q}/\!\sim$ as an object of $\mathcal{C}$ is entirely determined by $(\mathrm{U},\mathcal{Q},\sim)$. Therefore the equivalence classes of minimal morphisms form a set, since they can be indexed by a subset of $\mathcal{P}(\mathcal{P}(\mathrm{U})) \times \mathcal{P}(\mathcal{P}(\mathrm{U})\times\mathcal{P}(\mathrm{U}))$.

***LEMMA* (5.1.2).** *Let* $C \in \mathcal{C}$ *and let* V be a subuniverse of C *such that* $0_C = \mathrm{glb}_C\ 0V$. *Then* $\mathrm{cl}_C V = \{ \mathrm{glb}_C M \mid M \in \mathcal{M}(V) \}$.

Let $V' = \{ \mathrm{glb}_C M \mid M \in \mathcal{M}(V) \} \subset \mathrm{cl}_C V$. If $\mathrm{glb}\ M_1, \ldots, \mathrm{glb}\ M_p \in V'$ and $F \in A_p$, then

$$F(\mathrm{glb}\ M_1, \ldots, \mathrm{glb}\ M_p) = \mathrm{glb}(F(M_1, \ldots, M_p)) \in V',$$

so V' is a subuniverse of V. We have $0_C \in V'$. Now suppose $(\mathrm{glb}_C M_i)_{i \in I}$ is some indexed element of $\mathcal{M}(V')$. Let $M = \cup_{i \in I} M_i$. Then $M \in \mathcal{M}(V)$ and

$$\mathrm{glb}_C\ M = \mathrm{glb}_C\ \{ \mathrm{glb}_C\ M_i \mid i \in I \} \in V',$$

so $V' \in \mathcal{C}$, $V' \subset_C C$, so $V' = V$.

As the main benefit of the following theorem may only be an esthetic one, its proof will just be indicated. It works for many categories other than $\mathcal{U}$ and $\mathcal{C}$.

***THEOREM* (5.2).** *Any representable functor* $\mathcal{C} \to$ **sets** *or* $\mathcal{U} \to$ **sets** *has a canonical representative.*

If T is representable, let $A_T = (T, |\ |)$, where $|\ |: \mathcal{C} \to$ **sets** is the underlying set functor. If $C \in \mathcal{C}$ and $t \in TC$, let $t^\wedge: A_T \to |C|$ be given by $t^\wedge f = f_C t$, $f \in A_T$. If A_T is a set, in particular, if T is representable, than A_T has a unique structure of universe such that every $t^\wedge$ is a morphism of universes $A_T \to C$, and it is cohesive. If $T = (B, \ )_{\mathcal{C}}$, this representation being given by $t \in TB$, then $t^\wedge$ gives an isomorphism of A_T with B.

Our first application of (5.1) (and (5.2)) is to the functor $(U, \ )_{\mathcal{U}}$ itself. Let $U|\mathcal{C} \in \mathcal{C}$ be the canonical representative of $(U, \ )_{\mathcal{U}}$, i.e., $(U|\mathcal{C}, C) = (U, C|\mathcal{U})$ where $C|\mathcal{U}$ is the image of C in $\mathcal{U}$ under the forgetful functor $\mathcal{C}|\mathcal{U}: \mathcal{C} \to \mathcal{U}$. We shall let $\mathcal{U}|\mathcal{C}$ be the left adjoint of $\mathcal{C}|\mathcal{U}$ that we get in this way, $\mathcal{U}|\mathcal{C} \dashv \mathcal{C}|\mathcal{U}$.

We now have corollaries to (5.1) that are analogues of results previously proven for $\mathcal{U}$.

***COROLLARY* (5.3).** $\mathcal{C}$ *is cocomplete.*

Let $T: I \to \mathcal{C}$ be any diagram in $\mathcal{C}$. We need to show that $h : C \mapsto (T, C_I)$ is representable ([1], 8.1.3). By (1.4), $h' : \mathcal{U} \to$ **sets**, $h'(V) = (T|\mathcal{U}, V_I)$, where $T|\mathcal{U} = (\mathcal{C}|\mathcal{U})\circ T$, is represented by some $e \in (T|\mathcal{U}, U_I)$, where $U \in \mathcal{U}$. Then if $C \in \mathcal{C}$,

$$(T,C_I) = \{\, w \in (T|\mathcal{U},(C|\mathcal{U})_I) \mid \forall i \in I,\ w_i \in (Ti,C) \,\} =$$
$$\{\, f \in (U,C|\mathcal{U}) \circ \forall\ i \in I,\ fe_i \in \mathcal{C} \,\}.$$

This last expression T'C defines a subfunctor T' of $(U,\ )_{\mathcal{U}}$. Verification of the conditions of (5.1) using (3.3.1) is routine, so (5.3) follows.

***COROLLARY* (5.3.1).** *The forgetful functor* $|\ |_{\mathcal{C}}: \mathcal{C} \to$ **sets** *has a left adjoint* **sets**$|\mathcal{C}$.

This follows since (1.3) gives us an adjoint pair

$$U \dashv |\ |_{\mathcal{U}}.$$

***COROLLARY* (5.3.2).** *Let* $C \in \mathcal{C}$ *and let* E *be a set of equations on* C. *Then the functor* Sol E : $C \to$ **sets** *defined by*

$$(\mathrm{Sol\ E})(C') = \{\, f \in (C,C') \mid f \textit{ solves } E \,\}$$

has a representative C/E.

Indeed, Sol E $\subset (C,\ )_{\mathcal{U}}$ and satisfies the conditions of (5.1). We note that C/E and $(C|\mathcal{U})$/E can be different.

As we now have the same machinery in place for $\mathcal{C}$ that was established previously for $\mathcal{U}$, we can prove as before the following facts.

***PROPOSITION* (5.4).** *Let* $C \in \mathcal{C}$. *Then* $|\ |_{C-\mathcal{C}} : C-\mathcal{C} \to$ **sets** *has a left adjoint* $Y \mapsto C\bullet Y$.

***PROPOSITION* (5.5).** *Let* $C \in \mathcal{C}$. *Then* $(\ )_0 : C-\mathcal{C} \to$ **sets** *defined by* $U \mapsto U_0$ *has a left adjoint* $Y \mapsto C(Y)$.

Of course we must distinguish $C\bullet Y$ from $(C|\mathcal{U})\bullet Y$ and $C(Y)$ from $(C|\mathcal{U}_0)(Y)$.

6. LOCAL THEORY OF COHESIVE UNIVERSES.

Any $K_X \in C$, X a set. If $C \subset_C K_X$,

$$\{ \text{dom } c \mid c \in C\} = \{\text{dom } n \mid n \in 0C\}$$

is the set of open subsets for a topology on X. We shall call this *the topology defined by* C. If $x \in X$, define

$$x^\wedge = x^\wedge_C : |C| \to |A_0| \quad \text{by} \quad x^\wedge(c) = c(x),$$

defining $\qquad c(x) = \emptyset_0 \quad \text{if} \quad x \notin \text{dom } c.$

The following is trivial.

LEMMA (6.1). *With* C *as above,* $x^\wedge \in (C,A_0)_C$.

If $U \in U$, we define

$$\mathbf{S}U = \{ P \in (U,A_0) \mid P \text{ preserves glbs, glb } P(0U) = 0 \}.$$

In particular, if $C \in C$, then $\mathbf{S}C = \mathbf{S}(C|U)$ is just $(C,A_0)_C$. We call elements of $\mathbf{S}U$ *points* of U. If $f : U \to V$ in C, then $\mathbf{S}f : \mathbf{S}V \to \mathbf{S}U$ is defined by $Q \mapsto Q \circ f$.

Let $C \in C$, $P \in \mathbf{S}C$, and introduce a symbol $\emptyset_P$. Let

$$R_P = \text{colim} \{ C_n \mid n \in 0C, Pn = 0 \},$$

observing that $\{ n \in 0C \mid Pn = 0 \}$ is an upper semilattice, hence directed. If $c \in C$ and $P(0c) = 0$, we have a ring homomorphism $C_{0c} \to R_P$, since R_P is the colimit of all such C_n. Let $L_P = R_P \amalg \{\emptyset_P\}$ and define a map $P\_: C \to L_P$ by

$$\overline{P}(c) = \begin{cases} c|R_P & \text{if} \quad P(0c) = 0 \\ \emptyset_P & \text{if} \quad P(0c) = \emptyset_0. \end{cases}$$

LEMMA (6.2). *Let* $C \in C$, $P \in \mathbf{S}C$. *Then* L_P *has a unique structure of universe such that* $\overline{P} \in (C,L_P)_U$. *We have*

$$L_P \in \mathcal{U}_0, \qquad (L_P,A_0)_{\mathcal{U}_0} \neq \emptyset, \qquad \#0L_P = 2.$$

If f, $g \in C$ and $\bar{P}f = \bar{P}g$, then $Pf = Pg$. Also $\bar{P}$ is surjective. Thus we can write $P = Q\bar{P}$ for a unique set morphism $Q : |L_P| \to |A_0|$. To establish the first claim of (6.2), let $F \in A_h$, f, $g \in C^h$ and suppose $\bar{P}f_i = \bar{P}g_i$ for $1 \leq i \leq h$. We need to show $\bar{P}(F(f)) = \bar{P}(F(g))$. Now

$$\bar{P}(F(f)) = \emptyset_P \Leftrightarrow \bar{P}(F(g)) = \emptyset_P$$

since $\quad \bar{P}(F(f)) = \emptyset_P \Leftrightarrow P(F(f)) = \emptyset_0 \Leftrightarrow P(F(g)) = \emptyset_0.$

Suppose $\bar{P}(F(f)) \neq \emptyset_P$. Then we do not have $\bar{P}f_i = \emptyset_P$ or $\bar{P}g_i = \emptyset_P$ for any i. We have $n \in 0C$ with $Pn = 0$ and with $f_i + n = g_i + n$ for every i (by getting $f_i + n_i = g_i + n_i$ and taking $n = n_1 + \dots + n_h$). Then

$$\bar{P}(F(f)) = F(f)|R_P = (F(f) + n)|R_P = F(f_1+n, \dots, f_h+n)|R_P = \bar{P}(F(g)).$$

We have $\emptyset_P = \bar{P}(\emptyset(0_C)) \in 0L_P$. As the ring R_P contains only one solution of $n + n = n$, i.e., $n = 0_P$, $0L_P = \{ 0_P, \emptyset_P \}$. It is obvious that 0_P is an identity for L_P and that $Q \in (L_P,A_0)_{\mathcal{U}_0}$.

Any $L \in \mathcal{U}_0$ such that $(L,A_0)_{\mathcal{U}_0}f \neq \emptyset$ and such that $\#0L = 2$ will be called a *local universe.*

PROPOSITION (6.3). *Let* L *be a local universe. Then* $L \in C$, $(L,A_0)_{\mathcal{U}_0} = SL$ *and has exactly one element. Also* $\#\emptyset(C) = 1$.

Let $Q \in (L,A_0)_{\mathcal{U}_0}$, $\emptyset_L = \emptyset(0_L)$. As $Q(0_L) = 0$ is not a phantom, $0_L \neq \emptyset_L$, so $0L = \{ 0_L, \emptyset_L \}$. Let $R_L = \{ u \in L \mid 0u = 0_L \}$, and note that $L_{\emptyset L} = \{\emptyset_L\}$ by (4.1). The standard decomposition $\coprod_{n \in 0U} U_n$ of a universe U becomes $L = R_L \amalg \{\emptyset_L\}$ when $U = L$.

Any element of $M(L)$ must be a subset of a set $\{a,\emptyset_L\}$, $a \in R_L$, for no element of R_L can match with any other element of R_L. Thus L satisfies axiom (3.1.1) of a cohesive universe since $a \leq \emptyset_L$. Axiom (3.1.2) simplifies in this instance to the following:

(6.3.1) *If* $F \in M(A_h)$ *and* $a \in (R_L)^h$, *then*

$$(\mathrm{glb}\ F)(a_1, \dots, a_h) \neq \emptyset_L \Rightarrow F(a_1, \dots, a_h) \neq \{\emptyset_L\}.$$

We use (6.3.1) to show $L \in C$. We have evidently that (6.3.1) holds for A_0. If $a \in (R_L)^h$, then

$$(\text{glb } F)(a_1, \ldots, a_n) \neq \emptyset_L \Rightarrow (\text{glb } F)(Qa_1, \ldots, Qa_n) \neq \{\emptyset_0\} \Rightarrow$$
$$F(Qa_1, \ldots, Qa_n) \neq \{\emptyset_0\} \Rightarrow F(a_1, \ldots, a_n) \neq \{\emptyset_L\}.$$

It is evident that $(L,A_0)U_0 = SL$. Suppose $Q' \in SL$ but $Q' \neq Q$, say $Q'f \neq Qf$, $f \in L$. Let $Qf = a$, $Q'f = a'$, and choose $F, F' \in CA_1$ with $F(a) = 0 = F'(a')$ but $F + F' = \emptyset_1$, i.e., $(\text{dom } F) \cap (\text{dom } F') = \emptyset$. Then

$$\emptyset_L = \emptyset_1(f) = (F+F')(f) = 0_L + 0_L ,$$

a contradiction. This proves (6.3).

We define P_L by $SL = \{P_L\}$ if L is local.

PROPOSITION (6.4). *If* L *is a local universe,* R_L *is a local ring.*

Evidently $\{ u \in L \mid P_L u = 0 \}$ is a proper ideal J of R_L. Suppose $u \in R_L \setminus J$. Then $P_L(u) \neq 0, \emptyset_0$ so

$$P_L(u(1/u)) = P_L(u)(1/P_L(u)) \neq \emptyset_0.$$

Since $1_1 \leq z_1'(1/z_1')$ in A_1,

$$1_L = 1(u) \leq u{\bullet}(1/u) < \emptyset_L, \quad \text{so} \quad 1_L = u{\bullet}(1/u),$$

u is invertible in R_L. Thus the ideal J consists of exactly the non-invertible elements of R_L, so R_L is local.

It may come as a surprise that one does not need any special hypothesis on I in the following result.(*) By *ideal* of L we mean any ideal of R_L.

THEOREM (6.5). *Let* $L \in U$ *be local,* $I \neq R_L$ *an ideal of* L. *Define* $L/I = (R_L/I) \amalg \{\emptyset_*\}$, *where* $\emptyset_*$ *is a formal symbol. Then* L/I *has a unique structure of universe such that* $q = q_I\colon L \to L/I$, *given by*

$$q(x) = x + I \quad \text{if} \quad x \in R_L, \qquad q(\emptyset_L) = \emptyset_*,$$

is a morphism of universes. Furthermore, L/I *is local and* $q \in C$.

(*) Note however [9], §1, reference to Hadamard's Lemma.

To prove the first statement of (6.5), let u, $v \in L^n$, $F \in A_n$, and assume that $qu_j = qv_j$ for every j. We need to show that $q(F(u)) = q(F(v))$. Let $P = P_L$ and assume $F(u) = \emptyset_L$. Since $I \subset R_L$, $I \neq R_L$, $P(I) = \{0\}$ and $Pu_i = Pv_i$ for every i since either both sides equal $\emptyset_0$ or $u_i - v_i \in I$. Then also $\emptyset_0 = P(F(v))$, so

$$F(v) = \emptyset_L \ , \qquad q(F(u)) = \emptyset_* = q(F(v))$$

in this case. Thus we can assume all u_j, $v_j \in R_L$ and $P(F(u)) = P(F(v)) \in K$. We shall need the following

***LEMMA* (6.5.1).** *Let $G(z_1, \ldots, z_p, w_1, \ldots, w_q) \in A_{p+q}$ have domain $U \times V$ where $U \subset K^p$ is a convex open neighborhood of 0 and $V \subset K^q$ is open. Assume $G(0;w)$ is identically zero on V. Then G may be written*

$$G = \Sigma_{i=1}{}^p \, G_i(z;w) z_i$$

where each $G_i(z;w) \in A_{p+q}$ and has domain $U \times V$.

We have

$$G(z;w) = [G(tz;w)]_{t=0}{}^{t=1} = \int_0^1 (d/dt)(G(tz;w))\, dt$$

and the integrand is dt times

$$\Sigma_{i=1}{}^p \, (\partial G/\partial z_i)(tz;w) z_i.$$

Let

$$G_i(z;w) = \int_0^1 (\partial G/\partial z_i)(tz;w)\, dt.$$

Then $G = \Sigma_{i=1}{}^p \, G_i(z;w) z_i$ proving (6.5.1).

Let

$$H(z;w) = F((w/2) + (z/2)) - F((w/2) - (z/2)),$$

$$z = z_1, \ldots, z_n, \qquad w = w_1, \ldots, w_n.$$

Then

$$H(u-v;\, u+v) = H(u_1 - v_1, \ldots, u_n - v_n;\, u_1 + v_1, \ldots, u_n + v_n) = F(u) - F(v).$$

Define $a_j \in K$ by $Pu_j = a_j = Pv_j$. Then $F(a) = P(F(u)) \in K$, so

$$(0;\, 2a) = (0, \ldots, 0;\, 2a_1, \ldots, 2a_n) \in \text{dom } H,$$

and

$$H(0;2a) = P(H(u-v;\, u+v)) = 0.$$

Let $G \in A_{2n}$, $G(0;2a) = 0$, and suppose $G \succeq H$. Then

$$P(G(u-v;\ u+v)) = G(0;2a) \neq \emptyset_O.$$

Thus

$$\emptyset_L \neq G(u-v;\ u+v) \succeq H(u-v;\ u+v),$$

so

$$F(u) - F(v) = G(u-v;\ u+v).$$

Choose such a G with domain U×V as in (6.5.1). We get

$$F(u) - F(v) = \Sigma_{i=1}{}^{p}\, G_i(u-v;\ u+v)(u_i - v_i),$$

and $u_i - v_i \in I$ for every index i. Also $G_i(u-v;\ u+v) \in R_L$ (since $F(u) - F(v) \in R_L$), so $F(u) - F(v) \in I$, $q(F(u)) = q(F(v))$.

Evidently $\#0(L/I) = 2$. Also $Q(c + I) = P(c)$, $c \in R_L$, $Q(\emptyset_*) = \emptyset_O$ defines an element Q of $(L/I, A_O)_{U_O}$, so L/I is local. Obviously $Q \in C$.

Let U be a universe, E a set of equations on U and let $f \in (U,V)_U$ where $V \in \mathcal{U}$. We shall then let fE denote the set of equations

$$\{\ fu_1 = fu_2 \mid u_1 = u_2 \text{ is an equation in E } \}.$$

PROPOSITION (6.6). *Let* L *be a local universe,* E *a set of equations on* L *such that the set of equations* P_LE *holds. Let* I_E *be the ideal of* R_L *generated by all* $u-v \in R_L$ *such that the equation* $u = v$ *is in* E. *Then* $L/I_E \simeq L/E$. *In particular,* L/E *is local.*

Let $q : L \to L$ be the canonical morphism where $L' = L/I_E$. Then if $C \in C$,

$$(L',C) = \{\ f \in (L,C) \mid u,\ v \in L,\quad qu = qv \Rightarrow fu = fv\ \}.$$

Let $f \in (L,C)_C$. The following will complete the proof of (6.6).

LEMMA (6.6.1). $(u,\ v \in L,\ qu = qv \Rightarrow fu = fv) \Leftrightarrow f$E *holds.*

Assume $u, v \in L$, $qu = qv \Rightarrow fu = fv$ and let $u = v$ be an equation in E. If $u = \emptyset_L$ or $v = \emptyset_L$, as $P_L u = P_L v$, we have $u = \emptyset_L = v$, so $fu = fv$. Assume $u,\ v \neq \emptyset_L$. Then $u-v \in I_E$, so $qu = qv$, $fu = fv$. Thus fE holds. Assume conversely that fE holds, $u,\ v \in L$ and $qu = qv$. If $u = \emptyset_L$ or $v = \emptyset_L$, then $u = \emptyset_L = v$ by definition of q. Suppose $u,v \neq \emptyset_L$. Then $qu = qv \Rightarrow u-v \in I_L$. Write

$$u - v = \Sigma\ w_i(u_i - v_i)$$

where for each i, u_i, v_i, $w_i \in R_L$ and $u_i = v_i$ is an equation in E. Then

$$f(u-v) = \Sigma\ f(w_i)(f(u_i) - f(v_i))$$

and for all i,

$$0fu = 0fv = 0fu_i = 0fv_i = 0fw_i = 0_C.$$

Then $f(u) - f(v) = 0_C$, $f(u) = f(v)$. This proves (6.6.1) and (6.6).

Let L be any local universe and consider a map of sets h : $Y \to R_L$ where $P_L hy = 0$ for every $y \in Y$. We can choose h so that the induced morphism $p : A_o(Y) \to L$ in C, considering that $A_o \in C$, is surjective. The composition $P_L p$ is the element 0_Y of $SA_o(Y)$ that sends every element of Y to 0. Let $A_o\langle Y\rangle = L_{0_Y}$. From the lemma that follows, we shall have that p has a unique factorization $h\bar{0}_Y$ where h: $A_o(Y) \to L$. Let $I = h^{-1}0_L$. Then I is a proper ideal of $R_{A_o\langle Y\rangle}$ and h induces an isomorphism $\bar{h}$: $A_o\langle Y\rangle/I \to L$. We call $(Y,I,\bar{h})$ a *presentation* of L. We see thus that every local inverse has a presentation, i.e., is of the form $A_o\langle Y\rangle/I$.

It will help to understand $P \mapsto L_P$ as a functor. Let L, the *category of local universes*, be the full subcategory of C whose objects are the local universes. We note that $L \mapsto (L,P_L)$ gives us a natural inclusion $L \hookrightarrow C\text{-}A_o$.

LEMMA* *(6.7). Let* $U \in C$, $P \in SU$. *Then

$$(P,L) : (L_P,L)_L \longrightarrow ((U,P),\ L)_{C\text{-}A_o}$$

is a bijection.

This lemma basically says the following. Any solid arrow diagram in C

$$\begin{tikzcd}
 & A_o & \\
U \arrow[ur, "P"] \arrow[rr, "f"] \arrow[dr, "P"'] & & L \arrow[ul, "P_L"'] \\
 & L_P \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

whose upper part is commutative, can be completed with a unique dotted arrow g that makes the lower triangle commute. The uniqueness is clear since $\bar{P}$ is surjective. If $\bar{P}u = \emptyset_P$, then $Pu = \emptyset_O$ so $fu = \emptyset_L$. If $\bar{P}u \in R_P$ and $\bar{P}u = \bar{P}v$, $u, v \in U$, then $u + n = v + n$ for some $n \in 0U$ with $Pn = 0$. Then

$$fu = fu + fn = fv,$$

hence the existence of g.

It follows from (6.7) that if $h \in \langle(U,P), (V,Q)\rangle_{C\text{-}A_O}$, there exists a unique $L_h: L_P \to L_Q$ such that $\bar{Q}h = L_h\bar{P}$. Since P is determined by h and Q, i.e., $P = Qh$, it is often convenient to denote L_h as $h_Q: L_P \to L_Q$.

PROPOSITION (6.8). $A_O\langle\ \rangle \mapsto \mathfrak{m}\_$ *where* $\mathfrak{m}\_ : L \to$ **sets** *is given by*

$$\mathfrak{m}\_(L) = \mathfrak{m}_L = \textit{the maximal ideal of } R_L.$$

7. TOPOLOGICAL UNIVERSES.

There exist cohesive universes C that are of interest but with $\mathbf{S}C = \emptyset$, as for instance Example (8.3) below. Nevertheless our efforts for the rest of this paper will focus on those $C \in \mathcal{C}$ with the somewhat opposite property that given distinct $n, n' \in 0C$, there exists $P \in \mathbf{S}C$ such that $Pn \neq Pn'$. Most of the universes we have looked at so far have this property.

If $C \in \mathcal{C}$, let $L_C = \prod_{P \in \mathbf{S}C} L_P$ and let $\lambda_C: C \to L_C$ (in $\mathcal{C}$) be defined by $\lambda_C c = (\bar{P}c)_{P \in \mathbf{S}C}$. If $f : C \to C'$ in $\mathcal{C}$, we define

$$L_f : L_C \to L_{C'} \quad \text{by} \quad (L_f t)_{P'} = t_{P'f} \quad \text{if} \quad t \in L_C \quad \text{and} \quad P' \in \mathbf{S}C.$$

Then $\lambda\_: 1_{\mathcal{C}} \to L\_$ is a natural transformation.

THEOREM (7.1). *For* $C \in \mathcal{C}$, *the following conditions are equivalent:*

1) $\lambda_C : C \to L_C$ *is injective;*

2) If $n, n' \in 0C$, $n \neq n'$, *there is a* $P \in \mathbf{S}C$ *with* $Pn \neq Pn'$.

If 1 holds, there exists $P \in SC$ with $\bar{P}n \neq \bar{P}n'$. Then $Pn \neq Pn'$. Assume 2 holds. Let $u, v \in C$, $u \neq v$. If $0u \neq 0v$, take $P \in SC$ with $P(0u) \neq P(0v)$. Then $0\bar{P}u \neq 0\bar{P}v$, so $\bar{P}u \neq \bar{P}v$. Thus we can let $n = 0u = 0v$. Assume $\lambda_C u = \lambda_C v$. For each $P \in SC$ such that $Pn = 0$, $\bar{P}u = \bar{P}v$ so there exists $n_P \in 0C$ such that

$$P(n_P) = 0 \quad \text{and} \quad u + n_P = v + n_P.$$

By using $n_P + n$ instead, we can assume $n_P \geq n$. Let

$$w = \text{glb} \{ n_P \mid Pn = 0 \}.$$

If $P \in SC$, then

$$Pn = 0 \Rightarrow Pn_P = 0 \Rightarrow Pw = 0 \Rightarrow Pn = 0$$

since $n \leq w$, so $w = n$. Then

$$u = u + n = \text{glb} \{ u + n_P \mid Pn = 0 \} = v.$$

If $C \in \mathcal{C}$ and $\lambda_C: C \to L_C$ is injective, we shall say that C is a *topological universe*. The full subcategory of $\mathcal{C}$ supported by the topological universes will be denoted $\mathcal{K}$. In what follows, $\mathcal{K}$ will be our category of preference. However, as many interesting cohesive universes do not lie in $\mathcal{K}$, we shall be considering $\mathcal{C}$ occasionally.

Let $C \in \mathcal{C}$. If $n \in 0C$, let $U_n = \{ P \in SC \mid Pn = 0 \}$. Then

$$U_{n+n'} = U_n \cap U_{n'}, \qquad U_{\text{glb}\, M} = \bigcup \{ U_n \mid n \in M \}$$

if $n, n' \in 0C$, $M \subset 0C$. We see that $\{ U_n \mid n \in 0C \}$ is the family of open sets for a topology on SC ($U_0 = SC$, $U_{\emptyset_0} = \emptyset$). We shall henceforth consider that SC is a topological space (which is in fact a "sober space", cf. [5]). To say that $C \in \mathcal{K}$ is to say that $n \mapsto U_n$ is an order-reversing isomorphism of 0C with the family of open sets of SC. It is a fact (that will not be shown here) that if $C \in \mathcal{C}$ and there exists an order-reversing isomorphism of 0C with the family of all open sets of a topological space X (which need not be assumed to be SC), then $C \in \mathcal{K}$.

It will be helpful to describe directly $\text{im}\, \lambda_C \subset L_C$ when $C \in \mathcal{K}$, since that provides a useful alternative description of C. Call $f \in L_C$ *continuous* ($C \in \mathcal{C}$) if whenever $P \in SC$ and $f_P \neq \emptyset_P$, there exists $c \in C$

and an open neighborhood U of P in SC such that $Q \in U \Rightarrow f_Q = \overline{Q}c$, i.e., there exists $n \in 0C$ with $Pn = 0$ such that

$$Q \in \mathbf{S}C, \quad Qn = 0 \Rightarrow f_Q = \overline{Q}c.$$

The set of all continuous elements of L_C is seen to be a cohesive subuniverse of L_C that is topological. We denote it by $\mathbf{T}C$, and note that we in fact have a functor $\mathbf{T}$: $\mathcal{C} \to K$. The proof of the following is similar to the proof of its analogue in sheaf theory.

LEMMA (7.2). *Let* $C \in K$. *Then* λ_C *gives an isomorphism of* C *with* TC.

Thus if $C \in K$ and $C|\mathcal{C}$ is its image in $\mathcal{C}$, we have an isomorphism: $C \to \mathbf{T}(C|\mathcal{C})$.

Let $C \in K$, $X = \mathbf{S}C$. If

$$n \in 0C \text{ and } U_n = \{ x \in X \mid x(n) = 0 \},$$

define $\mathfrak{R}(U_n) = C_n$. This gives us a sheaf of rings $\mathfrak{R}$ on X such that $\mathfrak{R}_x$ is a local ring with residue class field K for each $x \in X$. However $(X,\mathfrak{R})$ is not merely a local ringed space, for it has a great deal more structure than that. We can apply theorems from sheaf theory to objects of K, but trying to reduce the study of K to a topic with sheaf theory is a little bit unnatural. For one thing, plugging elements of C^n into elements of A_n changes domains. Also, if $x \in X$, $\mathfrak{R}_x$ is the local ring of a local universe $L_x = \mathfrak{R}_x \amalg \{\emptyset_x\}$. Thus to talk about the additional structure on $\langle X, \mathfrak{R}\rangle$ we would at least need the theory of local universes.

From the observations of the preceding paragraph and sheaf theory, we can immediately conclude the following.

PROPOSITION (7.3). *Let* F: $U \to V$ *in* K. *Assume that* $\mathbf{S}f : \mathbf{S}V \to \mathbf{S}U$ *is a homeomorphism and that* $f_y : U_{yf} \to V_y$ *is an isomorphism for each* $y \in \mathbf{S}V$. *Then* f *is an isomorphism.*

The following corollaries of (7.1) are immediate:

PROPOSITION (7.4). *Suppose* $C \subset_c U \in K$. *Then* $C \in K$.

PROPOSITION (7.5). Let $C_i \in K$, $i \in I$. *Then* $\prod_{i \in I} C_i \in K$.

These two results imply K is complete.

Let $G\colon K \to C$ be the canonical inclusion $K \subset C$. Since $TG = 1_K$ by (7.2), the following will imply that K is cocomplete.

PROPOSITION (7.6). $T \dashv G$.

Let $C \in C$. Then $\lambda_C\colon C \to L_C$ induces $\alpha_C\colon C \to GTC$ (by restricting the range), and $\alpha : C \mapsto \alpha_C$ is a natural transformation $1_C \to GT$. If $U \in K$, then $\alpha_{GU} : GU \to GTGU$ is a bijection by (7.2). Also

$$G\colon (TGU,U) \to (GTGU,GU)$$

bijectively. Thus we can define

$$\beta_U\colon TGU \to U \quad \text{by} \quad G\beta_U = (\alpha_{GU})^{-1}.$$

Then $\beta\colon U \mapsto \beta_U$ is a natural transformation $TG \to 1_K$. Clearly, $G(\beta_U)\alpha_{GU} = 1_{GU}$. Thus (7.6) will follow once we show that

$$\beta_{TC}T\alpha_C = 1_{TC}, \quad \text{i.e.,} \quad G(\beta_{TC})GT\alpha_C = 1_{GTC},$$

or, since $G\beta_{TC} = (\alpha_{GTC})^{-1}$, that $GT\alpha_C = \alpha_{GTC}$. Now $GT\alpha_C$, α_{GTC} are in $(GTC, GTGTC)$ and we have $cl_C(\mathrm{im}\ \alpha_C) = GTC$, so $(GT\alpha_C)\alpha_C = \alpha_{GTC}\alpha_C$, which holds since α is a natural transformation, will do. This proves (7.6).

COROLLARY (7.6.1). L is cocomplete.

Let $\tau\colon L \to K$ be $L \mid K$ and let $D\colon I \to L$ be a diagram in L. Then $\#S(\mathrm{colim}\ \tau D) = 1$, so $\mathrm{colim}\ \tau D$ is local. Then

$$\tau^{-1}(\mathrm{colim}\ \tau D) = \mathrm{colim}\ D.$$

PROPOSITION (7.7). Let I *be a directed set,* $\{L_i \mid i \in I\}$ *a family of local universes directed by* I. *Assume* $L_i \to L_j$ *is an injection whenever* $i,j \in I$, $i < j$. *Let*

$$L = \mathrm{colim}_L \{ L_i \mid i \in I \}$$

and let $p_i : L_i \to L$ *be the canonical morphism. Then each* p_i *is an injection and* $L = \bigcup_{i \in I} \operatorname{im} p_i$.

Let $S = \operatorname{colim} \{ |L_i| \mid i \in I \}$. Then $|L_i| \subset S = \bigcup_{i \in I} |L_i|$. It follows that S has a unique structure of universe such that every inclusion $L_i \subset S$ is a morphism in U. Let L be S with this structure of universe. It is trivial that $L = \operatorname{colim}_U \{ L_i \mid i \in I \}$. As each L_i is local and the morphisms $L_i \to L_j$ are all injective, $\#0L = 2$, $\#\emptyset L = 1$, and L has an identity. As $(L, A_0) \neq \emptyset$, $L \in L$. It follows easily that $L = \operatorname{colim}_L \{ L_i \mid i \in I \}$.

COROLLARY *(7.7.1). Let S be any set. Then*

$$A_0\langle S \rangle = \bigcup \{ A_0\langle T \rangle \mid T \subset S \text{ is finite} \}.$$

This follows using (6.8).

Let $T: I \to K$ be any K-valued diagram, and let $C = \operatorname{colim} T$. To determine what C is in a more concrete way, we may start with

$$X = \mathbf{S}C = \lim \mathbf{S}Ti$$

and determine the local universes C_x, $x \in X$, where C_x denotes the localization L_x of C at x. (In the sequel we shall prefer this notation to the previously used L_x.) We have $C \subset \Pi_{x \in X} C_x$. Thus, to determine C, it will suffice to determine U and the compositions

$$\gamma_i : T_i \xrightarrow[\alpha_i]{} C \hookrightarrow U \qquad \text{for every} \quad i \in I.$$

Any $x \in X$ can be identified with $\{x_i\}_{i \in I}$, where $x_i = x\alpha_i$. Describing λ_i amounts to determining, for every $x \in X$, the maps $(\alpha_i)_x : (Ti)_{x_i} \to C_x$ defined by $(\alpha_i)_x \bar{x}_i = \bar{x}\alpha_i$. Lemma (7.8) below will allow this to be done in a routine manner. Given $x \in X$, define $T_x i = (Ti)_{x_i}$. Then T_x is a functor $T_x : I \to L$.

LEMMA *(7.8). With the above notation, (colim T)*$_x$ = colim (T_x) *for any* $x \in X$. *For any* $i \in I$, $(\alpha_i)_x$ *is the canonical morphism* $T_x i \to \operatorname{colim}(T_x)$,

Let $L \in L$, $\tau : L \to K$. Then

$$(C_x, L) = \{ f \in (C, \tau L) \mid P_L f = x \} =$$

$$\{ f \in (T, \tau L_I) \mid P_L f_i = x_i,\ i \in I \} = (T_x, L_I).$$

If $h \in (C_x, L)$, then under these identifications we have

$$h \mapsto hx \mapsto \{\, hx\alpha_i = h(\alpha_i)_x x \mid i \in I \,\} \mapsto \{\, h(\alpha_i)_x \mid i \in I \,\}.$$

PROPOSITION (7.9). *If* X, Y *are manifolds, then* $A_X \amalg A_Y = A_{X\times Y}$.

Let $U \subset X$, $V \subset Y$ be open with A_U (resp. A_V) isomorphic to $A_{U'}$ (resp. $A_{V'}$) for some U' (resp. V') open in K^r (resp. K^s), $r, s \in \mathbf{N}$. We have a commutative diagram

$$\begin{tikzcd}
A_X \amalg A_Y \arrow[r, "f"] \arrow[d] & A_{X\times Y} \arrow[d] \\
A_U \amalg A_V \arrow[r] & A_{U\times V} \\
A_r \amalg A_s \arrow[r] \arrow[u] & A_{r+s} \arrow[u]
\end{tikzcd}$$

where, for instance, $X\times Y \to X$ and $X\times Y \to Y$ induce f. Here the bottom arrow is an isomorphism and the isomorphisms

$$X\times Y \simeq \mathbf{S}A_{X\times Y} \simeq (\mathbf{S}A_X)\times(\mathbf{S}A_Y) \simeq \mathbf{S}(A_X \amalg A_Y)$$

are compatible with the isomorphism $\mathbf{S}A_{X\times Y} \to \mathbf{S}(A_X \amalg A_Y)$. By (7.3) it will suffice to show that if $x \in X$, $y \in Y$, then

$$f_{(x,y)}\colon (A_X \amalg A_Y)_{(x,y)'} \to (A_{X\times Y})_{(x,y)}$$

is an isomorphism. Pick U and V so that $x \in U$, $y \in V$, and localize the diagram at (x, y) and the corresponding point of K^{r+s}. Then all arrows in the diagram become isomorphisms and (7.9) follows.

8. SOBER SPACES AND THE SPACES SC.

We have seen that to "understand" $C \in K$, we need to know L_P for each $P \in$ SC. In §6 a wealth of detail was obtained on the L_P, so we look now at the spaces SC to complete the picture. The reader can refer to page 151 of [5] for the definitions that will be needed here.

***THEOREM* (8.1).** *Let* C $\in$ *C. Then every closed irreducible subset of* **S**C *has a unique generic point.*

***LEMMA* (8.1.1).** *Let* C $\in$ *C*, P, Q $\in$ **S**C. *Then the following are equivalent:*

1) If $c \in$ C, *then* $Pc = \emptyset_0 \Rightarrow Qc = \emptyset_0$;
2) If $c \in$ C, *then* $Qc \neq \emptyset_0 \Rightarrow Pc = Qc$;
3) Q $\in$ cl{P}.

Obviously 2 $\Rightarrow$ 1, so assume 1. If F $\in$ A_1 and $c \in$ C, then

$$Qc \in \text{dom } F \Leftrightarrow F(Qc) = Q(F(c)) \neq \emptyset_0.$$

Now assume 2 is false, so we have $Pc \neq Qc \neq \emptyset_0$. By taking F = $z_1{}'$, we see that

$$Qc \in \text{dom } z_1{}' \Rightarrow Pc \neq \emptyset_0.$$

Therefore we can now let F = $0_{K\setminus(P(c))}$. Then $Qc \in$ dom F, but

$$P(F(c)) = F(P(c)) = \emptyset_0,$$

contradicting what we proved above. The contrapositive of 1 means that any neighborhood of Q contains P, so clearly 1 $\Leftrightarrow$ 3.

If the equivalent properties of (8.1.1) hold, let us write P $\geqslant$ Q (since 1 suggests "domain" P is larger than "domain" Q). This gives an order on |**S**C| for which, because of 2 in (8.1.1), we have

$$P \leqslant Q \leqslant P \Rightarrow P = Q.$$

From 3 of (8.1.1) we see therefore that **S**C is a T_0-space, i.e., a closed irreducible subset of **S**C can have at most one generic point.

We shall always assume that Y $\neq \emptyset$ is part of the definition of "Y is irreducible". The following will establish (8.1).

***LEMMA* (8.1.2).** Y $\subset$ **S**C *closed and irreducible* $\Rightarrow$ Y *has a generic point.*

Let $c \in$ C, $Yc = \{ Pc \mid P \in Y \}$. We first show $Yc \in M(A_0)$, i.e., $\#(\text{KOY}c) \leq 1$. Let P, Q $\in$ Y and assume $a = Pc \neq Qc = b$, a, $b \in$ K. Choose f, $g \in 0A_1$ such that

$$f(a) = 0 = g(b), \quad f + g = \emptyset_1.$$

Then $$P \in U_{f(c)} \cap Y, \quad Q \in U_{g(c)} \cap Y$$

(cf. definitions of §7), so there exists $R \in U_{f(c)} \cap U_{g(c)} \cap Y$. Then

$$\emptyset_0 \neq R(f(c)) + R(g(c)) = R(\emptyset_1(c)) = \emptyset_0,$$

a contradiction. This proves (8.1.2).

We can now define $P_Y c = \text{glb } Yc$ to define $P_Y \in (|C|, |A_0|)$. We shall see that $P_Y \in$ **SC** and is a generic point of Y. Let $c \in C^n$, $F \in A_n$. Then $P_Y(F(c)) = \text{glb } (YF(c))$, and we claim

$$\text{glb } (YF(c)) = \text{glb } (F(Yc_1, \ldots, Yc_n)).$$

Write this claim as $u = v$. Since $YF(c) \subset F(Yc_1, \ldots, Yc_n)$, we must at least have $u \geq v$. Having $u > v$ can only come from having $YF(c) = \{\emptyset_0\}$, yet having $P_1, \ldots, P_n \in Y$ with $F(P_1 c_1, \ldots, Pc_n) \neq \emptyset_0$. In this case there will exist $P \in U_{oc_1} \cap \ldots \cap U_{oc_h} \cap Y$ since $U_{oc_i} \cap Y \neq \emptyset_0$ for every i. Then

$$PF(c) = F(P_1 c_1, \ldots, P_n c_n) \neq \emptyset_0$$

since $P, P_i \in Y \Rightarrow Pc_i = P_i c_i$ (as these are both in K) and we see that in fact $YF(c) \neq \{\emptyset_0\}$ in this case. Thus

$$P_Y(F(c)) = F(\text{glb } Yc_1, \ldots, \text{glb } Yc_n) = F(P_Y c_1, \ldots, P_Y c_n).$$

As $Y \neq \emptyset_0$, $P_Y(0_C) = 0$. Let $M \in \mathcal{M}(C)$. Then

$$P_Y(\text{glb } M) = \text{glb } Y(\text{glb } M) = \text{glb } YM = \text{glb } \{\text{glb } Ym \mid m \in M\} = \text{glb}(P_Y M).$$

Therefore $P_Y \in$ **SC**.

We finish (8.1) by showing that $\text{cl}\{P_Y\} = Y$. First $Y \leqslant P_Y$, i.e., $P \in Y \Rightarrow P \leqslant P_Y$. Indeed, if $c \in C$ and $Pc \neq \emptyset_0$, then

$$P_Y c \leq Pc < \emptyset_0 \quad \Rightarrow \quad P_Y c \neq \emptyset_0.$$

Now we only need $P_Y \in Y$. If not, as Y is closed, there exists $n \in 0C$ such that $P_Y n = 0$, $Yn = \{\emptyset_0\}$. But then $P_Y n \neq \text{glb } Yn$, a contradiction to the definition of P_Y.

In the terminology of [5], Theorem (8.1) says precisely that every **SC** is a *sober space*. We shall let **sob** denote the full subcategory of **top** that is supported by the sober spaces.

Let the topology of X be defined by $C \subset_c K_X$. If $P \in SC$, define $Y_P = \{ x \in X \mid x\hat{}_c \leqslant P \}$ (where $x\hat{}_c f = f(x)$, $f \in C$). Let $\bar{X}$ be the set of all closed irreducible subsets of X with its topology as defined on page 151 of [5] (but called $X\hat{}$ there).

THEOREM (8.2). With the above definitions, $P \mapsto Y_P$ *defines a homeomorphism* $Y\_: SC \to \bar{X}$.

***LEMMA** (8.2.1.).*

$$X \setminus Y_P = \bigcup \{ \operatorname{dom} f \mid f \in C, Pf = \emptyset_0 \} =$$

$$\bigcup \{ \operatorname{dom} n \mid n \in 0C \mid Pn = \emptyset_0 \} = \operatorname{dom} n_P$$

where

$$n_P = glb \, \{ n \in 0C \mid P_n = \emptyset_0 \}.$$

The second and third sets agree since $\operatorname{dom} f = \operatorname{dom} 0f$, and the last equality comes from

$$\operatorname{dom}(\operatorname{glb} M) = \bigcup \{ \operatorname{dom} m \mid m \in M \} \quad \text{if} \quad M \in \mathcal{M}(K_X).$$

Finally, if $x \in X$,

$$x \notin Y_P \Leftrightarrow x\hat{} \nleqslant P \Leftrightarrow x \in \bigcup \{ \operatorname{dom} f \mid f \in C, Pf = \emptyset_0 \}.$$

***LEMMA** (8.2.2). If* $P \in SC$, Y_P *is closed, irreducible and non-empty.*

If $Y_P = \emptyset$, $n_P = 0$, $\emptyset_0 = Pn_P = 0$, a contradiction. Also $X \setminus Y_P = \operatorname{dom} n_P$ is open. Let $f, g \in C$,

$$Y_P \cap \operatorname{dom} f \neq \emptyset \neq Y_P \cap \operatorname{dom} g.$$

Then $Pf, Pg \neq \emptyset_0$, so

$$P(f + g) = Pf + Pg \neq \emptyset_0,$$

so

$$Y_P \cap \operatorname{dom} f \cap \operatorname{dom} g = Y_P \cap \operatorname{dom} (f + g) \neq \emptyset_0.$$

Since $\operatorname{dom} f$, $\operatorname{dom} g$ are arbitrary open subsets of X, Y_P is irreducible.

Thus in fact $Y\_: SC \to \bar{X}$. Let $u = Y\_$. If $Y \in \bar{X}$, let

$$Y\hat{} = \{ x\hat{} \mid x \in Y \} \subset SC,$$

and define $v(Y) = P$ where P is chosen so that $cl\{P\} = cl(Y^\wedge)$ (cf. (8.1)). We shall show that u is a homeomorphism and $v = u^{-1}$. For $uv = 1_X$, let $Y \in \bar{X}$, $cl\{P\} = cl(Y^\wedge)$. We need to show $Y_P = Y$. If $x \in Y$, then $x^\wedge \in cl\{P\}$, so $Y \subset Y_P$. For $Y_P \subset Y$, let $x \in X\backslash Y$. Choose $n \in OC$ so that $n(x) = 0$, $Y \cap \mathrm{dom}\ n = \emptyset$. We shall have $x \notin Y_P$ if we show $Pn = \emptyset_0$ (as then $x^\wedge \not\leq P$). If $Pn \neq \emptyset_0$, i.e. $Pn = 0$, then $P \in U_n$, so $U_n \cap Y^\wedge \neq \emptyset$ (as $cl\{P\} = cl(Y^\wedge)$). Then there exists $x' \in Y$ with $x'^\wedge \in U_n$, i.e. $x'^\wedge(n) = 0 = n(x')$, contradicting $Y \cap \mathrm{dom}\ n = \emptyset$. To show $vu = 1_{SC}$, we let $P \in \mathbf{S}C$ and show $v(Y_P) = P$, i.e., $cl\{P\} = cl(Y_P^\wedge)$. Now

$$Y_P^\wedge = \{ x^\wedge \mid x^\wedge \leq P \} \subset cl\{P\},$$

so we only need to show $P \in cl(Y_P^\wedge)$. If $P \notin cl(Y_P^\wedge)$, there exists $n \in OC$ such that $Pn = 0$, $U_n \cap Y_P^\wedge = \emptyset$, i.e., $Y_P \cap \mathrm{dom}\ n = \emptyset$. But then $Pn = \emptyset_0$ by (8.2.1).

To show u is a homeomorphism, take a typical open subset U_n of $\mathbf{S}C$, $n \in OC$. Then

$$u(U_n) = \{ u(P) \mid P \in \mathbf{S}C,\ Pn = 0 \} = \{ Y_P \mid Pn = 0 \} = \\ \{ Y \in X \mid Y \cap \mathrm{dom}\ n \neq \emptyset \},$$

because $Pn = 0 \Leftrightarrow Y_P \cap \mathrm{dom}\ n \neq \emptyset$ by (8.2.1). But since dom n is a typical open set of X, $\{ Y \in \bar{X} \mid Y \cap \mathrm{dom}\ n \neq \emptyset \}$ is a typical open set of $\bar{X}$.

From (8.2) we see that if M is an admissible manifold, $x \mapsto x_{AM}{}^\wedge$ is a homeomorphism of M with $\mathbf{S}A_M$. We can now give our non-trivial example of a cohesive universe C with $\mathbf{S}C = \emptyset$.

EXAMPLE (8.3). Let I be the set of all open dense subsets of the manifold M and let $i \lesssim j$ mean $i \subset j$. Then I is an upper semilattice, and $i \mapsto A_i$ is a functor $A\_: I \to C$. Let $C = \mathrm{colim}\ A\_$. Then $\mathbf{S}C = \lim \mathbf{S}A = \emptyset$ since if $U \subset M$ is open dense and $x \in U$, then also $U\backslash\{x\}$ is open dense.

If $X \in \mathbf{sob}$, let

$$K_X = \{ f \in K_{|X|} \mid \mathrm{dom}\ f \text{ is open} \} \subset_c K_{|X|}.$$

Let $k_X \subset_c K_X$ consist of all those elements of K_X that are locally constant.

LEMMA (8.4). $X \mapsto k_X$ *defines a functor* $(\mathbf{sob})^{op} \to K$.

We note that k_X defines the topology of X. Since that topology is sober, when n, $p \in Ok_X$, $n \neq p$, there is an $x \in X$ such that $p(x) \neq n(x)$, i.e., $x^\wedge(p) \neq x^\wedge(n)$, so $k_X \in K$. It is easy to see that $X \mapsto k_X$ can be made into a subfunctor $k\_$ of $X \mapsto K_{|X|}$.

THEOREM (8.5). $k\_ \dashv \mathbf{S}$ *where* $\mathbf{S} : K \to (\mathbf{sob})^{op}$.

We note that (8.5) allows us to say that in **sob** we have

$$\mathbf{S}(\lim T) = \operatorname{colim} \mathbf{S}T$$

if T is any K-valued diagram. Moreover $X \mapsto \bar{X}$ is a left adjoint of the inclusion i : **sob** $\to$ **top** and so preserves colimits, so we shall have $\operatorname{colim} \mathbf{S}T = (\operatorname{colim} i\, \mathbf{S}T)^-$. Thus computing $\mathbf{S}(\lim T)$ is, at least in principle, no problem whatever. Of course $\mathbf{S}(\operatorname{colim} T) = \lim \mathbf{S}T$ quite trivially.

Let $X \in \mathbf{sob}$, $C \in K$. To prove (8.5), we need to show that $(k_X, C) \simeq (\mathbf{S}C, X)_{\mathbf{sob}}$. First we define u_X: $\mathbf{S}k_X \to X$ to be the composition

$$\mathbf{S}k_X \longrightarrow \bar{X} \longrightarrow X,$$

where the first arrow is the isomorphism given by (8.2) and the second inverts $\bar{X} \to X$, $x \mapsto \mathrm{cl}\{x\}$, which is a homeomorphism because X is sober. We also need to define $v_C : k_{\mathbf{S}C} \to C$. We do this by using the fact that $C \subset \prod_{P \in \mathbf{S}C} L_P = L_C$ as the set of continuous elements of L_C (cf. remarks preceding (7.2)). Let $c \in k_{\mathbf{S}C}$. Define $vc = v_C c$ by defining $(vc)_P \in L_P$ for each $P \in \mathbf{S}C$ as follows. If $c(P) = \emptyset_0$, let $(vc)_P = \emptyset_P$. If $c(P) = a \in K$, let $(vc)_P = a_? (0_P)$, i.e., $(vc)_P$ is the canonical image of a under $A_0 \to L_P$. If $U \subset \mathbf{S}C$ is the open set of those $Q \in \mathbf{S}C$ such that $c(Q) = a$; - remember c is locally constant - $(vc)_Q = Q(a_?(n))$ for every $Q \in U$ where $n \in OC$ is defined by the equation $Rn = 0$, $R \in \mathbf{S}C$. This shows vc is continuous, i.e., $vc \in C$. We note for pending use the easily proven fact that if $c \in C$, $P \in \mathbf{S}C$, then $P(vc) = c(P)$. The necessary proofs (intricate but routine) that u and v are natural transformations will be omitted. We need to show that

$$u_{\mathbf{S}C}\mathbf{S}v_C = 1_{\mathbf{S}C} \quad \text{if} \quad C \in K \quad \text{and} \quad v_{k_X} k_{u_X} = 1 \quad \text{if} \quad X \in \mathbf{sob}.$$

. Let P ϵ **SC**. We need to show $u_{SC}(Pv_C)$ = P. Now observe that if X ϵ **sob**,, $u_X = \phi_X^{-1}$ where $\phi_X(x) = x_{k_x}\hat{}$. Thus we only need $Pv_C = P\hat{}$, which is clear since

$$Pv_C = c(P) = P\hat{}(c).$$

Now let $c \in k_X$, $f = (v_{k_x} k_{u_x})(c) = v_{k_x}(cu_X)$. We need to show $f = c$, i.e., $\bar{P}f = \bar{P}c$ if P ϵ $\mathbf{S}k_X$. Since ϕ_X is a homeomorphism, P = $x\hat{}$ for some $x \in$ X. Observe that

$$(cu_X)(x\hat{}) = (cu_X(x\hat{})) = c(x).$$

If $(cu_X)(P) = \emptyset_O$, then $Pc = \emptyset_O$, so $\bar{P}(v_{k_x}(cu_X)) = \emptyset_P = \bar{P}c$. If $(cu_X)(P) = a \in K$, $\bar{P}(v_{k_x}(cu_X)) = a_1(0_P)$. Since c is locally constant and $Pc = c(x) = a$, $a_1(0_P) = \bar{P}c$. Thus $f = c$.

9. DERIVATIONS AND TANGENT SPACES.

Let V ϵ U-U, U ϵ U. If $u \in$ U and $vq \in V$, *define* $uv = (u|V)v = vu$. An element D of (|U|,|V|) is called an *admissible derivation* of V if the following axioms are satisfied:

1) $u \in$ U $\Rightarrow$ OD(u) = O$(u|V)$;
2) If $h > 0$, F ϵ A_h, $u \in U^h$, then

$$D(F(u)) = \Sigma_{i=1}^{h}((D_j F)(u))Du_j, \quad \text{where} \quad D_j F = \partial F/\partial z_j^h.$$

We can paraphrase this definition by saying that D "preserves domains" and "satisfies the chain rule". If f : U $\to$ V in U, we let

Ader f = {D ϵ (|U|,|V|)|D is an admissible derivation for f }

(= Ader(U,V) when we understand what f is).

LEMMA (9.1). *Let* V ϵ U-U *and* D ϵ Ader(U,V). *Then if* $f, g \in$ U, *we have*

$$D(f + g) = Df + Dg \quad \textit{and} \quad D(fg) = D(f)g + f\,D(g).$$

First

$$D(fg) = D((z_1{}^2 z_2{}^2)(f,g)) = z_2{}^2(f,g)\ Df + z_1{}^2(f,g)\ Dg =$$
$$(g + 0f)Df + (f + 0g)Dg = g\ Df + f\ Dg + (0Df)Df + (0Dg)Dg = gDf + fDg.$$

Also

$$D(f + g) = D((z_1{}^2 + z_2{}^2)(f,g)) = 1_2(f,g)\ Df + 1_2(f,g)\ Dg =$$
$$1_2(f|V,\ g|V)(Df + Dg) = Df + Dg + 0(f|V) + 0(g|V) = Df + Dg.$$

Let $U \in \mathcal{U}$. We shall construct $I_1 U \to U$ and a canonical element D_U of $\mathrm{Ader}(I_1 U, U)$ so that $I_1 U$ is like a "first order infinitesimal neighborhood" of U. Let

$$|I_1 U| = \{\ (u,v) \in U^2 \mid 0u = 0v\ \}.$$

We shall always write $(u,v) \in I_1 U$ as $u + vt = u + tv$ (picking a "symbol" t) and also any h-tuple $(u_1 + v_1 t,\ \ldots,\ u_h + v_h t)$ of $(I_1 U)^h$ as

$$u + vt = u + tv, \qquad u = (u_1,\ \ldots,\ u_h), \quad v = (v_1,\ \ldots,\ v_h).$$

If $F \in A_h$, define

$$F(u + vt) = F(u) + t\Sigma_{i=1}{}^h (D_j F)(u)v_j.$$

Since

$$0\Sigma_{j=1}{}^h (D_j F)(u)v_j = 0F(u) + 0v_1 + \ldots + 0v_h = 0F(u),$$

$F(u + vt) \in I_1 U$. To show $I_1 U \in \mathcal{U}$ we need to show it satisfies all seminal identities.

The identity (1.1.1) is trivial, so we look at those of type (1.1.2). Using the notation of (1.1.2), we must show that

$$[F(\ldots,\ F_i(z_{i*}),\ \ldots)](u + tv) = F(\ldots,\ F_i((u + tv)_{i*}),\ \ldots)$$

where $(u+tv)_{j*} = u_{j*} + tv_{j*}$. Write this equation as A = B. Let

$$G(z_1,\ \ldots,\ z_p) = F(\ldots,\ F_i(z_{i*}),\ \ldots),$$
$$G_i(z_1,\ \ldots,\ z_p) = (D_i F)(\ldots,\ F_i(z_{i*}),\ \ldots).$$

Calculation shows

$$B = G(u) + t\ \Sigma_{i=1}{}^h\ G_i(u)\ \Sigma_{j=1}{}^{p_i}\ (D_j\ F_i)(u_{i*})v_{ij}$$

and
$$A = G(u + tv) = G(u) + \Sigma_{j=1}{}^p\ (D_j\ G)(u)v_j.$$

Now

$$(D_j\ F)(z_1{}^p,\ \ldots,\ z_p{}^p) = \Sigma_{i=1}{}^h\ G_i(z_1{}^p,\ \ldots,\ z_p{}^p)\ \Sigma_{z_{ik}=z_j}\ (D_k F_i)(z_{i*}{}^p),$$

so the second summand of A is t times

$$\Sigma_{j=1}{}^{p}\, \Sigma_{i=1}{}^{n}\, G_i(u)\ (\Sigma_{x_{ik}=x_j}\ (D_k\ F_i)(u_{i*}))v_j.$$

A little effort shows this agrees with the coefficient of t in B.

It is evident that $p(u + tv) = u$ defines a canonical element $p = p_U$ of $(I_1 U,U)_U$ and $D(u + tv) = v$ a canonical element $D = D_U$ of $\mathrm{Ader}(I_1 U,U)$. If $f \in (U,U')_U$, define $I_1 f : I_1 U \to I_1 U'$ by

$$(I_1 f)(u + tv) = f(u) + t\ f(v).$$

We note the following, which shows that the functor I_1 supplies us with all possible examples of admissible derivations.

PROPOSITION (9.2). *Let* U, V $\in$ *U. Then*

$$(U,\ I_1 V) = \{(f,\ D)\ |\ f : U \to V,\ D \in \mathrm{Ader}\ f\ \}.$$

If g: U $\to$ I_1V, let Hg = $(p_V,\ d_V g)$. Evidently H is a one-one map of $(U,I_1 V)$ into

$$\{\ (f,\ D)\ |\ f : U \to V,\ D \in \mathrm{Ader}\ f\ \}.$$

If f : U $\to$ V, D $\in$ Ader f, $g(u) = f(u) + tD(u)$ is seen to define a preimage under H of $(f,$ D), so H is surjective.

LEMMA (9.3). *Let* U $\in$ *U*, *a*, *b* $\in$ I_1U. *If* $a \lesssim b$ *and* $pa = pb$, *then* $a = b$.

Indeed, $0pa = 0pb \Rightarrow 0a = 0b$, so $a = b$.

PROPOSITION (9.4). *Let* U $\in$ *C (resp. K). Then* I_1C $\in$ *C (resp. K). Also* $p_U \in C$.

If u, $v \in$ U, then (in I_1U) we have $0(u + tv) = 0u + t0v$. Also

$$(u_1 + tv_1) + (u_2 + tv_2) = (u_1 + u_2) + t(v_1 + v_2)$$

gives the addition in I_1U, so $(u_1 + tv_1)$, $(u_2 + tv_2)$ match iff $(u_1,\ u_2)$, (v_1, v_2) are matching pairs of elements of U. Also

$$u_1 + tv_1 \lesssim u_2 + tv_2 \ \Leftrightarrow\ u_1 \lesssim u_2 \text{ and } v_1 \lesssim v_2 .$$

Thus if $U \in C$, I_1U has glbs and p_U preserves glbs. To finish showing $I_1U \in C$, let $F \in \mathcal{M}(A_n)$, $M_1, \ldots, M_n \in \mathcal{M}(I_1U)$. Set $F = \text{glb}\ F$, $m_i = \text{glb}\ M_i$. Then

$$F(m_1, \ldots, m_n) \leq \text{glb}\ (F(M_1, \ldots, M_n)),$$

and

$$p(F(m_1, \ldots, m_n)) = F(pm_1, \ldots, pm_n) = \text{glb}\ (F(pM_1, \ldots, pM_n))) = \\ p(\text{glb}\ (F(M_1, \ldots, M_n))),$$

so we get equality using (9.3). Thus $I_1U \in C$ and clearly $p_U \in C$. Finally if $U \in K$, let $n, n' \in 0I_1U$ be distinct. Then $pn \neq pn'$, so $(Pp)n \neq (Pp)n'$ for some $P \in SU$. Therefore, as $Pp \in \mathbf{S}I_1U$, $I_1U \in K$.

Let $U, V \in C$. Then

$$(U,p_V)\colon (U,I_1V)_U \to (U,V)_U.$$

LEMMA (9.5). $(U,p_V)^{-1}(U,V)_C = (U,I_1V)C$ *if* $U, V \in C$.

(One could say that (U,p_V) "discovers" elements of $(U,I_1V)_C$.) Let $f \in (U,I_1V)_U$, $p_Vf \in (U,V)_C$. Then $f(0_U) = 0_V + tn$, $n \in 0V$. As $n = 0n = 0_V$, $f(0_U) = 0_{I_1V}$. *Let* $M \in \mathcal{M}(U)$. Then

$$f(\text{glb}\ M) \leq \text{glb}\ (fM) \quad \text{and} \quad p(f(\text{glb}\ M)) = p(\text{glb}\ f\,M),$$

so (9.5) follows from (9.3).

COROLLARY (9.5.1). *(9.2) holds with U replaced by C.*

COROLLARY (9.5.2). *Let* $F\colon U \to V$ *in* C, $D \in \text{Ader}\ f$, $M \in \mathcal{M}(U)$. *Then* $D(\text{glb}\ M) = \text{glb}\ DM$.

Let $N \in \mathcal{M}(I_1V)$. Then $D_V(\text{glb}\ N) \leq \text{glb}\ D_VN$. Also

$$0D_V(\text{glb}\ N) = 0p_V\text{glb}\ N = 0\ \text{glb}\ p_VN = \text{glb}\ 0p_VN = \\ \text{glb}\ 0D_VN = 0\ \text{glb}\ D_VN,$$

so

$$D_V(\text{glb}\ N) = \text{glb}\ D_VN.$$

Now write $D = D_Vh$, $h : U \to I_1V$ in C. Then

$$D\ \text{glb}\ M = D_V(h\ \text{glb}\ M) = \text{glb}\ D_VhM = \text{glb}\ DM.$$

***THEOREM* (9.6).** $I_1 : \mathcal{U} \to \mathcal{U}$ *or* $\mathcal{C} \to \mathcal{C}$ *has a left adjoint.*

If $C, C' \in \mathcal{C}$,

$$(C, I_1 C') = \{ (f, D) \mid f \in (C, C'),\ D \in \text{Ader } f \}.$$

We need to show that the functor $\mathcal{C} \to$ **sets** given by the latter expression has the form $(C, \ )$ for some $C \in \mathcal{C}$ (cf. [1], 16.4.5). We shall take $U = (C \bullet |C|)/E$ where E is a set of equations that we now describe.

Since $C \bullet |C| \in C - \mathcal{C}$, we have a morphism $\alpha \in (C, C \bullet |C|)$. We also have a canonical map $\beta : |C| \to |C \bullet |C||$. Let E consist of all equations $0\alpha c = 0\beta c$, $c \in C$ together with all equations

$$\beta(F(c)) = \Sigma_{j=1}{}^{n} \alpha((D_j F)(c))\beta c_j\ , \quad h > 0, \quad c \in C^n, \quad F \in A_n.$$

Then if $C' \in \mathcal{C}$, $(U, C') \simeq (C, I_1 C')$. Basically the same construction works for $\mathcal{U}$ in place of $\mathcal{C}$.

Denote $U \in \mathcal{C}$ (resp. $\mathcal{U}$) that was just constructed by $\tau C \in \mathcal{C}$ (resp. $\mathcal{U}$). The functor $\tau : \mathcal{C} \to \mathcal{C}$ is entirely analogous to the functor which sends an A-algebra B to $S_B(\Omega_{B/A})$ where $\Omega_{B/A}$ is the B-module of Kähler differentials of B over A and S_B denotes symmetric algebra. We note in particular the following.

***PROPOSITION* (9.7).** *Let* $S \in$ **sets**, $C \in \mathcal{C}$. *Then*

$$\tau(C(S)) = (\tau C)(S \amalg d\,S)$$

where $d\,S$ *is the set of all formal symbols* $\{ ds \mid s \in S \}$.

Let $U \in \mathcal{C}$. Then

$$((\tau C)(S \amalg d\,S), U) = \{ (i, f) \mid i \in (\tau C, U),\ f \in (S \amalg d\,S, U_0) \} =$$

$$\{ (i, g, h) \mid i \in (C, I_1 U),\ g \in (S, U_0),\ h \in (d\,S, U_0) \} =$$

$$(C(S), I_1 U) = (\tau(C(S)), U).$$

From (9.7) we get a principle for extending admissible derivations that is similar to one in algebra. Indeed assume we have

$$\mathcal{C} \xrightarrow{\ i\ } C(S) \xrightarrow{\ f\ } U$$

in *C*, where i is canonical, and also $D \in$ Ader fi. Then any $E \in$ Ader f such that $Ei = D$ can be defined uniquely by assigning, any way one pleases, the elements $E(ds) \in U_0$, for $s \in S$.

Let $f : U \to W$, $g: V \to W$ in *C* and consider the isomorphism

$$(U \amalg V,\ I_1 W) \simeq (U, I_1 W) \times (V, I_1 W).$$

By considering morphisms $U \amalg V \to I_1 W$ of the form

$$(f,g) + t\,H, \quad H \in \text{Ader}\,(f,g)$$

we get the following.

LEMMA (9.8). *With the above notation, let* $D \in$ Ader f, $E \in$ Ader g. *Then there exists a unique* $H \in$ Ader(f,g) *such that* $H|_U = D$, $H|_V = E$.

We shall denote H of (9.8) by (D,E). The following is proved as usual using (9.8) with $W = U' \amalg V'$.

LEMMA (9.10). *Let* $f : U \to U'$, $g: V \to V'$ *in C,* $D \in$ Ader f, $E \in$ Ader g. *Then there exists a unique* $D \amalg E \in$ Ader $f \amalg g$ *that makes the following diagram commute.*

$$\begin{tikzcd}
U \arrow[r, "D"] \arrow[d] & U' \arrow[d] \\
U \amalg V \arrow[r, "D \amalg E"] & U' \amalg V' \\
V \arrow[r, "E"'] \arrow[u] & V' \arrow[u]
\end{tikzcd}$$

The proof of the following result is analogous to that of (9.7).

LEMMA (9.11). *Let* $C \in C$ *and let* E *be a set of equations in* C. *The canonical map*

$$\tau C/(i_C E) \cup (d_C E) \to \tau(C/E)$$

where $i_C \in (C, \tau C)$ *and* $d_C \in$ Ader i_C *are canonical, is an isomorphism.*

10. INFINITESIMALS AND TAYLOR POLYNOMIALS.

If $\{ z_1, \ldots, z_h \}$ is a set of symbols and we consider that $A_0 \in C$, then $A_h \simeq A_0\langle z_1, \ldots, z_h\rangle$. We shall let $L_h = L_{0_h}$ - where 0_h is the origin of K^h. Then $L_h = A_0\langle z_1, \ldots, z_h\rangle$ (cf. (6.6.1)-(6.7)). We have $D_i = \delta/\delta z_i \in$ Ader A_h and for any $C \in C$, $0 \in$ Ader C defined by $c \mapsto 0c$. If $f \in A_h \amalg$ C, we write $D_i f$ for $(D_i \amalg 0)f$ (cf. (9.10)). When $f \in A_h$, we have

$$(D_i f)|A_h\amalg C = D_i(f\ |A_h\amalg C),$$

whereas if $c \in$ C,

$$D_i\ C|_{A_h}\amalg C) = 0(C|A_h\amalg C) = 0c,$$

since we consider that $C \in A_h \amalg C$. We use analogous conventions for L_h and $L_h \amalg C$.

Looking at L_h instead of A_h, and working exclusively in K, we note $S(L_h \amalg C) = SC$ (and similarly for any $L \in L$) by using $C \to L_h \amalg C$ or

$$(0_h, C)\colon L_h \amalg C \quad \to \quad A_0 \amalg C = C.$$

Thus $0C = 0(L_h \amalg C)$. We shall use these identifications repeatedly.

If $f \in L_h \amalg C$, $(0_h \amalg C)f \in C = A_0 \amalg C$ will be denoted $f(0)$. Sometimes we shall write $f \in L_h \amalg C$ or $f \in A_h \amalg C$ as $f(z,)$, pretending there is some sort of "variable" for C, and then we shall write $f(0)$ and $f(0,)$. That will make our notation more agreeable and also somewhat redolent of classical notation.

Let $f \in L_h \amalg C$ where $C \in K$. Following these conventions, we have $0f = 0f(0,) = 0f(z,)$. We shall say that f and $f(0)$ "have the same domain", and we shall follow the notational conventions

$$\text{dom } f = \text{dom } f(0) \subset X, \quad \text{where} \quad X = SC = S(L_h \amalg C).$$

If $r \in \mathbf{N}^h$ is any "multi-index", we define $D^r f = D_1^{r_1} \ldots D_h^{r_h} f$ as usual by appropriate repetition of the operators D_i. We note that

$$[D_i, D_j] = D_i\ D_j - D_j\ D_i \in \text{Ader}(L_h \amalg C)$$

clearly equals $0\amalg 0$, i.e., is the zero derivation. Therefore

$$f \in L_h \amalg C \ \Rightarrow\ D_i(D_j\ f) = D_j\ (D_i f),$$

so $D^r f$ *does not depend upon the order of the operators.*

Define $r\,! = r_1! \ldots r_h!$ if $r \in \mathbf{N}^h$, and let $|r| = r_1 + \ldots + r_h$. Then f has its *Taylor series*

$$P(f) = \Sigma_{r \in \mathbf{N}^h} (D^r f)(0)\,(z^r/r!),$$

where $z^r = z_1^{r_1} \ldots z_h^{r_h}$. In $P(f)$, the coefficients of z^r all lie in $C \subset L_r \amalg C$ and have exactly the same domain as f.

For $n \in \mathbf{N}$, let

$$P_n(f) = \Sigma_{0 < |r| < n} (D^r f)(0)\,(z^r/r!)$$

and let $R_n(f) = f - P_n(f)$. Then f, $P_n f$ and $R_n(f)$ all have the same domain. We shall develop a theory of integration that will allow us to write $R_n(f)$ as a familiar integral. In particular we shall see that $R_n(f)$ is a linear combination over $L_h \amalg C$ of the z^r, $|r| = n + 1$.

The following lemma, of vital importance for what follows, points out the naturalness of these definitions. Let $n \in \mathbf{N}$, and let $(z_1, \ldots, z_h)^n = (z)^n$ be the set of all linear combinations over $L_h \amalg C$ of the r for $r \in \mathbf{N}^h$, $|r| = n$. Set $(z)^1 = (z)$. An element of $(z)^n$ will be called an ***infinitesimal of order*** $\geq n$.

LEMMA *(10.1). Let* $f \in L_h \amalg C$, $C \in K$. *Suppose*

$$f = \Sigma_{r \in \mathbf{N}^h,\ |r| < n}\, c_r z^r + R$$

where $R \in (z)^n$ *and* $0c_r = 0f$ *for every* r. *Then* $c_r = (D^r f)(0)/r\,!$ *whenever* $|r| \leq n$.

We have $D^r f = c_r r\,! + T_r$, where $T_r \in (z)$, as we see by differentiating $|r|$ times. Thus $(D^r f)(0) = c_r r\,! + p$, where $p \in 0C$. Then $p \leq 0f$, so by adding $0f$ to both sides, we get $(D^r f)(0) = c_r r\,!$.

Let $C \in K$. Questions concerning $L_h \amalg C$ can often be resolved by considering the $L_h \amalg L$, $x \in X = SC$. Since we identify $(0_h, x)$ in $S(L_h \amalg C)$ with $x \in SC$, we have

$$(L_h \amalg C)_{(0,x)} = (L_h \amalg C)_x = L_h \amalg C_x.$$

From (6.8),

$$A_0\langle S \amalg T\rangle = A_0\langle S\rangle \amalg A_0\langle T\rangle$$

in L. Therefore

$$L_h = L_{h-1} \amalg L_1 = L_1 \amalg \ldots \amalg L_1 \quad (h \text{ summands}).$$

This allows us to use induction on h to deduce properties of the functor $L_h \amalg -$ from properties of $L_1 \amalg -$.

***PROPOSITION* (10.2).** *Let* $C \in K$, $f \in L_1 \amalg C$. *The following are equivalent:*

1) $f \in C$;

2) $df/dz = 0f$.

Obviously $1 \Rightarrow 2$. We prove $2 \Rightarrow 1$ by showing

$$2 \Rightarrow f = f(0) \quad (\Leftrightarrow f \in C).$$

We treat first the case where C is local.

***LEMMA* (10.2.1).** *Let* $S \in$ sets *and let* I *be an ideal of* $A_0\langle S\rangle$, $1 \notin I$. *Let* J *be the ideal of* $L_1 \amalg A_0\langle S\rangle$ *generated by* I. *Assume*

$$g \in L_1 \amalg A_0\langle S\rangle, \qquad g(0) = 0, \qquad dg/dz \in J.$$

Then

$$g \in zJ = \{ zj \mid j \in J \}.$$

By (7.7.1) we can assume that $n = \#S < \infty$. Choose $U \subset K$ a neighborhood of $0 \in K$ and $V \subset K^n$ a convex neighborhood of $\mathbf{0}$ such that $g = \bar{g}|L_1 \amalg L_n$ for some $\bar{g} \in A_{n+1}$ with dom $\bar{g} = U \times V$. By shrinking V, we can assume that $\bar{g}(0) = 0$. Working with $\bar{g}$ instead of g, we can carry out the calculation of (6.5.1) to get

$$g(z,\ ) = zu(z,\ ) \text{ where } \quad u(z,\ ) = \int_0^1 (Dg)(sz,\ )\, ds,$$

D denoting here differentiation with respect to the first variable. We can write

$$Dg = a_1(z,\ )x_1 + \ldots + a_w(z,\ )x_w \quad (w \in \mathbf{N})$$

where $x_1, \ldots, x_w \in I$ and the $a_i \in L_1 \amalg L_n$. Let $u_i = \int_0^1 a_i(sz,\ )\, ds$. Then

$$u(z,\ ) = \Sigma_{i=1}^{w} u_i x_i \in J \quad \text{and} \quad g = zu$$

proving (10.2.1).

To show 1 of (10.2) holds (assuming 2 and C local) let $A_0\langle S\rangle \twoheadrightarrow C$ with *kernel*

$$I = \{ f \in A_0\langle S\rangle \mid f|_C = 0 \}$$

be a presentation $I \hookrightarrow A_0\langle S\rangle \twoheadrightarrow C$ of C. By an easy deduction from (6.6), we have the presentation

$$J \hookrightarrow L_1 \amalg A_0\langle S\rangle \twoheadrightarrow L_1 \amalg C$$

of $L_1 \amalg C$, where J is the ideal of $L_1 \amalg A_0\langle S\rangle$ generated by I. Take a preimage p of f in $L_1 \amalg A_0\langle S\rangle$, and let $g = p - p(0, \ )$. *Then*

$$g|_{(L_1 \amalg C)} = f - f(0, \ ), \qquad g(0, \ ) = 0.$$

As
$$(dg/dz)|_{L_1 \amalg C} = df/dz = 0,$$

we have $dg/dz \in J$. By (10.2.1), $g \in zJ \subset J$, so $f = f(0, \ )$, proving $2 \Rightarrow 1$ when C is local.

For $C \in K$ arbitrary, let $X = \mathbf{S}C$, $x \in X$. Let $f_x = f|_{L_x} = \bar{x}(f)$. Then $df_x/dz = 0f_x$. Thus $f_x = f_x(0) = (f(0))_x$ by the local result. As this is so for all $x \in X$, $f = f(0)$ proving (10.2).

COROLLARY (10.2.2). Let $C \in K$, $h \in \mathbf{N}$, $f \in L_h \amalg C$. *Suppose* $D_i f = 0f$ *for* $i = 1, \ldots, h$. *Then* $f \in C$.

11. INTEGRATION OF ONE PARAMETER FAMILIES AND TAYLOR'S THEOREM.

For the rest of this paper "universe" will mean "object of K". If $C \in K$ and $f \in L_h \amalg C$, we know how to write f as $P_n(f) + R_n(f)$ where P_n denotes the "*n-th* degree Taylor polynomial" and R_n denotes "remainder" (cf. §10). The purpose of this section is to develop a theory of integration that will allow us to express R_n in one of the standard forms (cf. (11.3)).

PROPOSITION (11.1). *Let* $c \in C$, $C \in L$, $f \in L_1 \amalg C$. *Then there exists a unique* $g \in L_1 \amalg C$ *such that* $dg/dz = f$, $g(0) = c$.

Uniqueness is clear since from (10.2), if $g(0) = 0$ and $dg/dz = 0$, then $g = 0$. Pick a surjection $A_0\langle S\rangle \twoheadrightarrow C$, S a set. Then

$$L_1 \amalg A_0\langle S\rangle \longrightarrow L_1 \amalg C$$

is surjective by an easy deduction from (6.6). Choose $\bar{f} \in L_1 \amalg A_0\langle S\rangle$ a preimage of f, $\bar{c} \in A_0\langle S\rangle$ a preimage of c. As in the proof of (10.2.1), we can define $I = \int_0^z \bar{f}(s)\, ds$. Then $g = (\bar{c} + I)|_{L_1 \amalg C}$ satisfies the conclusion of (11.1).

A connected subset I of $\mathbf{R}$ will be called an *interval* if its interior $I^\circ \neq \emptyset$. Even if K = C, we shall consider that $I \subset K$ and let

$$\mathbf{M}_I = \mathrm{colim}_K \{ A_J \mid J \supset I, J \text{ open in } K \}.$$

We have therefore $A_1 \to \mathbf{M}_I$ canonically, and we let $t = z|_{M_I} = t_I$ (writing $A_1 = A_0(z)$ as usual). Our immediate goal is to develop a theory of integration with respect to the variable t for any universe $\mathbf{M}_I \amalg C$, $C \in K$.

Because

$$\tau \mathbf{M}_I = \mathrm{colim} \{ \tau A_J \mid J \supset I, \ J \text{ open in } K \}$$

(from $\tau \dashv I_1$) one sees easily that there is a unique d/dt in Ader $\mathbf{M}_I$ such that

$$(d/dt)(f|_{M_I}) = (df/dz) \quad \text{for every} \quad f \in A_1.$$

If $f \in \mathbf{M}_I \amalg C$, we shall denote $[(d/dt) \amalg 0](f)$ by df/dt. We have

$$\mathbf{SM}_I = \lim \{J \mid J \supset I, J \text{ open in } K\} = I$$

as topological spaces. The homeomorphism $I \simeq \mathbf{SM}_I$ sends $a \in I$ to $a\hat{} \in \mathbf{SM}_I$, where $a\hat{}(f|_{M_I}) = f(a)$ for every $f \in A_1$. We shall treat $a \mapsto a\hat{}$ as an identification. From (7.8) it follows routinely that if $a \in I$, $(\mathbf{M}_I)_a \simeq L_1$.

The next theorem will allow us to formulate a definition of $\int_a^\tau f(s)\, ds$, and to prove that it exists, when f is a global element of $\mathbf{M}_I \amalg C$, $C \in K$.

***THEOREM* (11.2).** *Let* I *be an interval,* $C \in K$. *Let* $a \in I$, *and let* c *be a global element of* C, f *a global element of* $\mathbf{M}_I \amalg C$. *Then there exists a unique* g *in* $\mathbf{M}_I \amalg C$ *such that* $dg/dt = f$, $g(a) = c$.

***LEMMA* (11.2.1).** *Let* $C \in L$, $f \in \mathbf{M}_I \amalg C$ *and assume* dom $f = U$ *where* U *is an interval relatively open in* I, $a \in U$. *Then there exists a unique* g *in* $\mathbf{M}_I \amalg C$ *such that* $dg/dt = f$, $g(a) = c$.

To show uniqueness, let

$$g,\ h \in M_I \amalg C, \quad dg/dt = f = dh/dt, \quad g(a) = c = h(a).$$

We have

$$I = \mathbf{S}(M_I) = \mathbf{S}(M_I \amalg C) \quad \text{and} \quad (M_I \amalg C)_b = (M_I)_b \amalg C \quad \text{if} \quad b \in I.$$

Thus by (11.1), since

$$g_a(a) = c = h_a(a) \quad \text{and} \quad dg_a/dt = f_a = dh_a/dt,$$

we have $g = h$ on a neighborhood of a. Let V the union of all intervals $W \subset U$ relatively open in I with $a \in W$, $g|_W = h|_W$. Then $V \neq \emptyset$ is a relatively open interval of I and $g|_V = h_V$. We need to show that V = U.

Suppose $V \neq U$. Then, by writing $V = (c,d) \cap I$ and looking at sketches of the possible cases, one sees that an endpoint b of V lies in U\V. Now g_b and $h_b + g(b) - h(b)$ both solve

$$du/dt = f_b, \quad u(b) = g(b).$$

Thus there exists W, an interval relatively open in U, such that

$$b \in W \quad \text{and} \quad h + g(b) - h(b) + 0_W = g + 0_W.$$

Pick an element b' of $W \cap V$. Then

$$h(b') + g(b) - h(b) = g(b') = h(b')$$

(since $g|_V = h|_V$), so

$$g(b) = h(b) \quad \text{and} \quad h + 0_W = g + 0_W.$$

Then

$$h + 0_{V\cup W} = g + 0_{V\cup W}\ ,$$

a contradiction since clearly V is the largest interval relatively open in U with $a \in V$, $g|_V = h|_V$. Thus V = U. We shall re-use (several times) the argument that was just made.

We need to show that g exists. By (11.1) we can solve

$$dg_a/dt = f_a, \quad g_a(a) = c.$$

Therefore we have an interval U' open in I and $g_{U'}$ in $M_I \amalg C$ such that

$$a \in U' = \operatorname{dom} g_{U'}\ , \quad dg_{U'}/dt = f|_{U'}, \quad g_{U'}(a) = c.$$

Let $\mathfrak{U}$ be the family of all intervals U' open in U for which such a $g_{U'}$ exists, and let $H = \{ g_{U'} \mid U' \in \mathfrak{U} \}$. By the already demonstrated uniqueness property, $H \in \mathcal{M}(M_I \amalg C)$. Let g = glb H and let U' = dom g, so that $U' = \bigcup \{ V \mid V \in \mathfrak{U} \}$. Then U' is the largest element of $\mathfrak{U}$.

We only need to show U' = U. If U' ≠ U, pick an endpoint $b \in U \setminus U'$ of U'. Let $h \in M_I \amalg C$ with $h_b(b) = 0$, $dh_b/dt = f_b$. By taking the domain V of h small enough, we can assume $V \subset U$, $dh/dt = f|_V$. Let $b' \in V \cap U'$, and consider $u = h - h(b') + g(b')$. We have

$$u(b') = g(b') \quad \text{and} \quad du/dt + 0_{U' \cap V} = f|_{U' \cap V} = dg/dt + 0_{U' \cap V},$$

so

$$u + 0_{U' \cap V} = g + 0_{U' \cap V}$$

by the already proven uniqueness statement. Then $\{ g, u \} \in \mathcal{M}(M_I \amalg C)$. Let $g' = g \wedge u$. By (9.5.2),

$$dg'/dt = f|_{U' \cap V}, \qquad g'(a) = c,$$

contradicting the maximality of U'.

To show uniqueness in (11.2) (so now $C \in K$ is not assumed to be local), let g, h be two solutions. Denote, for instance, $f|(M_I \amalg C_x)$ by f_x if $x \in X = SC$. Then g_x, h_x solve $du/dt = f_x$, $u(a) = c_x$, so $g_x = h_x$ by (11.2.1). Then, using (7.8), if $b \in I$,

$$g_{(b,x)} = (g_x)_b = (h_x)_b = h_{(b,x)}.$$

Since $b \in I$, $x \in X$ are arbitrary and $S(M_I \amalg C) = I \times X$, this shows $g = h$. If f is not global, but dom $f = J \times X$, $J \subset I$ an interval, this same argument will also show uniqueness.

To show g exists in (11.2), let $x \in X$ and apply (11.2.1) to C_x to solve

$$dg_x/dt = f_x\ , \qquad g_x(a) = c_x$$

with $g_x \in M_I \amalg C_x$. For $b \in I$, $x \in X$, let $g_{(b,x)} = (g_x)_b$. Now

$$g = \{ g_{(b,x)} \mid b \in I,\ x \in X \} \in \prod_{p \in I \times X} (M_I \amalg C)_p \supset M_I \amalg C$$

We can complete the proof of (11.2) by showing $g \in M_I \amalg C$.

Let $x \in X$. We can establish $g \in M_I \amalg C$ by showing that if $b \in I$, there is an interval J open in I and an $n \in 0C$ with the following properties:

1) $a, b \in J$ and $n(x) = 0$;

2) The equations $dh/dt = f|_{J\times\mathrm{dom}\, n}$, $h(a) = c + n$ have a solution h.

Indeed, assume this occurs. Then if $b \in$ J and $n(x') = 0$ $(x' \in X)$, $h_{x'} = (g_{x'})|_J$ by (11.2.1). Then $h_{(b',x')} = g_{(b',x')}$, and so the condition of "continuity" about an arbitrary (b,x) in I × X needed to show $g \in M_I \amalg C$ will hold.

Let U ⊂ I be the set of all b in I for which J and n exist as stated above. Then U is open in I and $a \in$ U. If U ≠ I, let b be an endpoint of U with $b \in$ I\U. There exists $n' \in$ 0C, V ⊂ I an interval relatively open in I such that $b \in$ V, $n'(x) = 0$, and such that we have h' with $dh'/dt = f|_{V\times\mathrm{dom}\, n'}$. Take $b' \in$ U∩V. Then, because $b' \in$ U, there exists a relatively open J" containing a, b', and there exists $h'' \in M_I \amalg C$, $n'' \in$ 0C, such that

$$n''(x) = 0, \quad dh''/dt = f|_{J''\times\mathrm{dom}\, n''}, \quad h''(a) = c + n''.$$

By replacing n', n'' by $n = n' + n''$, h'' by $h'' + n$, we can assume $n' = n = n''$. Then $h'' + h'(b') - h''(b')$ and h' match by uniqueness, since they have the same value at b'. Replacing h'' by $h'' - h'(b') + h''(b')$, we can assume that h',h'' match. Let J = J" ∪ V. Then $h = h' \wedge h''$ solves

$$dh/dt = f|_{J\times\mathrm{dom}\, n}, \quad h(a) = c + n.$$

But $b \in$ J contradicts the definition of U since $b \notin$ U. Therefore U = I, and so every $b \in$ I has the desired property. This proves (11.2).

Let I be an interval, $a \in$ I, f a global element of $M_I \amalg C$. We shall let $\int_0^t f(s)\, ds$ denote the unique g in $M_I \amalg C$ such that $g(a) = 0$, $dg/dt = f$. However, we shall sometimes use a "dummy variable" other than s in $\int_a^t f(s)\, ds$. The element g of (11.2) would now be written $c + \int_a^t f(s)\, ds$. The formula for integration by parts,

$$\text{(11.2.2)} \quad \int_a^t f\, dg = [gf]_a^t - \int_a^t g\, df, \qquad f, g \in (M_I \amalg C)_0,$$

where

$$dg = (dg/ds)ds \quad \text{and} \quad [h]_a^t = h - h(a),$$

can now be verified immediately.

With only a slight additional complexity of notation in the proof of (11.2), we can establish its conclusion if dom f = J × U, dom c = U, where $a \in$ J, J an interval open in I. Such a set J × U will be called *rectangular* about a. Thus if $f \in M_I \amalg C$ with domain J × U rectangular about a, we can define

$$g = \int_a^\tau f(s)\, ds \quad \text{by} \quad g(a) = 0_U, \quad dg/dt = f.$$

We note that (11.2.2) remains valid (even if dom $f \neq$ dom g) as long as dom f and dom g are rectangular about a. (Of course fg could then be a phantom.)

To write the remainder term for our Taylor's Theorem, we shall need to make sense of expressions $f(tz)$ for $f \in L_h \amalg C$, $C \in K$. Let U be any open neighborhood of 0 in K^h, and choose an open neighborhood J of [0,1] in K and an open neighborhood V of 0 in K^h such that JV $\subset$ U. We then have a commutative diagram

$$\begin{tikzcd}
A_U \arrow[r] \arrow[d] \arrow[dr, phantom, "(t-)"] & A_J \amalg A_V \arrow[d] \\
L_h \arrow[r] & M_I \amalg L_h
\end{tikzcd}$$

where the vertical arrows are the usual ones, and the top one comes from the multiplication map J $\times$ V $\to$ U. The bottom arrow $(t-)$ is independent of U, J, V since the composite $\urcorner$ is independent of (J, V) and since

$$L_h = \operatorname{colim} \{ A_U \mid 0 \in U, \ U \text{ open in } K^h \}.$$

Let $f = f(z) \in L_h$. We shall denote $(t-)f$ as $f(tz) = f(tz_1, \ldots, tz_h)$. If $f \in L_h \amalg C$, we let $f(tz) = f(tz, \ )$ denote $((t-) \amalg C)(f)$.

On L_{2h} written as

$$A_0\langle t_1, \ldots, t_h; z_1, \ldots, z_h\rangle = A_0\langle t; z\rangle$$

we can define

$$z\bullet\nabla_t = \Sigma_{j=1}{}^h z_j\, \partial/\partial t_j \in \text{Ader } L_{2h}.$$

We let $z\bullet\nabla t$ act on $A_0\langle t;z\rangle \amalg C$ as $(z\bullet\nabla_t) \amalg 0_C$. If $i \in [1,h]^n$, define

$$z^i = z_{i_1} \ \ldots \ z_{i_n} \in A_0\langle t; z\rangle \quad \text{and} \quad D_{i,t}f = (\partial^n/\partial t_{i_1} \ \ldots \ \partial t_{i_n})f,$$

$f \in A_0\langle t;z\rangle \amalg C$. If $r \in \mathbb{N}^h$, $|r| = n$, define

$$z^r = z_1{}^{r_1} \cdots z_h{}^{r_h}, \quad \text{and} \quad D_{r,t}f = (\partial^n/\partial t_{i_1}{}^{r_1} \cdots \partial t_{i_n}{}^{r_h})f.$$

We shall always let $(z\bullet\nabla_t)^0$ denote the appropriate identity operator (e.g., on $A_0\langle t;z\rangle$ or on $A_0\langle t;z\rangle \amalg C$). If f is in $A_0\langle t;z\rangle \amalg C$, the reader can verify that

$$(z.\nabla_t)^n f = \Sigma_{i\in[1,h]^n} z^i D_{i,t}f = \Sigma_{r\in\mathbb{N}^h, |r|=n} (n!/r!) z^r D_{r,t}f.$$

If $f = f(z) \in A_0\langle z\rangle \subset A_0\langle t;z\rangle$, define

$$(d^n f)(t;z) = (z\bullet\nabla_t)^n f(t) \quad \text{for} \quad n \in \mathbb{N} \quad (\text{so} \quad (d^0 f)(t;z) = f(t)\ ).$$

It is easy to see that

$$(d/dt)((d^n f)(tz;z)) = (d^{n+1}f)(tz;z)$$

and that the n-th Taylor polynomial $P_n(f)$ of f at 0 can be written

$$(P_n(f))(z) = \Sigma_{j=0}{}^{n}[(d^j f)(0;z)/j\,!\,].$$

Let $f \in L_n \amalg C$. We define $\int_0^1 f(tz)\,dt$ in $L_n \amalg C$ to be $[\int_0^t f(sz)\,ds](1)$.

THEOREM 11.3. Let $f \in L_n \amalg C$, $C \in K$, $n \in \mathbf{N}$. *Then* $f = P_n(f) + R_n(f)$ *where*

$$P_n(f) = \Sigma_{j=0}{}^{n}(d^j f/j\,!)(0;z)$$

and

$$R_n(f) = \int_0^1 ((1-t)^n/n\,!)(d^{n+1}f)(tz;z)\,dt.$$

For $n = 0$, we have

$$f - f(0) = [f(tz)]_{t=0}{}^{t=1} = \int_0^1 (d/dt)(f(tz))\,dt =$$
$$\int_0^1 (\Sigma_{j=1}{}^{h} z_j\,(D_j\,f)(tz))\,dt = \int_0^1 (df)(tz;z)\,dt = R_0(f).$$

Assume $n > 0$ and use induction. Then

$$R_{n-1}(f) = \int_0^1 (1-t)^{n-1}/(n-1)!\,(d^n f)(tz;z)\,dt =$$
$$\int_0^1 (d^n f)(tz;z)\,d(-(1-t)^n/n\,!) =$$
$$[-(1-t)^n/n\,!)d^n f(tz;z)]_0{}^1 + \int_0^1 ((1-t)^n/n!\,)d_t(d^n f(tz;z)) =$$
$$(1/n\,!)d^n f(0;z) + R_n(f).$$

Thus

$$f = P_{n-1}(f) + R_{n-1}(f) = P_n(f) + R_n(f)$$

proving (11.3).

The reader can verify, by examining our formulas for $(z\bullet\nabla_t)^n f$, that $R_n(f)$ is an infinitesimal of order $> n$.

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