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BICATEGORIES OF PARTIAL MAPS

by A. CARBONI

RÉSUMÉ. Différentes notions de bicatégories d'applications partielles sont discutées dans cet article. On caractérise les bicatégories d'applications partielles dans une catégorie exacte à gauche; cette théorie s'applique en particulier aux topos élémentaires. Enfin la catégorie régulière libre sur une catégorie exacte à gauche est décrite explicitement.

0. INTRODUCTION.

The notion of "cartesian bicategory" has been introduced in [C-V], where it is shown that it is flexible enough to give simple and meaningful characterizations of various bicategories of relations, including the additive ones, as well as of the bicategories of ordered objects and ideals defined in an exact category.

Recalling that a cartesian category (with a choice of products) is a symmetric monoidal category in which every object has a unique cocommutative comonoid structure for which every map is a homomorphism, the notion of a cartesian structure on a (locally ordered) bicategory B is the following:

$\otimes$) $\underline{B}$ is equipped with a tensor product $\otimes$: $\underline{B}\times\underline{B} \dashrightarrow \underline{B}$ which is a homomorphism of bicategories, coherently associative, commutative and with an identity I (coherence conditions are the Mac Lane ones, since we are considering locally ordered bicategories);

Δ) Every object is equipped with a unique cocommutative comonoid structure

$$\Delta_X: X \dashrightarrow X\otimes X \quad\text{and}\quad t_X: X \dashrightarrow I$$

for which every 1-cell $r: X \dashrightarrow Y$ is a lax homomorphism (= "lax natural transformation"):

$$\Delta_Y.r \leqslant (r\otimes r).\Delta_X \quad\text{and}\quad t_Y.r \leqslant t_X \;;$$

*) Comultiplications and counits have right adjoints $\Delta_X{}^*$ and $t_X{}^*$.

In [C-W] it is shown that the whole cartesian structure on a bicategory $\underline{B}$ is in fact unique (up to natural isomorphisms), so justifying the name of a "cartesian bicategory". In the same paper it has been observed that bicategories of partial maps defined in a left exact category have the same structure of a cartesian bicategory except that counit does not have a right adjoint; a stronger condition holds, because every 1-cell is in fact a strict comultiplication homomorphism. Such a remark clearly suggests the possibility to give a characterization of bicategories of partial maps in terms of the above modification of the notion of a cartesian bicategory.

Bicategories of partial maps have already been considered by various authors (including the writer [C]). However we think that to discuss partial maps in terms of the general language of relations developed in [C-W] can clarify various aspects of the subject.

In this paper we carry out this project, first discussing in Section 1 the weak notions of structures of bicategories of partial maps which naturally arise in Topology and, as recently pointed out [P-H], in recursion theory; in particular we investigate under which conditions they are in fact unique (up to natural isomorphisms). Observe that to investigate such a question, we really need to consider as a primitive notion the local order on arrows, because it does not seem to be definable in terms of the natural structure. Moreover, even when the local order is definable, as in the case of partial maps in left exact categories or in the case of relations, we stress that the main calculus in such simple cases can be enlightening when dealing with the general case of (enriched) categories and (enriched) profunctors, where certainly the local structure is not definable.

In Section 2 we discuss the full notion of bicategory of partial maps, i.e., of bicategories of partial maps definable in a left exact category, and we give a characterization of such bicategories. Since an elementary topos is a cartesian closed category with a partial map clasifier, clearly our theory of partial maps should apply to elementary topos too and we show how it does. Finally we give an explicit description of the free regular category on a left exact one.

1. BICATEGORIES OF PARTIAL MAPS.

1.1. DEFINITION. A ***structure of a bicategory of partial maps*** on a locally ordered bicategory B is given by:

$\otimes$) a tensor product $\otimes\colon \underline{B}\times\underline{B} \dashrightarrow \underline{B}$ which is a homomorphism of bicategories, coherently associative, commutative and with an identity I ;

Δ) a unique cocommutative comonoid structure

$$\Delta_X\colon X \dashrightarrow X\otimes X, \quad t_X\colon X \dashrightarrow I$$

on each object X for which each 1-cell is a *strict* comultiplication homomorphism and a *lax* counit homomorphism, i.e.,

$$\Delta_Y . r = (r\otimes r).\Delta_X \quad \text{and} \quad t_Y . r \leqslant t_X .$$

Clearly, if $\underline{E}$ is a left exact category with a choice of products, then defining a partial map to be an equivalence class of diagrams

$$X \xleftarrow{\;i\;} \cdot \xrightarrow{\;f\;} Y$$

where *i* is a mono, and defining composition by means of the pullback, we get a bicategory $\underline{B}$ = Par($\underline{E}$) having a structure of a bicategory of partial maps: the tensor in $\underline{B}$ is given by the product on $\underline{E}$, diagonal and terminal provide the coalgebra structure on each object and one can easily check that all the stated axioms are true.

Other examples of the structure of a bicategory of partial maps arise in Topology, considering partial maps with *open* domain or with *closed* domain (see [A-B] and [B-B]), and in recursion theory [P-H].

Another class of examples arises from any ringed topos ($\underline{E}$,R), defining a Zariski-open subobject of an object X as an inverse image along a map X $\dashrightarrow$ R of the subobject of invertibles of R (see [B-D]); then one can easily check that partial maps with Zariski-open domain compose and constitute a bicategory having a structure of a bicategory of partial maps.

In all these examples, the structure of a bicategory of partial maps is essentially unique. We now discuss under which conditions it is possible to prove that two structures of bicategories of partial maps on a bicategory $\underline{B}$ are naturally isomorphic.

We fisrt begin with the following

1.2. DEFINITION. Let $\underline{B}$ be a bicategory equipped with a structure of a bicategory of partial maps. Define projections p_X: $X\otimes Y \dashrightarrow X$ as $\rho.(1\otimes t_Y)$, where ρ is the isomorphism $X\otimes I \dashrightarrow X$ given by coherence conditions. Then call *domain* or *support* of a 1-cell $r: X \dashrightarrow Y$ the 1-cell $Dr: X \dashrightarrow X$ defined by

$$Dr = p_X.(r\otimes 1).\Delta_X.$$

Call *total* or *entire* a 1-cell r if $Dr = 1$.

1.3. *LEMMA.* *i)* $Dr \leqslant 1$;

ii) $D(Ds.r) = D(s.r)$ *; from which* $D(Dr) = Dr$;

iii) $r.Dr = r$;

iv) $Dr.Ds = Ds.Dr = D(Dr.Ds)$;

v) $Dr.Dr = Dr$;

vi) *r is total iff $t_Y.r = t_X$* ;

vii) if r, s are total, then $s.r$ is total; if $s.r$ is total, then r is total; identitites are total; thus total 1-cells determine a subcategory $\mathrm{Tot}(\underline{B})$ *of* $\underline{B}$;

viii) $\mathrm{Tot}(\underline{B})$ *is cartesian.*

PROOF. For the proof of ii)-v) we refer to [P-H]. As for i), it follows easily from the fact that every 1-cell is a lax counit homomorphism.

vi) If $t_Y.r = t_X$, then since $Dt_Y = 1$, from ii) we have

$$1 = Dt_Y = D(t_Y.r) = D(Dt_Y.r) = Dr \;;$$

conversely, if $Dr = 1$, then:

$$t_X = t_X.Dr = t_X.\rho.(t_Y\otimes 1).(r\otimes 1).\Delta_X = \rho.(1\otimes t_X).(t_Y\otimes 1).(r\otimes 1).\Delta_X =$$
$$= \rho.(t_Y.r\otimes t_X).\Delta_X \geqslant \rho.(t_Y.r\otimes t_Y.r).\Delta_X = \rho.\Delta_I.t_Y.r = t_Y.r.$$

vii) The first statement is obvious from ii). As for the second, if $s.r$ is entire, i.e. if $t_Z.s.r = t_X$, then

$$t_Y.r \geqslant t_Z.s.r = t_X \;;$$

finally $D1 = 1$ can be easily checked.

viii) First observe that $D(r\otimes s) = Dr\otimes Ds$ (a proof can be found in [R]), and that coherence isomorphisms, comultiplications and counits are total. Thus the tensor in $\underline{B}$ restricts to a tensor in Tot($\underline{B}$). Hence from vi) Tot($\underline{B}$) is a symmetric monoidal category in which every object has a unique cocommutative comonoid structure for which every map is a homomorphism; so Tot($\underline{B}$) is cartesian (see [Fo]).

Clearly in all models, even the weak ones, the subbicategory Tot($\underline{B}$) is merely a category, the order on total arrows is discrete. More than this, the monoidal structure on the bicategory is essentially unique. But this does not seem provable from our axioms, so that we are forced to look for a missing axiom, true in all the models mentioned above. The obvious axiom which is always true is the following:

(D) If $r \leqslant s$ and $\mathrm{D}r = \mathrm{D}s$, then $r = s$.

The defect of this axiom is that it is not an equation. However we can prove that it fits the job and we will see in the next section that in the case of partial maps defined in a left exact category it can be replaced by stronger *equational* axioms, which also characterize such bicategories.

1.4. ***LEMMA.*** *If* $\underline{B}$ *is a bicategory equipped with a structure of a bicategory of partial maps in which axiom (D) holds, then:*

i) $a \leqslant 1$ *iff* $a = \mathrm{D}a$ *;*

ii) the order on total arrows is discrete, i.e., Tot($\underline{B}$) *is merely a category;*

iii any other structure of a bicategory of partial maps on $\underline{B}$ *induces the same subcategory* Tot($\underline{B}$).

PROOF. i) If $a \leqslant 1$, then $a = a.\ \mathrm{D}a \leqslant \mathrm{D}a$; but since $\mathrm{D}a = \mathrm{D}(\mathrm{D}a)$, from axiom (D) we get $a = \mathrm{D}a$.

ii) is obvious.

iii) We need to show that if $\otimes'$, Δ_X', t'_X are the data for another structure of a bicategory of partial maps on $\underline{B}$ satisfying axiom (D), then calling $\mathrm{D}'r$ the induced notion of domain of an arrow r, one has that $\mathrm{D}'r = \mathrm{D}r$:

$$\mathrm{D}r = \mathrm{D}(r.\ \mathrm{D}'r) = \mathrm{D}(\mathrm{D}r.\ \mathrm{D}'r) = \mathrm{D}r.\ \mathrm{D}'r \leqslant \mathrm{D}'r,$$

and similarly $\mathrm{D}'r \leqslant \mathrm{D}r$.

Observe that we can now prove the true property of domains: Dr is the smallest coreflexive a such that $r.a = r$; for $r.\ Dr = r$ and, if $r.a = r$, $a \leq 1$, then

$$Dr = D(r.a) = D(Dr.a) \leq Da = a.$$

So, in presence of axiom (D) the tensor product on objects of $\underline{B}$ and on total arrows is characterized (up to a natural isomorphism) as the cartesian product on the subcategory Tot($\underline{B}$), which is the same, does not matter of which structure of bicategory of partial maps on $\underline{B}$ we use to define it. It remains to show that the tensor on general arrows of $\underline{B}$ is also unique up to natural isomorphism. This can be done by the following construction which goes back to [F] (see also [C-W] and [R]).

Let $\underline{B}$ be a locally ordered bicategory and let $\underline{A}$ be a class of idempotents of $\underline{B}$. Then the free splitting $\underline{A}^\wedge$ of idempotents of $\underline{A}$ still is a bicategory and if $\underline{A}$ contains identitites there is a canonical embedding $\underline{B} \dashrightarrow \underline{A}^\wedge$. If $\underline{B}$ is equipped with a structure of a bicategory of partial maps, then we can take $\underline{A}$ to be the class Dom($\underline{B}$) of idempotents of $\underline{B}$ given by domains of arrows; in particular, if axiom (D) holds, then this class coincides with the class Cor($\underline{B}$) of all coreflexives of $\underline{B}$ and we will denote Cor($\underline{B}$)$^\wedge$ simply as $\underline{B}^\wedge$. If the class $\underline{A}$ of idempotents is closed under tensor product and contains all the identities (Dom($\underline{B}$) is such), then $\underline{A}^\wedge$ is canonically equipped with a structure of bicategory of partial maps and there exists a strict monoidal homomorphism of bicategories $\underline{B} \dashrightarrow \underline{A}^\wedge$: just define the tensor on $\underline{A}^\wedge$ by means of the tensor on $\underline{B}$, the comultiplication on an idempotent $a\colon X \dashrightarrow X$ of $\underline{A}$ as $\Delta_X.a = (a \times a).\ \Delta_X$ and counit as $t_X.a$; then axioms for a structure of a bicategory of partial maps on $\underline{A}^\wedge$ can be easily checked. So, Tot($\underline{A}^\wedge$) is cartesian and the homomorphism $\underline{B} \dashrightarrow \underline{A}^\wedge$ restricts to a cartesian functor

$$\mathrm{Tot}(\underline{B}) \dashrightarrow \mathrm{Tot}(\underline{A}^\wedge).$$

When $\underline{A}$ = Dom($\underline{B}$), the Tot(Dom($\underline{A}^\wedge$)) turns out to be the category whose objects are domains (or all coreflexives, when axiom (D) holds) and whose arrows $a \dashrightarrow b$ are arrows r of $\underline{B}$ such that

$$a = Dr \quad \text{and} \quad b.r = r.$$

When $\underline{B}$ is a bicategory equipped with a structure of a bicategory of partial maps in which axiom (D) holds, then Tot($\underline{B}^\wedge$) enjoys further properties (see also [H] and [R]): if a is a coreflexive on an object X of $\underline{B}$, and if $x \leq a$, then x can be considered as a mono x:

$x \dashrightarrow a$ in Tot($\underline{B}^\wedge$); one can then prove that Tot($\underline{B}^\wedge$) has inverse images of such a class of monos; thus we can define the bicategory of partial maps Par(Tot($\underline{B}^\wedge$)) with respect to partial maps whose domains are in this class of monos. Observe that the only reason why Tot($\underline{B}^\wedge$) is not a left exact category is that we cannot prove the same for *all* monos in Tot($\underline{B}^\wedge$). One easily can see that there exists a canonical strictly monoidal embedding $\underline{B}^\wedge \dashrightarrow$ Par(Tot($\underline{B}^\wedge$)), which is in fact a *biequivalence*. Composing with the embeding $\underline{B} \dashrightarrow \underline{B}^\wedge$, we get that any bicategory equipped with a structure of a bicategory of partial maps can be strictly monoidally embededed in a "true" bicategory of partial maps. In particular, if axiom (D) holds and if coreflexives in $\underline{B}$, considered as idempotents, split, then $\underline{B}$ is *biequivalent* to a "true" bicategory of partial maps whose domains are completely determined by $\underline{B}$ as the coreflexives.

As for the problem of uniqueness (up to natural isomorphisms) of the structure of a bicategory of partial maps on $\underline{B}$ for which axiom (D) holds, we can now see that also the tensor of arrows on $\underline{B}$ is in fact determined as follows: if we identify an arrow r in $\underline{B}$ with the arrow $(Dr,r,1)$ of Tot($\underline{B}^\wedge$) (warning, this identification is *not* a functor), then the tensor product $r\otimes s$ of two arrows of $\underline{B}$ is identified with the *cartesian* product

$$(Dr,r,1)\otimes(Ds,s,1) = (Dr \otimes Ds,\ r\otimes s,1)$$

in Tot($\underline{B}^\wedge$).

From now on, we can call a bicategory equipped with a structure of a bicategory of partial maps for which axiom (D) holds simply a "bicategory of partial maps", since the structure is unique (up to natural isomorphisms) and since it can always be represented as a full subbicategory, closed under the tensor product, of a "true" bicategory of partial maps.

2. PARTIAL MAPS IN LEFT EXACT CATEGORIES.

When $\underline{E}$ is a left exact category, the bicategory of all partial maps definable in $\underline{E}$ enjoys a stronger property than axiom (D):

(*) Comultiplication Δ_X on each object X has a right adjoint $\Delta_X{}^*$ satisfying the following equations:

("discreteness") $\Delta.\Delta^* = (\Delta^*\otimes 1).(1\otimes\Delta)$;
("local projections") $\Delta^*.(r\otimes s).\Delta \leqslant r.$

(We forget the subscript on the Δ's and p's to simplify notations.)

As we will see soon, axiom (*) forces axiom (D), so that we can call a bicategory equipped with a structure of bicategory of partial maps for which axiom (*) holds simply "a bicategory of partial maps"; to avoid confusion we will call a bicategory of partial maps in which just axiom (D) holds a "weak" bicategory of partial maps.

2.1. ***LEMMA.*** *Let* $\underline{B}$ *be a bicategory of partial maps. Then:*

i) $\underline{B}$ *has local intersections* $r\cap s$, *such that*

$$r.(D(r\cap s).u = r.u \cap s.u \;;$$

ii)
$$r.(D(r\cap s)) = s.(D(r\cap s)) = r\cap s \;;$$

iii) Axiom (D) holds.

PROOF. i) The operation

$$r\cap s = \Delta^*.(r\otimes s).\Delta$$

is an associative, commutative and idempotent operation on each hom-poset such that

$$u \leqslant r\cap s \quad \text{iff} \quad u \leqslant r \quad \text{and} \quad u \leqslant s.$$

The stated preservation property holds because every arrow is a strict comultiplication homomorphism.

ii)
$$\begin{aligned} r.(D(r\cap s)) &= r.(D(s\cap r)) = r.p.(\Delta^*.(s\otimes r).\Delta\otimes 1).\Delta = \\ &p.(1\otimes r).[\Delta^*.(s\otimes r).\Delta\otimes 1].\Delta = p.[\Delta^*(s\otimes r).(\Delta\otimes 1)\otimes r].\Delta^* = \\ &p.(\Delta^*\otimes 1).(s\otimes r\otimes r).(\Delta\otimes 1).\Delta = p.(\Delta^*\otimes 1).(s\otimes r\otimes r).(1\otimes\Delta).\Delta = \\ &p.(\Delta^*\otimes 1).(s\otimes\Delta.r).\Delta = p.(\Delta^*\otimes 1).(1\otimes\Delta).(s\otimes r).\Delta = \text{(discreteness)} \\ &= p.\Delta.\Delta^*.(s\otimes r).\Delta = \Delta^*.(s\otimes r).\Delta = s\otimes r. \end{aligned}$$

Similarly, one can compute $s.(D(r\cap s))$ as $r\cap s$.

iii)
$$r = r.Dr = r.D(r\cap s) = s.D(r\cap s) = s.dr = s.Ds = s.$$

iv) Clearly $a.b \leqslant a\cap b$; moreover

$$a\cap b = (a\cap b).(a\cap b) \leqslant a\cap b.$$

2.2. DEFINITION. A bicategory of partial maps is *functionally complete* if coreflexives, considered as idempotents, split.

A splitting of a coreflexive a is in principle given by two arrows i, i^* such that $i.i^* = a$ and $i^*.i = 1$. Being $a \leqslant 1$, we have that $i \dashv i^*$; moreover, by Lemma 1.3 vii), i is total. Clearly, if $\underline{E}$ is a left exact category (with a choice of products), then $\underline{B} = \mathrm{Par}(\underline{E})$ is functionally complete.

2.3. ***THEOREM.*** *Let* $\underline{B}$ *be a functionally complete bicategory of partial maps. Then:*

i) $\underline{E} = \mathrm{Tot}(\underline{B})$ *is left exact;*

ii) there exists a faithful, strictly monoidal homomorphism of bicategories $\underline{B} \dashrightarrow \mathrm{Par}(\underline{E})$, *which is the identity on objects.*

PROOF. i) We just need to prove that $\underline{E} = \mathrm{Tot}(\underline{B})$ has equalizers. Let f, g be a pair of total arrows and consider $\mathrm{D}(f \cap g)$. Let i, i^* be a splitting of $\mathrm{D}(f \cap g)$. Then i is total and

$$f.i = f.(\mathrm{D}(f \cap g)).i = g.(\mathrm{D}(f \cap g)).i = g.i$$

(since, if i, i^* is a splitting of a coreflexive a, then $i = a.i$). If x is a total arrow such that $f.x = g.x$, then $h = i^*.x$ is total:

$$\mathrm{D}h = \mathrm{D}(i^*.x) = \mathrm{D}(\mathrm{D}i^*.x) = \mathrm{D}(\mathrm{D}(f \cap g).x) = \mathrm{D}(f.x \cap g.x) = \mathrm{D}(f.x) = 1;$$

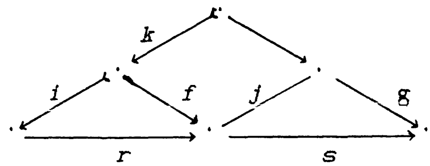
then, since $i.h = i.i^*.x \leqslant x$, we get $i.h = x$, by Lemma 2.1 iii). Uniqueness follows from $i^*.i = 1$.

ii) Given an arrow $r\colon X \dashrightarrow Y$ of $\underline{B}$, let i, i^* be a splitting of $\mathrm{D}r$. Then i is total and monic (since it can be cancelled). The arrow $f = r.i$ is also total:

$$\mathrm{D}f = \mathrm{D}(r.i) = \mathrm{D}(\mathrm{D}r.i) = \mathrm{D}(i.i^*.i) = \mathrm{D}i = 1.$$

Thus $\langle i,f \rangle$ is a partial map in $\underline{E}$. Any other splitting of $\mathrm{D}r$ gives rise to another partial map which is in the same equivalence class of $\langle i,r.i \rangle$. So, the correspondence on arrows is well defined. We just need to prove the functoriality of $r \mapsto \langle i,f \rangle$, which reduces to prove that in the following diagram where

$$j.j^* = \mathrm{D}s, \quad j^*.j = 1,$$



a splitting k, k^* of $D(j^*, r.i)$ is an inverse image of j along $r.i$ in $\underline{E} = \mathrm{Tot}(\underline{B})$.

To prove that the homomorphism $\underline{B} \dashrightarrow \mathrm{Par}(\underline{E})$ is a biequivalence, we need to consider an extra structure that bicategory of partial maps defined in left exact categories have, i.e., a *quasi-inverse* defined on partial maps $\langle i,f\rangle$ such that f is mono. Such partial maps can be equationally characterized in bicategory of partial maps of the form $\mathrm{Par}(\underline{E})$ as follows:

2.4. DEFINITION. i) In a bicategory of partial maps an arrow r is a *monic* if

$$\Delta^*.(r\otimes r) = r.\,\Delta^*.$$

One has that monics compose and identities are monics, so defining a subbicategory $\underline{M}$ of monics of $\underline{B}$.

ii) A *quasi-inverse* for a monic r is a 1-cell $r^\bullet$ such that

$$Dr = r^\bullet.r \quad \text{and} \quad Dr^\bullet = r.r^\bullet.$$

2.5. *LEMMA*. *In a bicategory of partial maps* $\underline{B}$ *with quasi-inverses for monics:*

i) quasi-inverses are unique;

ii) if $\underline{B}$ *is functionally complete, then an arrow* f *in* $\underline{E} = \mathrm{Tot}(\underline{B})$ *is mono in* $\underline{E}$ *iff it is a monic in* $\underline{B}$;

iii) if i, i^* *is a splitting of a coreflexive* a, *then* $i^* = i^\bullet$; *in particular* $\Delta^* = \Delta^\bullet$.

PROOF. i) If r' is an arrow such that $Dr = r'.r$ and $Dr' = r.r'$, then:

$$r' = r'.\,Dr' = r'.r.r' = Dr.r' = r^\bullet.r.r' \leqslant r^\bullet \;;$$

similarly $r^\bullet \leqslant r'$.

ii) From Lemma 2.4, it follows that a pullback of f, g in Tot($\underline{B}$) is given by $\langle p.i,\ p.\bar{\imath}\rangle$, where i, i^* is a splitting of $D(\Delta^*.(f\otimes g))$; so, if f is a mono in $\underline{E}$ = Tot($\underline{B}$), then ker(f) = Δ, thus $D(\Delta.i.(f\otimes f))$ = $\Delta.\ \Delta^*$; hence:

$$\Delta^*.(f\otimes f) = (\Delta^*.(f\otimes f)).D(\Delta^*.(f\otimes f)) = \Delta^*.(f\otimes f).\ \Delta.\ \Delta.i = f.\ \Delta^*\ ;$$

conversely, if $\Delta^*.(f\otimes f) = f.\ \Delta^*$, then

$$D(\Delta^*.(f\otimes f)) = D(f.\ \Delta^*) = D\Delta.i = \Delta.\ \Delta^*\ ;$$

thus ker(f) is Δ and f is mono.

iii) Since i is a mono in $\underline{E}$, then i is a monic in $\underline{B}$; so, $i^\bullet$ exists; since

$$Di = i^*.i = 1 \quad \text{and} \quad Di^* = i.i^*,$$

then from i) we get $i^* = i^\bullet$.

Clearly, if $\underline{B}$ is a functionally complete bicategory of partial maps such that every monic has a quasi-inverse and if $\langle i,f\rangle$ is a partial map in $\underline{E}$ = Tot($\underline{B}$), then $r = f.i$ is an arrow such that

$$Dr = Di^\bullet = i.i^\bullet \quad \text{and} \quad 1 = Di = i^\bullet.i\ ;$$

thus $\langle i,r.i\rangle = \langle i,f\rangle$ and the homomorphism $\underline{B} \dashrightarrow$ Par(Tot($\underline{B}$)) of Lemma 2.4 is a biequivalence. Summing up:

2.6. *COROLLARY (characterizing bicategories of partial maps defined in left exact categories). Functionally complete bicategories of partial maps in which every monic has a quasi-inverse characterize bicategories of partial maps defined in left exact categories with a choice of products.*

2.7. **DEFINITION**. A bicategory of partial maps is *total-closed* if for each pair of objects X and Y there exists an object $(X,Y)^\sim$ and a natural isomorphism

$$\underline{B}(-\otimes X,\ Y) \dashrightarrow \mathrm{Tot}(\underline{B})(-,(X,Y)^\sim).$$

2.8. COROLLARY *(characterizing bicategories of partial maps of elementary topos). Total-closed and functionally complete bicategories of partial maps having quasi-inverses of monics characterize bicategories of partial maps of elementary topos.*

PROOF. Recalling that an elementary topos is a left exact category $\underline{E}$ which is cartesian closed and has partial maps classifiers, clearly $\underline{B} = \mathrm{Par}(\underline{E})$ satisfies the hypotheses of the corollary. Conversely, if $\underline{B}$ is such a bicategory, then $\underline{E} = \mathrm{Tot}(\underline{B})$ is left exact and has partial maps classifiers defined as $X^{\sim} = (I,X)^{\sim}$; we need to show that $\underline{E}$ is cartesian closed. First observe that there exists a canonical total arrow $i: X \dashrightarrow X^{\sim}$ such that an arrow $U \dashrightarrow X$ is total iff the classifying total $U \dashrightarrow X^{\sim}$ factors (uniquely) through i. Now, to define the hom-objects (Y,Z) in $\mathrm{Tot}(\underline{B})$, let ϕ and ψ be the two total arrows $(Y,Z)^{\sim} \dashrightarrow (Y,Z)^{\sim}$ defined as follows: ϕ is the transpose of

$$i.\mathrm{val}: Y\otimes(Y,Z)^{\sim} \dashrightarrow Z \dashrightarrow Z^{\sim}$$

and ψ is the transpose of the classifying total arrow $Y\otimes(Y,Z)^{\sim} \dashrightarrow Z^{\sim}$ of the evaluation arrow. A straightforward checking proves that the equalizer of ϕ, ψ is the hom-object (Y,Z) in $\mathrm{Tot}(\underline{B})$.

2.9. REMARKS. i) If $\underline{B}$ and $\underline{C}$ are bicategories of partial maps, then a strictly monoidal homomorphism of bicategories $F: \underline{B} \dashrightarrow \underline{C}$ restricts to a cartesian functor $\mathrm{Tot}(\underline{B}) \dashrightarrow \mathrm{Tot}(\underline{C})$ and, if $\underline{B}$ and $\underline{C}$ are functionally complete, then it restricts to a left exact functor. So, if $\underline{E}$ and $\underline{F}$ are left exact categories, then there is an equivalence of categories between the category of strictly monoidal homomorphisms of bicategories $\mathrm{Par}(\underline{E}) \dashrightarrow \mathrm{Par}(\underline{F})$ and the category of left exact functors $\underline{E} \dashrightarrow \underline{F}$.

ii) There are functionally complete bicategories of partial maps in which not every monic has a quasi-inverse. An example is given by considering the bicategory of partial maps with closed domain between Hausdorff spaces: certainly such a bicategory is a bicategory of partial maps, since Δ has an adjoint which satisfies all stated equations; however any proper dense subspace Q of a space R is a monic $Q \dashrightarrow R$ which does not have a quasi-inverse. On the other hand, Theorem 2.3 ensures that $\mathrm{Par}(\mathrm{Tot}(\underline{B}))$ exists and has of course quasi-inverses of monics, so that $\underline{B} \dashrightarrow \mathrm{Par}(\mathrm{Tot}(\underline{B}))$ is the free functionally complete bicategory of partial maps with quasi-inverses of monics.

iii) According to the basic principle of categorical logic that a theory is a small category with a specified kind of properties and a model in a (possibly) large category having the same kind of properties is a functor which preserves these properties, we would define a left exact theory simply as a small left exact category. However, for the classical notions of (multisorted, intuitionistic) theories such a principle of categorical logic is justified by the fact that there is a precise correspondence between (multisorted, intuitionistic) theories and a certain class of categories. The main point in proving such a correspondence is that from a theory $\underline{T}$ we can construct a *bicategory of concepts* (= equivalence classes of formulas) $\underline{B}(\underline{T})$ which can be proved to be a bicategory of relations; then using the theory of relations, we can prove that the category of maps of the free functional completion of $\underline{B}(\underline{T})$ is a category $\underline{C}(\underline{T})$ (the "classifying category" of $\underline{T}$) having the property that models of $\underline{T}$ correspond to an appropriate class of functors out of $\underline{C}(\underline{T})$ (see [M-R], [F], [C-W]). So, reversing the above process, we could define a (multisorted, intuitionistic) theory as a *syntactical presentation* of a bicategory of relations. In the same spirit, we could define a left exact (higher order) theory $\underline{T}$ as a syntactical presentation of a (total-closed) bicategory $\underline{B}(\underline{T})$ of partial maps and a model of $\underline{T}$ in a left exact category (in a topos) $\underline{E}$ as a monoidal homomorphism of bicategories $\underline{B}(\underline{T}) \dashrightarrow \mathrm{Par}(\underline{E})$ (hom-objects preserving). Syntactical presentations of bicategories of partial maps have been discussed by various authors (e.g., [CO]). To prove the existence of the classifying category of a left exact (higher order) theory in the same way as for ordinary theories, we need to construct the free functional completion of a (total-closed) bicategory of partial maps.

2.11. ***LEMMA.*** *Let $\underline{B}$ be a bicategory of partial maps and let $\underline{B}\hat{}$ be the splitting (as a category) of the class of idempotents given by coreflexives. Then $\underline{B}\hat{}$ is again a bicategory of partial maps, which is the free functional completion of $\underline{B}$. Moreover, if $\underline{B}$ is total-closed, then $\underline{B}\hat{}$ is also total-closed.*

PROOF. As we noticed at the end of last section, $\underline{B}\hat{}$ is a "weak" bicategory of partial maps. Recall that $\underline{B}\hat{}$ has as objects coreflexives $a\colon X \dashrightarrow X$ of $\underline{B}$ and as arrows $r\colon a \dashrightarrow b$ the arrows r of $\underline{B}$ such that $r.a = r = r.b$; comultiplication on each object a is given by $\Delta_a = (a\otimes a).\,\Delta$ and counit is $t_a = t.a$. We just need to prove that axiom (*) lifts to $\underline{B}\hat{}$. First observe that, if $a\colon X \dashrightarrow X$ is a coreflexive, then $\Delta^*.(a\otimes a) = a.\,\Delta^*$. For

$$a.\ \Delta^* = \mathrm{D}a.\ \Delta^* = p.(a\otimes 1).\ \Delta.\ \Delta^* = p.\ \Delta.\ \Delta^*.(a\otimes 1)$$

which is equal (since composition of coreflexives is commutative) to

$$\Delta^*.(a\otimes 1) \geqslant a.\ \Delta^* \ ;$$

the other inclusion follows from $\Delta \dashv \Delta^*$. So, $a.\ \Delta^*$ is an arrow

$$\Delta_a^* = a.\ \Delta^*:\ a\otimes a \dashrightarrow a$$

in $\underline{B}\hat{}$ and one has that $\Delta_a \dashv \Delta_a^*$. Being $\Delta_a^*.(r\otimes s) \leqslant r$, it just remains to prove the discreteness axiom:

$$(a\otimes(a.\ \Delta^*)).((\Delta.a)\otimes a) = (a\otimes a).(1\otimes \Delta^*).(\Delta\otimes 1).(a\otimes a) =$$
$$(a\otimes a).\ \Delta.\ \Delta^*.(a\otimes a) = \Delta_a.\ \Delta_a^*.$$

So, $\underline{B}\hat{}$ is again a bicategory of partial maps and it is functionally complete: if $b: a \dashrightarrow a$ is a coreflexive of $\underline{B}\hat{}$, then it is also a coreflexive of $\underline{B}$ and $b: b \dashrightarrow a$, $b: a \dashrightarrow b$ provide a splitting of b in $\underline{B}\hat{}$. The universal property of $\underline{B}\hat{}$ is now immediate.

To prove that $\underline{B}\hat{}$ is total-closed if $\underline{B}$ is, let b and c be coreflexives on Y and Z respectively; define the hom-object in $\underline{B}\hat{}$ as the coreflexive $(b,c)\tilde{}\,\tilde{}$: $(Y,Z)\tilde{} \dashrightarrow (Y,Z)\tilde{}$ given by intersecting the identity with the total arrow defined as the transpose of

$$b.\mathrm{val}.(1\otimes c):\ (Y,Z)\tilde{}\otimes Y \dashrightarrow (Y,Z)\tilde{}\otimes Y \dashrightarrow Z \dashrightarrow Z\ ;$$

it is not hard to check that this object of $\underline{B}\hat{}$ has the appropriate universal property, using the existence of a *total* arrow

$$((Y,Z)\tilde{}\,\tilde{})\tilde{} \to (Y,Z)\tilde{}\,\tilde{}$$

(see Corollary 2.8 for notations) defined as the one corresponding to

$$((Y,Z)\tilde{}\,\tilde{})\tilde{}\otimes Y \to (Y,Z)\tilde{}\otimes Y \to Z.$$

Thus, for any bicategory of partial maps $\underline{B}$, the category Tot$(\underline{B}\hat{})$ is a left exact category and, if $\underline{B}$ is total-closed, it is an elementary topos.

If $\underline{T}$ is a left exact (higher order) theory, then the construction of $\underline{B}(\underline{T})\hat{}$ amounts to taking as new objects (equivalence classes of) formulas $\phi(x)$ of $\mathbb{T}$ and as arrows $r(x,y)$: $\phi(x) \dashrightarrow \psi(y)$ the (equivalence classes of) formulas $r(x,y)$ such that

$$r(x,y) \vdash_{\mathbb{T}} \phi(x).\ \Delta.\psi(y).$$

To finish, let us mention the connections between bicategories of partial maps and bicategories of relations. We recall that these last ones are defined as cartesian bicategories (see the Introduction) in which every object satisfies the "discreteness" axiom (see [C-W]). Given such a bicategory $\underline{R}$, we define the "Horn part" of $\underline{R}$ as the subbicategory H($\underline{R}$) with the same objects and with arrows just the comultiplication homomorphisms. Observe that Map($\underline{R}$) = Tot(H($\underline{R}$)). The definition of H($\underline{R}$) extends to a functor H from the category of bicategories of relations to the category of bicategories of partial maps. We can describe the left adjoint F to H, when H is restricted to functionally complete bicategories, in the following way: if $\underline{B}$ is a functionally complete bicategory of partial maps, then $\underline{E}$ = Tot($\underline{B}$) is a left exact category; thus we can form the bicategory Span($\underline{E}$) (see [C-W-K]) and we can make it locally ordered; it is immediate to see that the resulting bicategory R($\underline{B}$) is in fact a bicategory of relations; clearly R($\underline{B}$) is not functionally complete; define F($\underline{B}$) to be the free functional completion of R($\underline{B}$); then it is easy to check the universal property of F($\underline{B}$). Starting with a left exact category $\underline{E}$, then maps of the free functional completion of the bicategory Span($\underline{E}$), made locally ordered, provides an explicit description of the free regular category on $\underline{E}$.

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