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ON A SYNTHETIC PROOF OF THE AMBROSE-PALAIS-SINGER THEOREM FOR INFINITESIMALLY LINEAR SPACES

by Marta BUNGE

RÉSUMÉ. Dans cet article, on examine la preuve synthétique (c'est-à-dire dans le cadre de la Géométrie Différentielle Synthétique) du théorème d'Ambrose, Palais et Singer (*Anais da Acad. Bras. de Ciencias* 32, 1960, 163-178) donnée par Bunge et Sawyer (*Cahiers Top. et Géom. Diff.* XXV-3, 1984, 221-257), à la lumière des résultats plus récents sur le même sujet. En particulier, dans un travail de Moerdijk et Reyes (Rapport des recherches, *DMS* 85-3, *Univ. de Montréal*, 1985), on donne une version synthétique du même théorème, valable pour tout objet infinitésimalement linéaire au sens de Bergeron (Rapport des recherches, *DMS* 80-12, *Univ. de Montréal*, 1980) et, par la suite, avec des conséquences classiques nouvelles. Ici, en utilisant des diagrammes d'algèbres de Weil du type de ceux trouvés par Kock et Lavendhomme (*Cahiers Top. et Géom. Diff.* XXV-3, 1984, 311-324), on démontre qu'avec très peu de modifications, la preuve synthétique que nous avions donnée au début (avec Sawyer) est aussi générale que celle de Moerdijk et Reyes et qu'en plus, elle garde un rapport plus étroit avec la preuve classique. Pour justifier cette dernière affirmation, on rend explicite ici le processus de passage "du local à l'infinitésimal" qui nous a permis d'obtenir le résultat plus général à partir du cas classique, et qui pourrait bien servir d'inspiration pour des situations analogues.

INTRODUCTION.

A theorem of W. Ambrose, R.S. Palais and I.M. Singer [1] establishes a bijective correspondence between torsion free affine connections on a finite dimensional smooth manifold M, and sprays on M. Exploiting the infinitesimal aspects of the main notions involved in that theorem, a proof was given in [5] in the context of Synthetic

Differential Geometry (cf. [10]). In this proof, the assumptions made on an object M of a model of SDG (i.e., of a "space" M) - although of an infinitesimal nature - were all suggested directly by analogous properties of the manifold M employed in the classical proof in [1] (or in [19]). Several of those properties of M were known consequences of "infinitesimal linearity", a property which holds for the ring R of line type and has good closure properties (finite limits, exponentiation by arbitrary exponents, etale descent) (cf [2, ɛ]). Other properties assumed of M ("iterated tangent bundle property", "property of existence of the exponential map") were introduced in [5] for the specific purposes of the proof.

In an attempt to deduce the classical theorem from [5], A. Kock [11] replaced the exponential map property by a "ray property" on M, while adding the more restrictive assumption that, for some $n > 0$ and every $p \in M$, the fiber T_pM at p of the tangent bundle of M, be isomorphic to R^n. In that paper [11] it was remarked that the iterated tangent bundle property was a consequence of infinitesimal linearity. This was followed up by a paper of A. Kock and R. Lavendhomme [12] in which, among other things, the ray property is deduced from infinitesimal lineraity.

Motivated by the above results, I. Moerdijk and G.E. Reyes [16] produced a synthetic proof of the Ambrose-Palais-Singer Theorem under the sole assumption of infinitesimal linearity, while exploiting its generality to exhibit versions of the theorem for objects other than those covered by the original theorem, e.g., manifolds with singularities and spaces $C^\infty(M, N)$ of smooth maps between manifolds. This was achieved by interpreting the synthetic theorem in fully well adapted models of SDG constructed via C^∞-rings, such as the toposes **F** or **G** (cf [10]). However, in the process, the link with the classical proof had been severed to the extent of provoking the remark (cf. [17], Introduction) that "the original proof proceeds by locally integrating the spray (...) and cannot be simply generalized to the case of spaces $C^\infty(M,N)$ ".

It is the purpose of this note to show that there is a simple generalization of the classical proof of the Ambrose-Palais-Singer Theorem to infinitesimally linear objects, and that is already implicit in [5]. Specifically, from the exponential map property in its infinitesimal form we extract a similar property which is then shown to follow from infinitesimal linearity. The diagram of Weil algebras that we employ for this purpose is basically that found by Kock and Lavendhomme [12] for the ray property and which was later employed in the proof by Moerdijk and Reyes [16]. Thus, by proceeding from the local to the infinitesimal - an approach taken in [5] and carried here a step further - it is possible to extract a simple proof of the Ambrose-Palais-Singer Theorem from the classical proof, but which is valid in a far greater generality. (A similar phenomenon of the passage "from the local to the infinitesimal" was exhibited in [4]; infinitesimal versus local integration of vector fields on an infinitesimally linear object is discussed further in [3].)

The contents are as follows. In *Section* 1 we review briefly the basic assumptions of Synthetic Differential Geometry [10] including the notion of infinitesimal linearity in the (strong) sense of Bergeron [2], centring our attention in its role in the passage from connections to sprays performed in [5] in the synthetic context. Although we shall ultimately reduce all of our assumptions to that of infinitesimal linearity, this portion of the proof of the Ambrose-Palais-Singer Theorem - including the injectivity of the correspondence for torsion-free connections - only requires basic infinitesimal linearity and the Euclideaness of the tangent bundle. In *Section* 2, we begin by reviewing the passage from sprays to connections as done in [5] both under local and infinitesimal assumptions. The local context, being only of a motivational nature needs only be described in very general terms: its consideration is needed to justify our assumption of the existence of an exponential map associated to a spray, where the latter is cast in an infinitesimal, rather than usual, local, form. This brings us to the main purpose here, which is to show that the infinitesimal context of [5], with just a minor modification which is thoroughly natural in view of the proof, follows from infinitesimal linearity and therefore yields the Ambrose-Palais-Singer Theorem in the generality first encountered in [16]. However, unlike the proof of [16] the proof given in [5] closely parallels the classical proof. The "method" we employ, which is to proceed from the local to the infinitesimal as dictated by the proof itself, could well be imitated in similar circumstances in order to obtain theorems of greater generality and conceptual simplicity than their classical analogues.

1. FROM CONNECTIONS TO SPRAYS.

The basic framework employed here is that of SDG (Synthetic Differential Geometry) given by an elementary topos **E** with a commutative ring R in **E**, satisfying (cf. [10]):

1.1. AXIOM (Kock-Lawvere). For each Weil algebra W, the morphism

$$\alpha: R[W] \to R^{Spec_R(W)}$$

defined, for any given presentation

$$W = Q[x_1,\ldots,x_n]/(k_1,\ldots,k_n) ,$$

by evaluating any representative $h \in R[x_1,\ldots,x_n]$ of an element of R[W] at the elements $(x_1,\ldots,x_n) \in Spec_R(W) \subset R^n$, is (well defined and) an isomorphism.

We recall (cf. [2]):

1.2. DEFINITION. An object M of E is said to be *infinitesimally linear* if given any finite limit $\{W \longrightarrow W_i\}_i$ of Weil algebras, the corresponding diagram in E, obtained by applying the functor $Spec_R(-) \circ M^{(-)}$, i.e., the diagram

$$\{M^{Spec_R(W)} \to M^{Spec_R(W_i)}\}_i$$

is a finite limit in E.

It is well known (cf. [10]) that R is infinitesimally linear by virtue of the Kock-Lawvere Axiom, and that the class of such objects in E is closed under finite limits and exponentiation by arbitrary objects. Among the consequences of infinitesimal linearity are the basic (or original) assumption called by the same name, as well as Euclideaness of the tangent bundle of M. These properties are the sole ones needed in establishing the results contained in this section.

We now recall the following (cf. [9]):

1.3. DEFINITION. A *connection* on M is a morphism

$$\nabla: M^D \times_M M^D \longrightarrow (M^D)^D$$

satisfying:

(i) $\nabla(v_1,v_2)(d, 0) = v_1(d); \quad \nabla(v_1, v_2)(0,d) = v_2(d);$
(ii) $\nabla(\lambda \odot v_1, v_2) = \alpha \circ \nabla(v_1, v_2) \; ; \quad \nabla(v_1, \lambda \odot v_2) = \lambda \odot \nabla(v_1, v_2)$

where $(v_1, v_2) \in M^D \times_M M^D$, $d \in D$ and $\lambda \in R$.

A connection ∇ is called *torsion free* if it satisfies:

$$\nabla(v_1, v_2)(d_2, d_1) = \nabla(v_1, v_2)(d_1, d_2),$$

for all $(v_1, v_2) \in M^D \times_M M^D$, $d_1, d_2 \in D$.

In order to define covariant differentiation and geodesics relative to a connection, the notion of a "connection map", introduced by Patterson [18], is more suggestive. Consider the "horizontal map" $H: M^D \times_M M^D \to (M^D)^D$, given by the rule

$$(v_1, v_2) \mapsto [d \mapsto v_1 \oplus (d \odot v_2)].$$

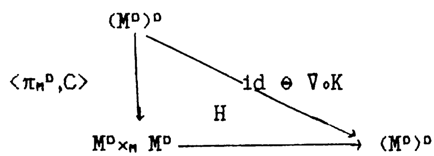
H can be shown (cf. [9, 5]) to be the kernel of the morphism

$$K = \langle (\pi_M)^D, \pi_{(M^D)} \rangle : (M^D)^D \to M^D \times_M M^D.$$

Notice that, by definition, a connection ∇ on M is a splitting of K, i.e., that $K \circ \nabla = \mathrm{id}(M^D \times_M M^D)$ and therefore that

$$K \circ (\mathrm{id}(M^D)^D \ominus \nabla \circ K) = 0.$$

It follows that there exists a unique morphism $C: (M^D)^D \to M^D$, such that



(*)

commutes. The commutativity says, for $t \in (M^D)^D$, that

$$H(t(0), C(t)) = t \ominus \nabla((\pi_M)^D(t), t(0)).$$

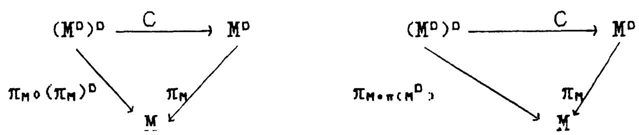
It implies the equation

$$C \circ \nabla = 0 \qquad (**)$$

as follows (H is mono):

$$H \circ \langle \pi_M{}^D, C \rangle \circ \nabla = (\mathrm{id} \ominus \nabla \circ K) \circ \nabla = \nabla \ominus (\nabla \circ K \circ \nabla) = 0.$$

1.4. DEFINITION. A *connection map* on M is a morphism $C: (M^D)^D \to M^D$ satisfying the commutativity of the diagrams



and such that C is linear for both structures of modules on the iterated tangent bundle of M.

A connection map C is said to be *torsion free* if it satisfies $C \circ \Sigma_M = C$ where Σ_M is the symmetry map.

1.5. PROPOSITION. *Let* M *be infinitesimally linear. Then, the data for a connection on* M *is equivalent to the data for a connection map on* M*, a correspondence which is given by the commutative diagram* (*). *Further, if* ∇ *and* C *correspond to each other in this way,* ∇ *is torsion free iff* C *is torsion free.*

A proof of the above is routine (and can be found in [5]). Using covariant differentiation (cf. [5]), the notion of a geodesic may be defined. For a "curve" α: R $\to$ M, α is a geodesic for C iff $C \circ \alpha'' = 0$ and in terms of the data for a connection given by ∇, α is a geodesic for ∇ iff $\nabla(\alpha', \alpha') = \alpha''$.

We now recall (cf. [5]):

1.6. DEFINITION. A *spray* on M is a morphism S: $M^D \to (M^D)^D$ satisfying:

(i) $\pi_{(M^D)} \circ S = id$ and $(\pi_M)^D \circ S = id$;

(ii) $S(\lambda \odot v) = \lambda \circ (\lambda \odot S(v))$ for any $v \in M^D$, $\lambda \in R$.

If ∇ is any connection on M, a spray S is said to be *geodesic spray for* ∇ whenever it is the case that for every curve α: R $\to$ M, α' is an integral curve for S (i.e., α is a solution of the second-order differential equation by S) iff α is a geodesic (relative to ∇).

1.7. THEOREM. *Let* M *be infinitesimally linear. Given a connection* ∇ *on* M, *there is associated a geodesic spray* S_∇ *for* ∇, *by the rule* $\nabla \mapsto S_\nabla$, *with* $S_\nabla(v) = \nabla(v,v)$. *In addition, if* ∇ and $\nabla^\wedge$ *are torsion free connections on* M, *and* $S_\nabla = S_{\nabla^\wedge}$, *then* $\nabla = \nabla^\wedge$.

PROOF. We refer the reader to [5] except for the injectivity of the correspondence for torsion-free connections, not included in [5]. This is now given below under the weak assumptions indicated at the beginning of this section, in spite of the blank assumption made of infinitesimal linearity from which those follow.

If ∇ and $\nabla^\wedge$ are torsion free connections and $S_\nabla = S_{\nabla^\wedge}$, then $\nabla = \nabla^\wedge$. To prove it, we work instead with the associated connection maps C and $C^\wedge$, which are torsion free (Proposition 1.5). In turn, to prove that $C = C^\wedge$ it is enough to prove that

$$H(t(0),\ C(t)) = H(t(0),\ C^\wedge(t))$$

for every $t \in (M^D)^D$ - as H is mono. For any $t \in (M^D)^D$ such that $(\pi_M)^D(t) = \pi_{(M^D)}(t)$, the above is immediate, as then:

$$H(t(0),\ C(t)) = H((\pi_{M^D})(t),\ C(t)) = H((\pi_M)^D(t),\ C(t)) =$$
$$t \ominus (\nabla \circ K)(t) = t \ominus (((\pi_M)^D(t), \pi_{(M^D)}(t)) = t \ominus S_\nabla(\pi_{(M^D)}(t))\ ;$$

similarly

$$H(t(0), C^\wedge(t)) = t \ominus S_{\nabla^\wedge}(\pi_{(M^D)}(t)),$$

so that the result would follow from the assumption that $S_\nabla = S_{\nabla^\wedge}$. The rest of the proof consists in reducing the general case to this special case, i.e., to find, given now an arbitrary $t \in (M^D)^D$, some $\tau \in (M^D)^D$ such that $(\pi_M)^D(\tau) = \pi_{(M^D)}(\tau)$, and for which

$$C(t) = C(\tau) \quad \text{and} \quad C^\wedge(t) = C^\wedge(\tau).$$

Letting

$$\tau = t \ominus \nabla(v, v \ominus t(0)), \quad \text{where} \quad v = (\pi_M)^D(t),$$

it is clear that $C(t) = C(\tau)$, by linearity of C and (**). Indeed,

$$C(t) = C(t \oplus \nabla(v, v \ominus t(0)) \ominus C \circ \nabla(v, v \ominus t(0)) = C(\tau).$$

Also, τ satisfies the required conditions, for

$$(\pi_M)^D(t \oplus \nabla(v, v \ominus t(0)) = (\pi_M)^D(t) \oplus (\pi_M)^D(\nabla(v, v \ominus t(0))) = v \oplus v = v,$$

and

$$\pi_{(M^D)}(t \oplus \nabla(v, v \ominus t(0))) = \pi_{(M^D)}(t) \oplus \pi_{(M^D)}(\nabla(v, v \ominus t(0))) = t(0) \oplus (v \ominus t(0))) = v.$$

It remains to prove that $C^\wedge(t) = C^\wedge(\tau)$. Clearly, this would follow readily from a condition such as

$$C^\wedge \circ \nabla = 0 \tag{***}$$

which happens to be true simply because $S_\nabla = S_{\nabla^*}$, and because both ∇ and $C^\wedge$ are torsion free. To prove it, we first observe that one may regard $C^\wedge \circ \nabla : M^D \times_M M^D \to M^D$ as a bilinear form, and that it vanishes on the diagonal, i.e., for $\phi(v,w) = (C^\wedge.\nabla)(v,w)$, we have $\phi(v,v) = 0$. Indeed:

$$\phi(v,v) = C^\wedge(\nabla(v,v)) = C^\wedge(S_\nabla(v)) = C^\wedge(S_{\nabla^*}(v)) =$$
$$= C^\wedge(\nabla^\wedge(v,v)) = C^\wedge \circ \nabla^\wedge(v,v) = 0.$$

It follows that $\phi(v,w) = 0$ for all v,w. Indeed:

$$0 = \phi(v+w,v+w) = \phi(v,v) + \phi(v,w) + \phi(w,v) + \phi(w,w) = \phi(v,w) + \phi(w,v),$$

so

$$\phi(v,w) = -\phi(w,v).$$

Hence

$$2.\phi(v,w) = 0,$$

hence $\phi(v,w) = 0$, for all v, w (since $2 \in \mathbf{Q}$ is invertible and R is a $\mathbf{Q}$-algebra). This finishes the proof.

2. FROM SPRAYS TO CONNECTIONS.

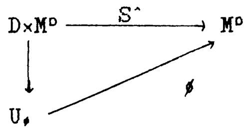
In the classical proof of the Ambrose-Palais-Singer Theorem (cf. [1, 19]), the passage from a spray to a torsion free connection of which the spray is a geodesic spray, is guaranteed by the local integrability of sprays, in turn a consequence of the corresponding theorem on the local existence of solutions to second order differential equations (cf. [14]).

As a motivational device only, we begin by assuming the "local" integrability of sprays, expressible by means of any given *topological structure on* **E** in the sense of [3], i.e., of the data consisting of a subobject $O(X) \subset \Omega^X$, for each object X, closed under finite infima (including the empty one) and under arbitrary suprema in Ω^X, satisfying, in addition, the condition of continuity for every $f \in Y^X$: i.e., if $U \in O(Y)$ then $f^{-1}(U) \in O(X)$. A further assumption (true, e.g., of the Penon or intrinsic topological structure, cf. [10]) will be that *any open of* X *contain the monad of each of its points*, i.e., that for every $U \in O(X)$ and $x \in X$, if $x \in U$ then $\neg\neg\{x\} \subset U$. As is usual in SDG, R will be assumed to be a *field of fractions* (cf. [10, 8]), from which it follows that $D \subset \neg\neg\{0\}$.

We now give the following, for O any topological structure on **E** as above:

2.1. DEFINITION. Let S be a spray on M. An *O-local flow for* S is a morphism $\phi: U_\phi \to M^D$, $\langle[1],0_M\rangle \in U_\phi \in O(R\times M)$, such that

(i) $D\times M^D$ O U_ϕ, and



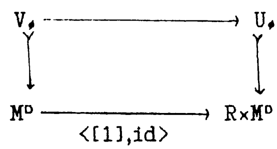
commutes;

(ii) $\phi(\lambda+\lambda', v) = \phi(\lambda,\phi(\lambda', v))$ ("flow equation");

(iii) $\phi(\lambda, \lambda'\odot v) = \lambda'\odot\phi(\lambda.\lambda', v)$ ("homogeneity"),

the last two for every $\lambda, \lambda' \in R$, $v \in M^D$, such that (say) the left hand-side of each equation is defined (so is the other and it holds).

Since O is a topological structure on the topos E, taking the pullback:



gives an object $V_\phi \in O(M^D)$ such that, for all $p \in M$, $0_p \in v$. Then, for $d_1,\ldots d_n \in D$ and $v_1,\ldots,v_n \in M^D$ such that $\pi_M(v_1) = \ldots = \pi_M(v_n) = p$, it follows that

$$(d_1\odot v_1) \oplus \ldots \oplus (d_n\odot v_n) \in V_\phi.$$

2.2. PROPOSITION. *("Local exponential map"). Let* (E,R) *be a model of SDG with* R *a field of fractions. Let* O *be a topological structure on* E, *such that every* O*-open contains the monad of any one of its points. Let us assume given an O-local flow* ϕ *of* S, *for a spray* S *on* M. *Then, there exists a "local exponential map" for* S, *i.e., a morphism* $\exp_\phi: V_\phi \to M$ *with* $0_M \in V_\phi \in O(M^D)$, *such that*

$$\exp_\phi(d \odot v) = v(d) \qquad (*).$$

The above is easily shown (cf. [5]). So is the next Proposition, whose proof we recall in detail since it will be by examining it that the "exponential map property" - to be shown a consequence of infinitesimal linearity - will be extracted.

2.3. PROPOSITION. *Under the same assumptions as in Proposition 2.2, for S a spray on M, the formula*

$$\nabla_S(v_1,v_2)(d_1,d_2) = \exp_\#(d_1 \odot v_1 \oplus d_2 \odot v_2)$$

defines a torsion free connection ∇_S *on* M *such that the geodesic spray associated uniquely to* ∇_S *by Theorem 1.7 is* S *itself.*

PROOF. ∇_S *is a connection.* - We verify (i) and (ii) in the definition of a connection (Definition 2.1). By (*) in Proposition 1.4, we get (i):

$$\nabla_S(v_1,v_2)(d,0) = \exp_\#((d \odot v_1)\oplus(0\odot v_2)) = \exp_\#(d\odot v_1) = v_1(d)$$

and

$$\nabla_S(v_1,v_2)(0,d) = \exp_\#((0\odot v_1)\oplus(d \odot v_2)) = \exp_\#(d \odot v_2) = v_2(d).$$

As for the homogeneity conditions on ∇_S, expressed in (ii), they just follow from the definitions:

$$(\lambda\bullet\nabla_S(v_1,v_2))(d_1,d_2) = \nabla_S(v_1,v_2)(\lambda\bullet d_1,d_2) =$$
$$\exp_\#((\lambda\bullet d_1)\odot v_1 \oplus d_2\odot v_2)) = \exp_\#(d_1\odot(\lambda\odot v_1)\oplus(d_2\odot v_2)) = \nabla_S(\lambda\odot v_1,v_2)(d_1,d_2)$$

and

$$(\lambda\odot\nabla_S(v_1,v_2))(d_1,d_2) = \nabla_S(v_1,v_2)(d_1,\lambda\bullet d_2) =$$
$$\exp_\#((d_1\odot v_1)\oplus(\lambda\bullet d_2)\odot v_2)) = \exp_\#((d_1\odot v_1)\oplus(d_2\odot(\lambda\odot v_2)) = \nabla_S(v_1,\lambda\odot v_2)(d_1,d_2).$$

(2) ∇_S *is torsion free.* -

$$\nabla_S(v_1,v_2)(d_2,d_1) = \exp_\#((d_2\odot v_1)\oplus(d_1\odot v_2)) =$$
$$\exp_\#((d_1\odot v_2)\oplus(d_2\odot v_1)) = \nabla_S(v_2,v_1)(d_1,d_2).$$

(3) $S_{\nabla_S} = S$. -

$$S_{\nabla_S}(v)(d_1)(d_2) = \nabla_S(v,v)(d_1,d_2) = \exp_\#((d_1\odot v)\oplus(d_2\odot v)) = \exp_\#((d_1+d_2)\odot v)$$
$$= \pi_M(\phi(1,(d_1+d_2)\odot v)) = \pi_M(\phi(d_1+d_2,v)) = \pi_M(\phi(d_2,\phi(d_1,v)) =$$
$$\pi_M(S\hat{}(d_2,S\hat{}(d_1,v)) = (\pi_M\bullet S\hat{})(S\hat{}(d_1,v)(d_2)) = S\hat{}(d_1,v)(d_2) = S(v)(d_1)(d_2).$$

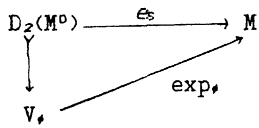
A closer look at the above proof indicates immediately that all we need of the map $\exp_\#$: $V_\# \to M$ is its restriction to the subobject $D_2(M^D)$ of $V_\#$ given by the image of the morphism

$$\gamma\colon D\times D\times(M^D\times_M M^D) \longrightarrow M^D$$

whose rule is given by:

$$(d_1,d_2,(v_1,v_2)) \mapsto (d_1\odot v_1)\oplus(d_2\odot v_2)\ ;$$

notice that $D_2(M^D) = \mathrm{Im}(\gamma) \subset V_\phi$, by Lemma 3.3. I.e., we could do with a map e_S: $D_2(M^D) \to M$ such that



commutes.

This observation was made already in [5]. Here we go one step further in the same direction by assuming that the desired exponential map be defined directly on the object $D \times D \times (M^D \times_M M^D)$ rather than on the quotient given by the image of γ, but restoring the properties of need in the proof of the above Proposition as in the following definition, directly suggested by the above considerations.

2.4 DEFINITION. An object M is said to have the *exponential map property* if for any spray S on M there exists a morphism

$$e_S\colon D \times D \times (M^D \times_M M^D) \to M$$

satisfying the conditions (for all $v \in M^D$, $(v_1,v_2) \in M^D \times_M M^D$, d, d_1, $d_2 \in D$, and $\lambda \in R$):

EXP(1) (i) $e_S(d,0,(v_1,v_2)) = v_1(d)$;
(ii) $e_S(0,d,(v_1,v_2)) = v_2(d)$;
(iii) $e_S(\lambda \cdot d_1, d_2, (v_1,v_2)) = e_S(d_1,d_2,(\lambda \odot v_1, v_2))$;
(iv) $e_S(d_1, \lambda \cdot d_2, (v_1,v_2)) = e_S(d_1,d_2,(v_1, \lambda \odot v_2))$

and

EXP(2) $e_S(d_1,d_2,(v,v)) = S(v)(d_1,d_2)$.

The following proposition is obvious, but it will be seen later to contain a superfluous hypothesis: indeed, it will be shown that the exponential map property is, in fact, a consequence of infinitesimal linearity.

2.5. PROPOSITION. *Let M be an infinitesimally linear object satisfying the exponential map property (as in Definition 2.4). Let S be a spray on M, and let* e_S: $D \times D \times (M^D \times_M M^D) \to M$ *be as in Definition 2.4. Then the formula*

$$\nabla_S(v_1,v_2)(d_1,d_2) = e_S(d_1,d_2,(v_1,v_2))$$

defines a torsion free connection ∇_S *on* M, *with* S *as its unique associated geodesic spray.*

Before stating and proving our last theorem linking our treatment in [5] to the general result obtained in [16] with the help also of results established in [11] and [12], we recall the following theorem proven in [5] giving an alternative characterization of sprays on infinitesimally linear objects. We refer to [5] for the proof, where infinitesimal linearity is used in an essential way.

2.6. THEOREM. *Let* M *be an infinitesimally linear object. A spray on* M *is given, equivalently, by the data consisting of a "spray map"* σ: $M^D \to M^{D_2}$, *where* $D_2 = [\![x \in R \mid x^3 = 0]\!]$, *satisfying:*

(i) $M^u \circ \sigma = \mathrm{id}$, *where* $u: D \hookrightarrow D_2$ *is the inclusion, i.e., for* $d \in D$, $v \in M^D$, $\sigma(v)(d) = v(d)$;

(ii) $\sigma(\lambda \odot v) = \lambda \odot \sigma(v)$, *for any* $v \in M^D$, $\lambda \in R$.

We now prove our main result in this section

2.7. PROPOSITION. *("Exponential map property"). Let* M *be infinitesimally linear. Then,* M *satisfies the exponential map property (in the sense of Definition 2.4).*

PROOF. The diagram of Weil algebras given by:

$$Q[\epsilon,\alpha] \xrightarrow{\ f\ } Q[\epsilon,\alpha,\chi] \underset{h}{\overset{g}{\rightrightarrows}} Q[\epsilon,\alpha,\chi,\mu]$$

where

$\epsilon^2 = \alpha^2 = \chi^3 = \mu^3 = 0$, $f(\epsilon) = \epsilon.\chi$, $f(\alpha) = \alpha.\chi$, $g(\epsilon) = \epsilon.\mu$, $g(\alpha) = \alpha.\mu$, $g(\chi) = \chi$, $h(\epsilon) = \epsilon.\chi$, $h(\alpha) = \alpha.\chi$, $h(\chi) = \mu$

is an equalizer, as it is easily checked. It follows that the diagram

$$(*) \qquad M^{D\times D} \xrightarrow{\ M^{f^\wedge}\ } M^{D\times D\times D_2} \underset{M^{h^\wedge}}{\overset{M^{g^\wedge}}{\rightrightarrows}} M^{D\times D\times D_2\times D}$$

where

$$f^\wedge(d_1,d_2,\delta) = (\delta.d_1,\delta.d_2), \quad g^\wedge(d_1,d_2,\delta_1,\delta_2) = (\delta_1.d_1,\delta_1.d_2,\delta_2),$$
$$h^\wedge(d_1,d_2,\delta_1,\delta_2) = (\delta_2.d_1,\delta_2.d_2,\delta_1)$$

for d_1, $d_2 \in D$ and δ_1, $\delta_2 \in D_2$, is an equalizer in E. In fact, since the class of infinitesimally linear objects is closed under exponentials by arbitrary exponents, also the more complicated diagram

$$(**) \qquad (M^L)^{D\times D} \xrightarrow{id^{f^*}} (M^L)^{D\times D\times D_2} \overset{id^{g^*}}{\underset{id^{h^*}}{\rightrightarrows}} (M^L)^{D\times D\times D_2\times D_2}$$

where $L = M^D\times_M M^D$, is an equalizer in E. Consider the morphism

$$e_r\colon D\times D\times D_2\times L \longrightarrow M$$

given by

$$e_r(d_1,d_2,\delta,(v_1,v_2)) = \sigma((d_1\odot v_1)\oplus(d_2\odot v_2))(\delta),$$

where σ is the spray map associated to a given spray S on M, and for which we wish to define an "exponential map" S, as in Definition 2.4. It is easily verified that $e_r \in (M^L)^{D\times D\times D_2}$ is equalized by the two morphisms of the equalizer diagram. This follows from:

$$\begin{aligned}
e_r(\delta_1.d_1,\ \delta_1.d_2,\delta_2,(v_1,v_2)) &= \sigma((\delta_1.d_1)\odot v_1\oplus(\delta_1.d_2)\odot v_2)(\delta_2) = \\
\sigma(\delta_1\odot(d_1\odot v_1)\oplus(d_2\odot v_2))(\delta_2) &= (\delta_1\odot\sigma((d_1\odot v_1)\oplus(d_2\odot v_2)))(\delta_2) = \\
\sigma((d_1\odot v_1)\odot(d_2\odot v_2))(\delta_1.\delta_2) &= \sigma((d_1\odot v_1)\oplus(d_2\odot v_2))(\delta_2.\delta_1) = \\
\sigma((\delta_2.d_1)\odot v_1\oplus(\delta_2.d_2)\odot v_2)(\delta_1) &= e_r(\delta_2.d_1,\delta_2.d_2,\delta_1,(v_1,v_2)).
\end{aligned}$$

Therefore, there exists a unique (global section) $e_s \in (M^L)^{D\times D}$ satisfying, when regarded as a morphism

$$e_s\colon D\times D\times L \longrightarrow M,$$

the condition: for d_1, $d_2 \in D$, $\delta \in D_2$, $(v_1,v_2) \in L$,

$$e_s(\delta.d_1,\delta.d_2,(v_1,v_2)) = e_r(d_1,d_2,\delta,(v_1,v_2)).$$

We now verify the conditions EXP(1) and EXP(2). We begin by EXP(2). Consider, for $v \in M^D$,

$$D\times D \overset{\phi}{\underset{\psi}{\rightrightarrows}} M$$

given by:

$$\phi(d_1,d_2) = e_s(d_1,d_2,(v,v)) \quad\text{and}\quad \psi(d_1,d_2) = S(v)(d_1)(d_2).$$

To show they are equal, on account of (*), it is enough to verify that they are equalized by $M^{f^\wedge}$, if ϕ, ψ are regarded as global elements of $M^{D\times D}$. Now

$$M^{f^\wedge}(\phi)(d_1,d_2,\delta) = e_s(\delta.d_1,\delta.d_2,(v,v)) = e_r(d_1,d_2,\delta,(v,v)) =$$
$$\sigma((d_1\odot v)\oplus(d_2\odot v))(\delta) = \sigma((d_1+d_2)\odot v)(\delta) = \sigma(v)((d_1+d_2).\delta) =$$
$$\sigma(v)(\delta.d_1+\delta.d_2) = S(v)(\delta.d_1)(\delta.d_2) = M^{f^\wedge}(\psi)(d_1,d_2,\delta).$$

This establishes EXP(2). We now turn to EXP(1). To show (i) ((ii) is similar), consider, for a given $(v_1,v_2) \in M^D\times_M M^D$, the two morphisms ϕ, ψ: $D\times D \to M$ given by

$$\phi(d_1,d_2) = e_s(d_1,0,(v_1,v_2)) \quad\text{and}\quad \psi(d_1,d_2) = v_1(d_1).$$

Now

$$M^{f^\wedge}(\phi)(d_1,d_2,\delta) = e_s(\delta.d_1,\delta.0,(v_1,v_2)) = e_r(d_1,0,\delta,(v,v)) =$$
$$\sigma((d_1\odot v_1)\oplus(0\odot v_2))(\delta) = \sigma(d_1\odot v_1)(\delta) = \sigma(v_1)(d_1.\delta) = \sigma(v_1)(\delta.d_1) =$$
$$v_1(\delta.d_1) = M^{f^\wedge}(\psi)(d_1,d_2,\delta),$$

where we have used the fact that, for $d \in D$ and $\delta \in D_2$, $\delta.d \in D$. This shows (i), and similarly, (ii) holds.

To show (iii) and (iv) we need to consider the equalizer (**) again, at least, the fact that the equalizer map is mono. Consider the following morphisms ϕ, ψ: $R \to (M^L)^{D\times D}$ defined as follows:

$$\phi(\lambda)(d_1,d_2)(v_1,v_2) = e_s(d_1,\lambda.d_2,(v_1,v_2))$$

while

$$\psi(\lambda)(d_1,d_2)(v_1,v_2) = e_s(d_1,d_2,(v_1,\lambda\odot v_2)).$$

Now

$$(M^L)^{f^\wedge}(\phi)(\lambda)(d_1,d_2,\delta)(v_1,v_2) = \phi(\lambda)(\delta.d_1,\delta.d_2)(v_1,v_2) =$$
$$e_s(\delta.d_1,\lambda.(\delta.d_2),(v_1,v_2)) = e_s(\delta.d_1,\delta.(\lambda.d_2),(v_1,v_2)) =$$
$$e_s(d_1,\lambda.d_2,\delta,(v_1,v_2)) = \sigma((d_1\odot v_1)\oplus(\lambda.d_2)\odot v_2)(\delta) =$$
$$\sigma((d_1\odot v_1)\oplus d_2\odot(\lambda\odot v_2))(\delta) = e_r(d_1,d_2,\delta,(v_1,\lambda\odot v_2)) =$$
$$e_s(\delta.d_1,\delta.d_2,(v_1,\lambda\odot v_2)) = \psi(\lambda)(\delta.d_1,\delta.d_2)(v_1,v_2) =$$
$$(M^L)^{f^\wedge}(\psi)(\lambda)(d_1,d_2,\delta)(v_1,v_2).$$

This shows that ϕ and ψ are equal, hence (iv) holds, and (iii) is similarly proved.

It remains to establish (v). In this case, let ϕ and ψ be global sections of $(M^L)^{D\times D}$ be defined by:

$$\phi(d_1,d_2)(v_1,v_2) = e_s(d_1,d_2,(v_2,v_1)),$$

whereas

$$\psi(d_1,d_2)(v_1,v_2) = e_S(d_1,d_2,(v_1,v_2)),$$

and the fact that they are equal follows, just as in the previous cases, by observing that for $\delta \in D_2$,

$$e_S(\delta.d_1,\delta.d_2,(v_1,v_2)) = \sigma((d_1\odot v_1)\oplus(d_2\odot v_2))(\delta) =$$
$$\sigma((d_2\odot v_1)\oplus(d_1\odot v_2))(\delta) = e_S(\delta.d_2,\delta.d_1,(v_1,v_2)).$$

This completes the proof.

2.8. **THEOREM** [16] *(Ambrose-Palais-Singer Theorem for infinitesimally linear spaces"). Let* (E,R) *be any model of SDG and let* M *be any infinitesimally linear object in* E. *There is a bijective correspondence between torsion free connections* ∇ *on* M *and sprays* S *on* M, *given by the rule* $\nabla \mapsto S_\nabla$, *with* $S_\nabla(v) = \nabla(v,v)$.

PROOF. By Theorem 1.7, there is such a correspondence, and it is injective on torsion free connections. By Proposition 2.7, M satisfies the exponential map property, therefore, by Proposition 2.5, the correspondence is also surjective, hence a bijection.

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