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REMARKS ON DUALITY FOR Σ-GROUPS
I. PRODUCTS AND COPRODUCTS

by Denis HIGGS[1]

RÉSUMÉ. En 1957, Wylie a introduit la notion de Σ-groupe (i.e., un groupe abélien avec certaines sommes infinies) afin d'obtenir une théorie de dualité satisfaisante pour différents groupes de chaînes infinies. Dans cet article, on rappelle les théorèmes de Wylie relatifs à cette dualité, en les généralisant de différentes façons: on considère un certain Σ "tight" défini sur les ensembles Hom(A,B) de Σ-groupes, ainsi que les Σ ponctuels, ou "lax", étudiés par Wylie; on prouve que le produit de Σ-groupes réflexifs est toujours réflexif, quelque soit le Σ-groupe B utilisé pour former Hom(-,B); et on donne des conditions Σ-théoriques sur B qui assurent que les résultats de Wylie pour Hom(-,T), où T est le groupe cercle, s'étendent à Hom(-,B).

1. INTRODUCTION

In 1957, Wylie [6] introduced the notion of an abelian group fortified by a non-finite additive structure in order to be able to define duality relations of the familiar kind between various chain and cochain groups on infinite cell complexes. Let us call such an enriched abelian group a Σ-group ([2,3]; in [6], the term "congregation" is used). It turns out that if A and B are Σ-groups then the set Hom(A,B) of all Σ-preserving functions from A to B can be given the structure of a Σ-group in two natural ways, leading to Σ-groups $[A,B]_t$ and $[A,B]_l$ (t = "tight", l = "lax").

Let T be the circle group with its usual Σ-structure (see below). Wylie proved that the dual with respect to T of the Φ-product of a family of Σ-groups is isomorphic to the Φ^*-product of their duals and deduced that Φ-products of re-

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flexive Σ-groups are reflexive provided that $\Phi^{**} = \Phi$ (Φ-products are defined in §2, and Φ^* in §4; Wylie considered only lax duality but his arguments go through essentially unchanged for tight duality).

In the present paper, the notions which we shall need are introduced in §2; in §3 we prove that all products (in the usual sense) of reflexive Σ-groups are reflexive irrespective of the Σ-group B with respect to which duals are taken, and we recall Wylie's Theorems in §4, showing that the circle group T may be replaced by any Σ-group B satisfying a certain condition (W) in the case of tight duality and a somewhat more restrictive condition (W') in the case of lax duality.

I thank Bob Paré, who independently had the idea of using axiomatic infinite sums to handle infinite-dimensional duality and whose remarks to me thereon encouraged me to return to the topic, and also Shaun Wylie, without whose pioneering work the present paper would largely evaporate.

2. PRELIMINARIES.

A *Σ-group* is an abelian group A together with a class S of series (that is, families) of elements of A and a function Σ: $S \to A$ satisfying axioms (Σ1) to (Σ4) below. First we mention some terminology and notation.

A series $(x_i : i \in I)$ which is in S is said to be *summable* and $\Sigma(x_i : i \in I)$ is its *sum*. As is customary, we write $\Sigma_{i \in I} x_i$, or just $\Sigma_i x_i$, for both the series $(x_i : i \in I)$ and, if it is in S, for its sum. A *subseries* of a series $(x_i : i \in I)$ is a series of the form $(x_i : i \in J)$ where $J \subset I$.

Here are the axioms:

(Σ1) Each finite series $\Sigma_{i \in I}^n x_i$ is summable and its sum = $x_1 + \ldots + x_n$;

(Σ2) If a series $\Sigma_{i \in I} x_i$ can be partitioned into finitely many summable subseries $\Sigma_{i \in I_j} x_i$, $j = 1, \ldots, n$, then the whole series is summable and its sum = $\Sigma_{j=1}^n (\Sigma_{i \in I_j} x_i)$;

(Σ3) If a summable series $\Sigma_{i \in I} x_i$ is partitioned into arbitrarily many finite subseries $\Sigma_{i \in I_j} x_i$ for $j \in J$, then the series

$\Sigma_{j\in J}(\Sigma_{i\in I_j}x_i)$ is summable and its sum $= \Sigma_{i\in I}x_i$;

(Σ4) The summability of a series is unaffected by the insertion/deletion of arbitrarily many zero terms.

This definition of Σ-group is equivalent to Wylie's but is more general than that used in [2] and [3]; I am indebted to Isidore Fleischer for suggesting the present version to me.

A series $\Sigma_i x_i$ on a Σ-group A will be said to be *subsummable* if it is a subseries of summable series on A; this is easily seen to be equivalent to saying that the series $\Sigma_i x_i + \Sigma_i(-x_i)$ is summable (necessarily to 0).

Many Σ-groups (but not all) are obtained as follows (Bourbaki [1], Chapter III, §5). Let A be a Hausdorff abelian group (that is, a topological abelian group whose topology is Hausdorff). Then a Σ-group structure on A is defined by putting $\Sigma_{i\in I}x_i = x$ iff, for each neighborhood U of 0 in A, there exists a finite subset I_0 of I such that

$$x - \Sigma_{i\in I_1}x_i \in U \text{ for every finite subset } I_1 \text{ of } I \text{ containing } I_0.$$

In particular, let A be a metric abelian group. Then, as is well-known, we may suppose that the topology on A is determined by a norm on A, that is, a function $\|-\|: A \to \mathbb{R}$ such that

(i) $\|x\| \geq 0$ for all $x \in A$, with equality iff $x = 0$,

and (ii) $\|x-y\| \leq \|x\| + \|y\|$ for all $x,y \in A$.

It is convenient to record here the easily verified fact that if $\Sigma_n x_n$ is a series on A for which $\Sigma_n\|x_n\| < \infty$ then $\Sigma_n x_n$ is subsummable in A (of course, it is not true in general that if $\Sigma_n\|x_n\| < \infty$ then $\Sigma_n x_n$ is actually summable in A, nor that if $\Sigma_n x_n$ is summable then $\Sigma_n\|x_n\| < \infty$).

A *morphism* from a Σ-group A to a Σ-group B is a function $f: A \to B$ such that

$$\Sigma_i f(x_i) = f(x) \text{ in B whenever } \Sigma_i x_i = x \text{ in A.}$$

The set Hom(A,B) of all morphisms from A to B is evidently an abelian group under pointwise addition and it can be given the structure of a Σ-group in two natural ways: Let f_j, $j \in J$, and f be in Hom(A,B). Then we say that $\Sigma_j f_j = f$ in the *tight* sense if

$$\Sigma_{j,i} f_j(x_i) = f(x) \text{ in B whenever } \Sigma_i x_i = x \text{ in A,}$$

and we say that $\Sigma_j f_j = f$ in *lax* sense if

$$\Sigma_j f_j(x) = f(x) \text{ in B for all } x \text{ in A.}$$

It is straightforward to verify that both of these definitions lead to Σ-group structures on Hom(A,B). For the tight Σ, we denote the resulting Σ-group by $[A,B]_t$ — or, more usually, by just [A, B] — and for the lax Σ, by $[A,B]_l$. We refer to [A,B] and $[A,B]_l$ as the tight and lax B-*duals* of A respectively. We discuss tight duals primarily; most of the results are of a sufficiently general nature that they carry over unchanged to the lax case, the proofs going through largely or completely unchanged too.

The internal hom functors $[-,-]$ and $[-,-]_l$ enjoy various predictable properties. For example, a morphism $f\colon A \to B$ induces morphisms

$$[f,C]\colon [B,C] \to [A,C] \text{ and } [C,f]\colon [C,A] \to [C,B],$$

and similarly for $[-,-]_l$ ((2.2) below provides another example). Of particular relevance here are the natural morphisms

$$\eta = \eta_{A,B}\colon A \to [[A,B],B]$$

defined by $\eta(x)(f) = f(x)$ for all $x \in A$ and $f \in [A,B]$; likewise for $[-,-]_l$. If $\eta\colon A \to [[A,B],B]$ is an isomorphism then we say that A is *tightly* B-*reflexive*, the lax version being defined similarly. The following proposition is well-known, and very elementary.

(2.1) *If* A *is tightly* B-*reflexive then* [A,B] *and all retracts of* A *are tightly* B-*reflexive, and likewise for lax reflexivity.*

PROOF. $[\eta_{A,B},B]$ is the inverse of $\eta_{[A,B],B}$. If $f\colon R \to A$ and $g\colon A \to R$ are morphisms such that $g \circ f = 1$ then

$$g \circ \eta^{-1}_{A,B} \circ [[f,B],B]$$

is the inverse of $\eta_{R,B}$.

Let A_λ, $\lambda \in \Lambda$, be a family of Σ-groups and let Φ be an ideal on Λ such that every finite subset of Λ is in Φ (that is, Φ is a dense ideal on Λ). Then the Φ-product $\Pi^\Phi A_\lambda$ (more precisely, $\Pi^\Phi_{\lambda\in\Lambda} A_\lambda$) of the A_λ's is the Σ-group constructed as follows: the elements of $\Pi^\Phi A_\lambda$ are those elements x of the cartesian product $\Pi_{\lambda\in\Lambda}A_\lambda$ with

$$\mathrm{supp}(x) = \{\lambda\in\Lambda \mid x(\lambda) \neq 0\} \in \Phi,$$

and $\Sigma_i x_i = x$ in $\Pi^\Phi A_\lambda$ iff

(i) $\Sigma_i x_i(\lambda) = x(\lambda)$ for each $\lambda\in\Lambda$,

and (ii) $\bigcup_i \mathrm{supp}(x_i) \in \Phi$.

This notion is due to Wylie ([6], §5). For each $\mu\in\Lambda$, define morphisms

$$\pi_\mu : \Pi^\Phi A_\lambda \longrightarrow A_\mu,\ \varepsilon_\mu : A_\mu \longrightarrow \Pi^\Phi A_\lambda, \text{ and } e_\mu : \Pi^\Phi A_\lambda \longrightarrow \Pi^\Phi A_\lambda$$

by

$$\pi_\mu(x) = x(\mu),\ \varepsilon_\mu(a)(\lambda) = \delta_{\lambda\mu}a, \text{ and } e_\mu = \varepsilon_\mu\circ\pi_\mu$$

respectively, and if $f_\lambda : A_\lambda \to B_\lambda$, $\lambda\in\Lambda$, is a family of morphisms, define the morphism

$$\Pi^\Phi f_\lambda : \Pi^\Phi A_\lambda \longrightarrow \Pi^\Phi B_\lambda \text{ by } (\Pi^\Phi f_\lambda)(x)(\mu) = f_\mu(x(\mu)).$$

If we take $\Phi = P(\Lambda) =$ the power set of Λ then $\Pi^\Phi A_\lambda$, along with the π_λ's, gives the product ΠA_λ of the A_λ's in the category of Σ-groups, whereas if we take $\Phi = P_{fin}(\Lambda) =$ the ideal of all finite subsets of Λ then $\Pi^\Phi A_\lambda$, along with the ε_λ's, gives the coproduct $\amalg A_\lambda$ of the A_λ's in this category (Brunker [2], 4.2.2, 4.2.3).

The following isomorphisms are obtained by taking $\Phi = P_{fin}(\Lambda)$ in the isomorphisms of [6], Theorem 4, and (4.2) below. We discuss them here because they are needed in §3 and because, in contrast to the case of arbitrary Φ, they hold for all Σ-groups B.

(2.2) *Let* A_λ, $\lambda\in\Lambda$, *and* B *be Σ-groups. Then*

$$[\amalg A_\lambda, B] \approx \Pi[A_\lambda, B],$$

and similarly for $[-,-]_l$.

PROOF. We discuss the tight case only; the proof for the lax case is similar. Define

$$f\colon [\amalg A_\lambda, B] \longrightarrow \Pi[A_\lambda, B]$$

by $f(z)(\lambda) = z\circ\varepsilon_\lambda$ for all $z\in[\amalg A_\lambda, B]$ and $\lambda\in\Lambda$,

and define

$$g\colon \Pi[A_\lambda, B] \longrightarrow [\amalg A_\lambda, B]$$

by $g(y)(x) = \Sigma_\lambda y(\lambda)(x(\lambda))$ for all $y\in\Pi[A_\lambda, B]$ and $x\in\amalg A_\lambda$.

We show that f and g are morphisms which are inverse to each other.

To verify that f is a morphism, note first that, for z in $[\amalg A_\lambda, B]$,

$$f(z)(\lambda) = z \circ \varepsilon_\lambda \in [A_\lambda, B]$$

for each λ so that $f(z) \in \Pi[A_\lambda, B]$. Let $\Sigma_k z_k = z$ in $[\amalg A_\lambda, B]$; we want $\Sigma_k f(z_k) = f(z)$ in $\Pi[A_\lambda, B]$, that is,

$$\Sigma_k f(z_k)(\lambda) = f(z)(\lambda)$$

in $[A_\lambda, B]$ for each λ. So let $\Sigma_l a_l = a$ in A_λ. Then $\Sigma_l \varepsilon_\lambda(a_l) = \varepsilon_\lambda(a)$ in $\amalg A_\lambda$ and thus, since $\Sigma_k z_k = z$ in $[\amalg A_\lambda, B]$,

$$\Sigma_{k,l} z_k(\varepsilon_\lambda(a_l)) = z(\varepsilon_\lambda(a))$$

in B. But this last equation says that

$$\Sigma_{k,l} f(z_k)(\lambda)(a_l) = f(z)(\lambda)(a)$$

as required.

In the definition of g, the equation

$$g(y)(x) = \Sigma_\lambda y(\lambda)(x(\lambda))$$

makes sense since the sum on the right-hand side is essentially finite (supp(x) is finite). To see that $g(y) \in [\amalg A_\lambda, B]$, let $\Sigma_i x_i = x$ in $\amalg A_\lambda$, so that

(i) $\Sigma_i x_i(\lambda) = x(\lambda)$ in A_λ for each λ,

and (ii) $\bigcup_i \mathrm{supp}(x_i)$ is finite.

Now for each λ, $y(\lambda) \in [A_\lambda, B]$ and hence

$$\Sigma_i y(\lambda)(x_i(\lambda)) = y(\lambda)(x(\lambda))$$

by (i). By (ii), there are only finitely many values of λ for which the series $\Sigma_i y(\lambda)(x_i(\lambda))$ is not identically zero, from which it follows that

$$\Sigma_i(\Sigma_\lambda y(\lambda)(x_i(\lambda))) = \Sigma_\lambda(\Sigma_i y(\lambda)(x_i(\lambda))).$$

In view of the equation we obtained a moment ago, the right-hand side here equals $\Sigma_\lambda y(\lambda)(x(\lambda))$. We thus have

$$\Sigma_i g(y)(x_i) = g(y)(x),$$

as desired. To verify that g is itself a morphism now amounts to showing that if $\Sigma_j y_j = y$ in $\Pi[A_\lambda, B]$ and $\Sigma_i x_i = x$ in $\amalg A_\lambda$ then

$$\Sigma_{j,i} g(y_j)(x_i) = g(y)(x)$$

in B; the argument is nearly the same as the one just given and we omit it.

Finally, to see that f and g are inverse to each other, first let $z \in [\coprod A_\lambda, B]$. Then for each $x \in \coprod A_\lambda$ we have

$$\begin{aligned} g(f(z))(x) &= \Sigma_\lambda f(z)(\lambda)(x(\lambda)) = \Sigma_\lambda z(\varepsilon_\lambda(x(\lambda))) \\ &= z(\Sigma_\lambda \varepsilon_\lambda(x(\lambda))) = z(x) \end{aligned}$$

and so $g \circ f = 1$. On the other hand, if $y \in \Pi[A_\lambda, B]$ then for each $\lambda \in \Lambda$ and $a \in A_\lambda$ we have

$$f(g(y))(\lambda)(a) = g(y)(\varepsilon_\lambda(a)) = \Sigma_\mu y(\mu)(\varepsilon_\lambda(a)(\mu)) = y(\lambda)(a),$$

and so $f \circ g = 1$.

3. REFLEXIVITY OF PRODUCTS.

Let A_λ, $\lambda \in \Lambda$, and B be Σ-groups. Since $\pi_\lambda \circ \varepsilon_\lambda = 1$ for each λ, it follows from (2.1) that if ΠA_λ (or any $\Pi^\Phi A_\lambda$) is tightly (laxly) B-reflexive then so is every A_λ. The main purpose of this section is to prove the converse result. (Thus we obtain an analogue for Σ-groups of the Theorem of Kaplan [4] that the product of topological abelian groups, each of which is reflexive in the sense of Pontrjagin duality, in itself reflexive.) As in the previous section, we discuss only the tight case in detail.

If A_λ, $\lambda \in \Lambda$, and B, C are Σ-groups, the morphisms

$$[[\pi_\lambda, B], C]: [[\Pi A_\lambda, B], C] \longrightarrow [[A_\lambda, B], C], \quad \lambda \in \Lambda,$$

determine a morphism

$$\vartheta: [[\Pi A_\lambda, B], C] \longrightarrow \Pi[[A_\lambda, B], C];$$

in terms of elements,

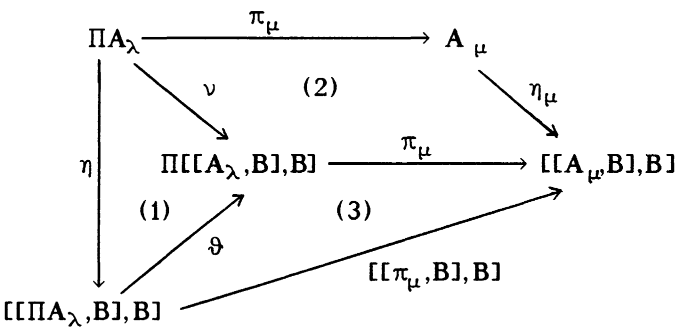
$$\vartheta(h)(\lambda)(f) = h(f \circ \pi_\lambda)$$

for all $h \in [[\Pi A_\lambda, B], C]$, $\lambda \in \Lambda$ and $f \in [A_\lambda, B]$.

(3.1) *The following diagram commutes:*

$$\begin{array}{ccc} & \Pi A_\lambda & \\ {\scriptstyle \eta} \swarrow & (1) & \searrow {\scriptstyle \Pi \eta_\lambda} \\ [[\Pi A_\lambda, B], B] & \xrightarrow{\quad \vartheta \quad} & \Pi[[A_\lambda, B], B]. \end{array}$$

PROOF. Let $\mu \in \Lambda$. Then in the diagram



where ν stands for $\Pi\eta_\lambda$, the outside square commutes by the naturality of $[-,-]$, the square labelled (2) commutes by the definition of ν, and the triangle labelled (3) commutes by the definition of ϑ. It follows that $\pi_\mu \circ \vartheta \circ \eta = \pi_\mu \circ \nu$ for all $\mu \in \Lambda$, and hence $\vartheta \circ \eta = \nu$ (as can also be verified by a direct calculation, of course).

(3.2) *For any Σ-group* B, *products of tightly* B*-reflexive Σ-groups are tightly* B*-reflexive, and likewise for lax reflexivity.*

PROOF. Let A_λ, $\lambda \in \Lambda$, be tightly B-reflexive. Referring to the commutative triangle (1) in the preceding result, we note that since each η_λ is an isomorphism, so also is $\nu = \Pi\eta_\lambda$. Hence from $\vartheta \circ \eta = \nu$ we obtain

$$\nu^{-1} \circ \vartheta \circ \eta = 1 \text{ and } \vartheta \circ \eta \circ \nu^{-1} = 1,$$

so that

$$\vartheta \circ \eta \circ \nu^{-1} \circ \vartheta = \vartheta.$$

Suppose we have proved, as we shortly will, that ϑ is a monomorphism in the category of Σ-groups. Then from the last of the above equations we will have $\eta \circ \nu^{-1} \circ \vartheta = 1$ which, coupled with $\nu^{-1} \circ \vartheta \circ \eta = 1$, implies that η is an isomorphism, hence that ΠA_λ is tightly B-reflexive.

That ϑ is indeed always a monomorphism is a consequence of (3.4) below, for which we need:

(3.3) *Let* A_λ, $\lambda \in \Lambda$, *be Σ-groups. Then* $\Sigma_\lambda e_\lambda = 1$ *in* $[\Pi A_\lambda, \Pi A_\lambda]$.

PROOF. We have to show that if $\Sigma_i x_i = x$ in ΠA_λ then $\Sigma_{\lambda,i} e_\lambda(x_i) = x$ in ΠA_λ, that is,

$$\Sigma_{\lambda,i} e_\lambda(x_i)(\mu) = x(\mu)$$

in A_μ for all μ. But $e_\lambda(x_i)(\mu) = \delta_{\lambda\mu} x_i(\mu)$. Hence

$$\Sigma_{\lambda,i} e_\lambda(x_i)(\mu) = \Sigma_i x_i(\mu) = x(\mu) .$$

(3.4) *Let* A_λ, $\lambda \in \Lambda$, *and* B,C *be* Σ*-groups. Then*

$$\vartheta: [[\Pi A_\lambda, B], C] \longrightarrow \Pi[[A_\lambda, B], C]$$

is injective.

PROOF. Let $\vartheta(h) = 0$, $h \in [[\Pi A_\lambda, B], C]$. For any $z \in [\Pi A_\lambda, B]$ we can write $z = \Sigma_\lambda z \circ e_\lambda$ by (3.3) and hence have

$$h(z) = h(\Sigma_\lambda z \circ e_\lambda) = \Sigma_\lambda h(z \circ e_\lambda) = \Sigma_\lambda \vartheta(h)(\lambda)(z \circ \varepsilon_\lambda)$$

by the elementwise definition of ϑ given above. Therefore we obtain $h(z) = 0$ for all $z \in [\Pi A_\lambda, B]$ and so $h = 0$. Thus ϑ is injective.

It is not generally true that coproducts of B-reflexive Σ-groups are B-reflexive in either the tight or the lax sense and we now give an example to show this. Again we prove things explicitly for tight duality only.

Let $\mathbb{Z}_p$ be the additive group of p-adic integers and let $\mathbb{Z}_p$ carry the Σ-structure associated with the usual metric topology on $\mathbb{Z}_p$.

(3.5) $\mathbb{Z}_p$ *is tightly and laxly* $\mathbb{Z}_p$*-reflexive.*

PROOF. The morphism

$$\alpha: \mathbb{Z}_p \to [\mathbb{Z}_p, \mathbb{Z}_p] \text{ defined by } \alpha(y)(x) = yx$$

is an isomorphism, with α^{-1} given by $\alpha^{-1}(f) = f(1)$ (this is already well-known to be the case when $[\mathbb{Z}_p, \mathbb{Z}_p]$ is interpreted in the purely algebraic sense). It follows that

$$\eta: \mathbb{Z}_p \to [[\mathbb{Z}_p, \mathbb{Z}_p], \mathbb{Z}_p] = \mathbb{Z}_p \xrightarrow{\alpha} [\mathbb{Z}_p, \mathbb{Z}_p] \xrightarrow{[\alpha^{-1}, \mathbb{Z}_p]} [[\mathbb{Z}_p, \mathbb{Z}_p], \mathbb{Z}_p]$$

is also an isomorphism.

(3.6) *The coproduct of countable many copies of $\mathbb{Z}_p$ is neither tightly nor laxly $\mathbb{Z}_p$-reflexive.*

PROOF. Let Π and $\amalg$ denote the product and coproduct respectively of countably many copies of the Σ-group indicated. By (2.2), we have

$$[\amalg \mathbb{Z}_p, \mathbb{Z}_p] \approx \Pi[\mathbb{Z}_p, \mathbb{Z}_p].$$

Since $[\mathbb{Z}_p, \mathbb{Z}_p] \approx \mathbb{Z}_p$, we obtain $[\amalg \mathbb{Z}_p, \mathbb{Z}_p] \approx \Pi \mathbb{Z}_p$, so that

$$[[\amalg \mathbb{Z}_p, \mathbb{Z}_p], \mathbb{Z}_p] \approx [\Pi \mathbb{Z}_p, \mathbb{Z}_p].$$

Let

$$\gamma\colon \amalg \mathbb{Z}_p \longrightarrow [\Pi \mathbb{Z}_p, \mathbb{Z}_p] = \amalg \mathbb{Z}_p \xrightarrow{\eta} [[\amalg \mathbb{Z}_p, \mathbb{Z}_p], \mathbb{Z}_p] \approx [\Pi \mathbb{Z}_p, \mathbb{Z}_p];$$

it is then easily checked that

$$\gamma(y)(x) = \Sigma_n y(n)\, x(n) \text{ for all } y \in \amalg \mathbb{Z}_p \text{ and } x \in \Pi \mathbb{Z}_p.$$

Therefore, if we define $z \in [\Pi \mathbb{Z}_p, \mathbb{Z}_p]$ by $z(x) = \Sigma_n p^n x(n)$, there is clearly no $y \in \amalg \mathbb{Z}_p$ with $\gamma(y) = z$. Since γ is thus not an isomorphism, neither is η.

Let A_λ, $\lambda \in \Lambda$, and B be Σ-groups. The above example shows that, even if each A_λ (and hence also ΠA_λ) is tightly B-reflexive then it is not necessarily the case that $[\Pi A_\lambda, B] \approx \amalg [A_\lambda, B]$. Specifically, the proof of (3.6) shows that the canonical morphism

$$g\colon \amalg[\mathbb{Z}_p, \mathbb{Z}_p] \longrightarrow [\Pi \mathbb{Z}_p, \mathbb{Z}_p]$$

(defined as in the proof of (2.2)) is not surjective. In fact, $[\Pi \mathbb{Z}_p, \mathbb{Z}_p]$ and $\amalg[\mathbb{Z}_p, \mathbb{Z}_p]$ are not isomorphic at all: the former is tightly $\mathbb{Z}_p$-reflexive by (3.2) and (2.1), whereas the latter, being isomorphic to $\amalg \mathbb{Z}_p$, is not.

4. WYLIE'S RESULTS ON T-DUALITY.

Wylie considered duality with respect to the circle group T, where T carries the Σ-structure associated with its standard metric topology. In order to isolate what appear to be the essential Σ-theoretic properties of T in this context, we introduce the following two conditions on an arbitrary Σ-group B:

(W) For all $b_n \in B$, $n \in \mathbb{N}$, with $b_n \neq 0$ for all n, there exist $r_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $\Sigma_n r_n b_n$ is not subsummable in B;

(W') For all $b_{mn} \in B$, $m,n \in \mathbb{N}$, with $b_{nn} \neq 0$ and $b_{mn} = 0$ for $m > n$, there exist $r_m \in \mathbb{N}$, $m \in \mathbb{N}$, such that $\Sigma_n(\Sigma_{m=1}^{n} r_m b_{mn})$ is not subsummable in B.

In the case of a Hausdorff abelian group B, these two conditions are related to the condition that B is a NSS group, a topological group being an NSS group if there exists a neighbourhood of the identity containing no non-trivial subgroup (see, for example, Morris [5], Chapter 8; NSS = "no small subgroups").

(4.1) (a) (W') *implies* (W) *for all* Σ*-groups.*
(b) *Every NSS Hausdorff abelian group satisfies* (W').
(c) *A metric abelian group satisfying* (W) *is an NSS group.*

PROOF. (a) Take $b_{nn} = b_n$, $b_{mn} = 0$ for $m \neq n$.

(b) Let B be an NSS Hausdorff abelian group and let U be a symmetric neighbourhood of 0 containing no non-trivial subgroup of B. Let V be a symmetric neighbourhood of 0 such that $V+V \subset U$. We first show that if $a,b \in B$ and $b \neq 0$ then there exists an $r \in \mathbb{N}$ such that $a+rb \notin V$. Suppose that this does not hold. Then for all $r \in \mathbb{N}$,

$$rb = (a+(r+1)b) - (a+b) \in V+V \subset U ,$$

and so the cyclic subgroup generated by b is contained in U, a contradiction. Now let $b_{mn} \in B$ be as in (W'). Then there exists an $r_1 \in \mathbb{N}$ such that $r_1 b_{11} \notin V$, there exists an $r_2 \in \mathbb{N}$ such that $r_1 b_{12} + r_2 b_{22} \notin V$, and so on. It follows that $\Sigma_n(\Sigma_{m=1}^{n} r_m b_{mn})$ is not subsummable in B.

(c) Let B be a metric abelian group and suppose that for every $\varepsilon > 0$ there exists a non-trivial subgroup of B contained in

$$U_\varepsilon = \{x \in B \mid \|x\| < \varepsilon\}.$$

Let $\Sigma_n \varepsilon_n$ be a convergent series of positive numbers and for each n let b_n be a non-zero element of B such that the cyclic subgroup generated by b_n is contained in U_{ε_n}. Then, for all choices of $r_n \in \mathbb{N}$, $\Sigma_n r_n b_n$ satisfies

$$\Sigma_n \|r_n b_n\| < \Sigma_n \varepsilon_n < \infty$$

and is thus subsummable in B. Hence B does not satisfy (W).

I have been unable to determine whether (W) always implies (W').

In order to formulate Wylie's main results concerning duality, the following notion is needed: if Φ is a dense ideal on a set Λ then Φ^* consists of all subsets Y of Λ such that $X \cap Y$ is finite for all $X \in \Phi$; Φ^* is evidently itself a dense ideal on Λ.

(4.2) *Let* A_λ, $\lambda \in \Lambda$, *and* B *be* Σ*-groups and let* Φ *be a dense ideal on* Λ.

(a) *If* B *satisfies* (W), *then* $[\Pi^\Phi A_\lambda, B] \approx \Pi^{\Phi^*}[A_\lambda, B]$.

(b) *If* B *satisfies* (W') *then* $[\Pi^\Phi A_\lambda, B]_l \approx \Pi^{\Phi^*}[A_\lambda, B]_l$.

PROOF. An instance of this result has already been proved in (2.2), namely that with $\Phi = P_{fin}(\Lambda)$ (for which no condition on B was required). The proof in the general case follows the same lines, in particular the formulæ defining the morphisms

$$f\colon [\Pi^\Phi A_\lambda, B] \longrightarrow \Pi^{\Phi^*}[A_\lambda, B] \text{ and } g\colon \Pi^{\Phi^*}[A_\lambda, B] \longrightarrow [\Pi^\Phi A_\lambda, B]$$

are exactly the same as before. It will be sufficient to mention how the relevant steps in the earlier proof are to be augmented; again we consider explicitly only the case of tight duality, with one exception. The specific points which require attention are as follows.

(i) If $z \in [\Pi^\Phi A_\lambda, B]$ then $f(z) \in \Pi^{\Phi^*}[A_\lambda, B]$: we now require $\mathrm{supp}(f(z)) \in \Phi^*$. Suppose that this is not so, let $X \in \Phi$ be such that $X \cap \mathrm{supp}(f(z))$ is infinite, and let λ_n, $n \in \mathbb{N}$, be distinct elements of $X \cap \mathrm{supp}(f(z))$. Then for each n, $f(z)(\lambda_n) \neq 0$ and hence we may choose $a_n \in A_{\lambda_n}$ such that $b_n = f(z)(\lambda_n)(a_n) \neq 0$. Since B satisfies (W), there exist $r_n \in \mathbb{N}$ such that $\Sigma_n r_n b_n$ is not subsummable in B. However

$$\{\lambda_n \mid n \in \mathbb{N}\} \subset X \in \Phi$$

and so $\Sigma_n \varepsilon_{\lambda_n}(r_n a_n) \in \Pi^\Phi A_\lambda$, whence

$$\Sigma_n z(\varepsilon_{\lambda_n}(r_n a_n)) = \Sigma_n r_n b_n$$

is summable in B, a contradiction.

(ii) If $\Sigma_k z_k = z$ in $[\Pi^\Phi A_\lambda, B]$ then $\Sigma_k f(z_k) = f(z)$ in $\Pi^{\Phi^*}[A_\lambda, B]$: we require that

$$\bigcup_k \mathrm{supp}(f(z_k)) \in \Phi^*$$

(also that $\Sigma_k f(z_k)(\lambda) = f(z)(\lambda)$ for all λ — for this, the previous argument applies unchanged). Suppose that we have $\bigcup_k \mathrm{supp}(f(z_k)) \notin \Phi^*$ and let $X \in \Phi$ be such that $X \cap \bigcup_k \mathrm{supp}(f(z_k))$ is infinite. Since each $X_k = X \cap \mathrm{supp}(f(z_k))$ is finite, we can find distinct k_n and distinct λ_n, $n \in \mathbb{N}$, such that $\lambda_n \in X_{k_n}$ for each n. Then $f(z_{k_n})(\lambda_n) \neq 0$ and we may choose $a_n \in A_{\lambda_n}$ such that

$$b_n = f(z_{k_n})(\lambda_n)(a_n) \neq 0 .$$

By (W), there exist $r_n \in \mathbb{N}$ such that $\Sigma_n r_n b_n$ is not subsummable in B. However,

$$\{\lambda_n \mid n \in \mathbb{N}\} \subset X \in \Phi$$

and so $\Sigma_n \varepsilon_{\lambda_n}(r_n a_n)$ is summable in $\Pi^{\Phi} A_\lambda$, which, coupled with the fact that $\Sigma_k z_k$ is summable in $[\Pi^{\Phi} A_\lambda, B]$, means that $\Sigma_{k,n} z_k(\varepsilon_{\lambda_n}(r_n a_n))$ is summable in B. Since $\Sigma_n r_n b_n$ is a subseries of this last series (take $k = k_n$), we have a contradiction.

(iii) If $y \in \Pi^{\Phi^*}[A_\lambda, B]$ and $x \in \Pi^{\Phi} A_\lambda$, then the sum $\Sigma_\lambda y(\lambda)(x(\lambda))$ (used to define g) is essentially finite: $\mathrm{supp}(y) \in \Phi^*$ and $\mathrm{supp}(x) \in \Phi$ implies $\mathrm{supp}(y) \cap \mathrm{supp}(x)$ is finite.

(iv) If $y \in \Pi^{\Phi^*}[A_\lambda, B]$ and $\Sigma_i x_i = x$ in $\Pi^{\Phi} A_\lambda$ then there are only finitely many values of λ for which the series $\Sigma_i y(\lambda)(x_i(\lambda))$ is not identically zero:

$$\mathrm{supp}(y) \in \Phi^* \text{ and } \bigcup_i \mathrm{supp}(x_i) \in \Phi$$

implies $\mathrm{supp}(y) \cap \bigcup_i \mathrm{supp}(x_i)$ is finite.

(v) If $\Sigma_j y_j = y$ in $\Pi^{\Phi^*}[A_\lambda, B]$ and $\Sigma_i x_i = x$ in $\Pi^{\Phi} A_\lambda$ then

$$\Sigma_{j,i}\, g(y_j)(x_i) = g(y)(x)$$

in B: as in the proof of (2.2), this step is nearly the same as (iv) and we omit it again.

The details of the argument for the lax case (b) are similar, except for step (ii) above, where the fact that now $\Sigma_k z_k = z$ is only given to hold in the lax sense necessitates the use of the stronger condition (W') in place of (W), together with the more delicate argument employed by Wylie, in order to obtain $\bigcup_k \mathrm{supp}(f(z_k)) \in \Phi^*$. Suppose that $\bigcup_k \mathrm{supp}(f(z_k)) \notin \Phi^*$ and let X, X_k, k_n and λ_n be as in (ii), where we now insist in addition that $\lambda_m \notin \bigcup_{n<m} X_{k_n}$ for each m (this is possible since each X_k

is finite whereas $\bigcup_k X_k$ is infinite). Then

$$f(z_{k_n})(\lambda_n) \neq 0 \text{ and } f(z_{k_n})(\lambda_m) = 0 \text{ for } m > n.$$

Again choose $a_n \in A_{\lambda_n}$ so that $f(z_{k_n})(\lambda_n)(a_n) \neq 0$. Then the elements $b_{mn} = f(z_{k_n})(\lambda_m)(a_m)$ of B satisfy the hypothesis of (W') and so there exist $r_m \in \mathbb{N}$ such that $\Sigma_n(\Sigma_{m=1}^n r_m b_{mn})$ is not subsummable in B. However, evaluating $\Sigma_k z_k$ at the element $\Sigma_m \varepsilon_{\lambda_m}(r_m a_m)$ of $\Pi^\Phi A_\lambda$ shows that $\Sigma_k z_k(\Sigma_m \varepsilon_{\lambda_m} r_m a_m)$ is summable in B. Since this last series has

$$\Sigma_n z_{k_n}(\Sigma_m \varepsilon_{\lambda_m}(r_m a_m)) = \Sigma_n(\Sigma_{m=1}^n r_m b_{mn})$$

as a subseries, our sought-for contradiction is obtained.

(4.2) has the following corollaries.

(4.3) *Let* A_λ, $\lambda \in \Lambda$, *and* B *be* Σ*-groups and let* Φ *be a dense ideal on* Λ *such that* $\Phi^{**} = \Phi$.

(a) *Let* B *satisfy* (W). *If each* A_λ *is tightly* B*-reflexive then so is* $\Pi^\Phi A_\lambda$.

(b) *Let* B *satisfy* (W'). *If each* A_λ *is laxly* B*-reflexive then so is* $\Pi^\Phi A_\lambda$.

PROOF. It is sufficient to note that

$$\eta: \Pi^\Phi A_\lambda \longrightarrow [[\Pi^\Phi A_\lambda, B], B] =$$

$$\Pi^\Phi A_\lambda \xrightarrow{\Pi^\Phi \eta_\lambda} \Pi^\Phi[[A_\lambda, B], B] \approx [\Pi^{\Phi^*}[A_\lambda, B], B] \approx [[\Pi^\Phi A_\lambda, B], B]$$

where the first isomorphism here is the appropriate version of the isomorphism g of the preceding proof and the second is induced by the isomorphism f of that proof (and where $[-,-]$ is read as $[-,-]_l$ for (b)).

(4.4) *Let* A_λ, $\lambda \in \Lambda$, *and* B *be* Σ*-groups.*

(a) *Let* B *satisfy* (W). *Then* $[\Pi A_\lambda, B] \approx \amalg[A_\lambda, B]$ *and if each* A_λ *is tightly* B*-reflexive then so is* $\amalg A_\lambda$.

(b) *Let* B *satisfy* (W'). *Then* $[\Pi A_\lambda, B]_l \approx \amalg[A_\lambda, B]_l$ *and if each* A_λ *is laxly* B*-reflexive then so is* $\amalg A_\lambda$.

PROOF. The isomorphisms here follow from (4.2) and the fact that $P(\Lambda)^* = P_{fin}(\Lambda)$: the reflexivity statements follow from (4.3) and the fact that $P_{fin}(\Lambda)^{**} = P_{fin}(\Lambda)$.

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