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GLUEING ENRICHED MODULES AND COMPOSITION OF AUTOMATA[1]

by S. KASANGIAN and R. ROSEBRUGH

RÉSUMÉ. Le recollement de modules entre catégories enrichies appropriées conduit à des constructions pour les opérations régulières sur les langages qui interviennent comme comportements d'automates non-déterministes considérés comme catégories enrichies. On obtient une condition suffisante pour qu'une paire d'automates et un module définissent un automate par recollement.

1. INTRODUCTION.

The dynamics of non-deterministic automata have been fruitfully described as categories enriched in a monoidal category ([1, 2, 4], and see [3] for the (bicategory-enriched) extension to tree automata). Automata are then described as certain composable pairs of modules with behaviour given by composition of modules. The purpose of this paper is to show how the regular operations on behaviours in a monoid may all be obtained using constructions of automata which result from a categorical tool, namely glueing of modules.

We first review the definitions and basic results needed. Let X be a monoid with identity e. The category (qua preorder) of subsets of X, here denoted $X^{\sim}$, has a (biclosed, see [2]) monoidal structure defined by

$$Y \otimes Z = \{yz \mid y \in Y,\ z \in Z\}, \quad I = \{e\}.$$

An $X^{\sim}$-enriched category $\underline{Q}$ can be viewed as the dynamics of a non-deterministic automaton with input monoid X as follows. The objects of $\underline{Q}$ are states of the automaton. If q and q' are objects, $\underline{Q}(q,q')$ in $X^{\sim}$ is the set of elements of X which act on q with result q'. Composition and identity in $\underline{Q}$ make the action associative and unitary. Conversely, a non-deterministic dynamics determines an $X^{\sim}$-enriched category.

Recall that if $\underline{Q}'$ and $\underline{Q}$ are $X^{\sim}$-categories then a module

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from $\underline{Q}'$ to $\underline{Q}$, Φ: $\underline{Q}' \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}$ consists of an object $\Phi(q,q')$ in $X^\sim$ for each pair q, q' in $\underline{Q}\times\underline{Q}'$, and associative and unitary left and right actions

$$\underline{Q}(p,q)\otimes\Phi(q,q') \to \Phi(p,q') \text{ and } \Phi(q,q')\otimes\underline{Q}(q',r') \to \Phi(q,r')$$

for p in $\underline{Q}$ and r' in $\underline{Q}'$. Modules can be composed according to the well-known coend formula which here, since $X^\sim$ is partially ordered, reduces to a sum. Further, an $X^\sim$-functor $F:\underline{Q}\to\underline{Q}'$ induces a pair of adjoint modules

$$F_*: \underline{Q} \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}' \text{ and } F^*: \underline{Q}' \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}$$

(see e.g. [8] for notations and terminology).

A set of initial states I for a non-deterministic automaton with states the objects of $\underline{Q}$ and input monoid X determines an $X^\sim$-module I: $\underline{Q} \longrightarrow\!\!\!\!\!\!+\;\; \underline{1}$ ($\underline{1}$ is the terminal $X^\sim$-category) by defining $I(*,q)$ to be the set of all inputs which reach q from an initial state ($*$ is the unique object of $\underline{1}$). Similarly, terminal states T determine an $X^\sim$-module $T: \underline{1} \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}$. Motivated by these constructions, a ***(generalized) automaton*** is defined to be a triple $\underline{Q}$,I,T) where $\underline{Q}$ is an $X^\sim$-category, and $I: \underline{Q} \longrightarrow\!\!\!\!\!\!+\;\; \underline{1}$ and $T: \underline{1} \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}$ are $X^\sim$-modules. The *behaviour*, denoted $\beta(\underline{Q},I,T)$, of the (generalized) automaton ($\underline{Q}$,I,T) is the object of $X^\sim$ (= module from $\underline{1}$ to $\underline{1}$) given by the composite module IT.

In what follows, we will be interested in dynamics obtained by glueing certain modules. We first describe the glueing of modules as applied to the enriched categories of interest. If $\underline{Q}'$ and $\underline{Q}$ are $X^\sim$-categories and $\Phi: \underline{Q}' \longrightarrow\!\!\!\!\!\!+\;\; \underline{Q}$ is an $X^\sim$-module, we obtain a new $X^\sim$-category $\underline{\Phi}$ as follows: the objects of $\underline{\Phi}$ are the disjoint union of those of $\underline{Q}'$ and $\underline{Q}$;

$$\underline{\Phi}(q_1,q_2) = \begin{cases} \underline{Q}'(q_1,q_2) & \text{if } q_1,\ q_2 \text{ are in } \underline{Q}' \\ \underline{Q}(q_1,q_2) & \text{if } q_1,\ q_2 \text{ are in } \underline{Q} \\ \underline{\Phi}(q_1,q_2) & \text{if } q_1 \text{ in } \underline{Q},\ q_2 \text{ in } \underline{Q} \\ \varnothing & \text{otherwise } (q_1 \text{ in } \underline{Q}',\ q_2 \text{ in } \underline{Q}) \end{cases}$$

composition in $\underline{\Phi}$ is given by that in $\underline{Q}'$, $\underline{Q}$ and by the actions of Φ on $\underline{Q}$ and $\underline{Q}'$. Moreover, there are $X^\sim$-functors

$$J: \underline{Q}' \longrightarrow \underline{\Phi} \longleftarrow \underline{Q}: K$$

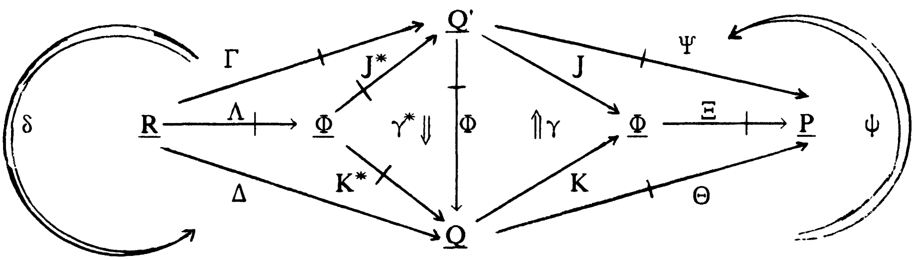
given by inclusion, and a transformation $\gamma: K\Phi \Rightarrow J$ which is universal, i.e., composing with γ determines an isomorphism

$$X^\sim\text{-CAT}(\underline{\Phi},\underline{P}) \longrightarrow X^\sim\text{-CAT}(\underline{Q}',\underline{P})(\Phi-,-).$$

The adjoint modules J^* and K^* and transformations γ^* are dually universal for modules to $\underline{\Phi}$. In more detail, and referring to the diagram below, for each pair of modules $\Psi: \underline{Q}' \to \underline{P}$, $\Theta: \underline{Q} \to \underline{P}$ and transformations $\psi: \Theta\Phi \Rightarrow \Psi$ there is a unique

$$\Xi: \underline{\Phi} \to \underline{P} \text{ such that } \Psi \approx \Xi J,\ \Theta \approx \Xi K \text{ and } \Xi\gamma = \psi.$$

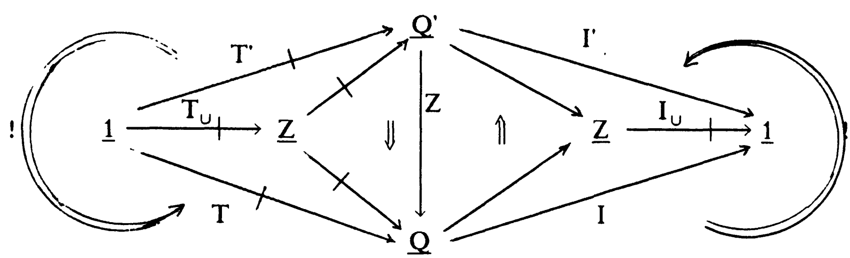
Moreover, the correspondence extends to transformations $\Xi \Rightarrow \Xi'$ say, and if Ψ and Θ are $X^{\sim}$-functors so is Ξ. On the other side, the data Γ, Δ and δ correspond to modules $\Lambda: \underline{R} \to \underline{\Phi}$ with $\gamma^*\Lambda = \delta$ (although here the correspondence does *not* restrict to $X^{\sim}$-functors).



2. GLUEING AUTOMATA.

Since our interest is in the case when $\underline{Q}$, $\underline{Q}'$ and $\underline{\Phi}$ are the dynamics of automata, we will usually take $\underline{P} = \underline{R} = \underline{1}$. Our first example of a regular operation is union of behaviours. The module used is the most trivial, namely the zero-module Z, defined by $Z(q,q') = \emptyset$ for all objects q in $\underline{Q}$, q' in $\underline{Q}'$. Note that then $\underline{Z}$ is the direct sum of $\underline{Q}$ and $\underline{Q}'$ in the bicategory of $X^{\sim}$-modules. Moreover Z is a *local* zero object, i.e., its pre- or post-composite with any module is the corresponding zero-module, so there are unique transformations to and from any composite with Z.

DEFINITION 1. Let $(\underline{Q},I,T)$ and $(\underline{Q}',I',T')$ be automata. Define a new automaton $(\underline{Q}\oplus\underline{Q}', I_U, T_U)$ by $\underline{Q}\oplus\underline{Q}' = \underline{Z}$, for $Z: \underline{Q} \to \underline{Q}'$, and I_U and T_U defined by the following diagram



PROPOSITION 1. *The behaviour of* $(\underline{Q}\oplus\underline{Q}', I_U, T_U)$ *is*

$$\beta(\underline{Q}\oplus\underline{Q}', I_U, T_U) = \beta(\underline{Q}, I, T) \cup \beta(\underline{Q}', I', T').$$

PROOF. The behaviour $\beta(\underline{Q}\oplus\underline{Q}', I_U, T_U)$ is by definition

$$\begin{aligned}\sum_{q \text{ in } \underline{Q}\oplus\underline{Q}'} I_U(*,q)T_U(q,*) &= \sum_{q \text{ in } \underline{Q}} I_U(*,q)T_U(q,*) \cup \sum_{q \text{ in } \underline{Q}'} I_U(*,q)T_U(q,*) \\ &= \sum_{q \text{ in } \underline{Q}} I(*,q)T(q,*) \cup \sum_{q' \text{ in } \underline{Q}'} I'(*,q)T'(q,*) \\ &= \beta(\underline{Q}, I, T) \cup \beta(\underline{Q}', I', T').\end{aligned}$$

The second equality uses the definition of I_U and T_U as morphisms out of, and to, a direct sum.

The reader will have observed that the $X^{\sim}$-category $\underline{Z}$ did not appear in the calculation of the above proposition. In fact, any $\Phi: \underline{Q} \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; \underline{Q}'$ for which the required transformations $I'\Phi \Rightarrow I$ and $\Phi T \Rightarrow T'$ exist (there can be at most one transformation) will induce I_U and T_U making (Φ, I_U, T_U) an automaton with behaviour $\beta(\underline{Q}, I, T) \cup \beta(\underline{Q}', I', T')$. However $\underline{Z}$ is easily seen to be the initial (in $X^{\sim}$-categories) dynamics with induced behaviour as above.

The next regular operation we consider is concatenation of languages. In this case we ought, intuitively, to join the output from the first automaton to the input for the second. This is precisely what we do. That is, let $(\underline{Q}, I, T)$ and $(\underline{Q}', I', T')$ be X-automata. Define $\Phi = TI'$.

PROPOSITION 2. *The automaton* $(\underline{\Phi}, IK^*, JT')$ *has behaviour*

$$\beta(\underline{Q}, I, T)\, \beta(\underline{Q}', I', T')$$

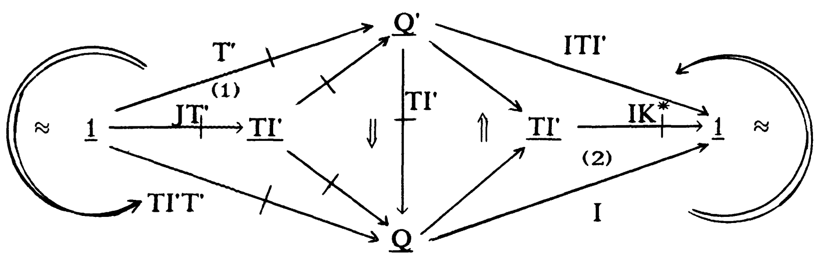
(with the notation above, K *and* J *as at the end of the last section for* $\Phi = TI'$*).*

PROOF. The behaviour of $(\underline{\Phi}, IK^*, JT')$ is by definition $IK^*JT'(*,*)$. The computation is easy if we recall that $K^*J = TI'$, so that

$$\begin{aligned}\beta(\underline{TI'}, IK^*, JT') &= ITI'T'(*,*) = \sum_{q \in \underline{1}} IT(*,q)I'T'(q,*) \\ &= IT(*,*)I'T'(*,*) = \beta(\underline{Q}, I, T)\, \beta(\underline{Q}', I', T').\end{aligned}$$

We remark first that the diagram defining the automaton $(\underline{\Phi}, IK^*, JT')$ above is the following one, where the indicated isomorphic transformations use $I^*J = TI'$, and (1) and (2) commute since

$$J^*J \approx \underline{Q}' \text{ and } K^*K \approx \underline{Q}.$$

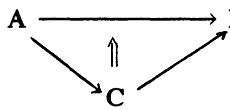


As the remarks after Proposition 1 show, we get only one behaviour by using only I, I', T and T' to define modules to and from $\underline{\Phi}$. Here we have used transformations

$$\Phi T' \Rightarrow T\Psi \text{ and } I\Phi \Rightarrow \Xi I' \text{ for } \Psi = I'T' \text{ and } \Xi = IT.$$

We shall see a similar case below.

We should also remark that automata have been viewed as a special case of *gamuts* (i.e., diagrams of the form



in a bicategory of modules [4]); that gamuts as above correspond to cofibrations from B to A [7,6] and that the "series composition" of automata we have just described is a special case of the composition of gamuts (which corresponds to the composition of cofibrations [6]).

The third regular operation is Kleene star. Let $(\underline{Q},I,T)$ be an automaton. The module Φ: $\underline{Q} \mathrel{-\!\!\!+\!\!\!\rightarrow} \underline{Q}$ is defined by $T\beta(\underline{Q},I,T)^*I+1$, where 1 is the constant module with value $\{e\} \in X^{\sim}$ and + is taken in modules from $\underline{Q}$ to $\underline{Q}$ (where it is defined by union).

PROPOSITION 3. *For* Φ *as defined above, the behaviour* $\beta(\underline{\Phi},IK^*,JT)$ *is* $\beta(\underline{Q},I,T)^+$.

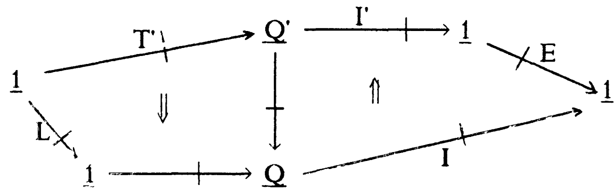
PROOF. Again, we observe that

$$\beta(\underline{\Phi},IK^*,JT) = IK^*JT(*,*) = I\underline{\Phi}T(*,*) = I(T\beta(\underline{Q},I,T)^*I+1)T(*,*)$$
$$= (IT\beta(\underline{Q},I,T)^*IT+IT)(*,*) = \beta(\underline{Q},I,T)^{++} \cup \beta(\underline{Q},I,T) = \beta(\underline{Q},I,T)^+.$$

Finally, we can obtain $\beta(\underline{Q},I,T)^*$ as the union of 1 and $\beta(\underline{Q},I,T)^+$.

All examples above are of a common type. There are X-automata $(\underline{Q},I,T)$ and $(\underline{Q}',I',T')$ together with $\Phi:\underline{Q}' \longrightarrow\!\!\!\!\!\!+\ \ \underline{Q}$. We obtain an X-automaton with dynamics $\underline{\Phi}$ when there are languages L and E, and transformations $\Phi T' \Rightarrow TL$ and $I\Phi \Rightarrow EI'$ as in

(*)



As the results above show, such L and E are by no means unique. We will first seek sufficient conditions on the modules I, I', Γ, T' and Φ to guarantee that at least one pair L, E does exist, and then describe the behaviours which result.

We first consider a sufficient condition for the existence of E. Since $X^{\sim}$ is partially ordered, it is enough that for all q' in $\underline{Q}$ we have $I\circ\Phi(*,q') \subset E\circ I'(*,q')$. Using the definition of composition for $X^{\sim}$-modules, we thus require that

$$\sum_{q \text{ in } \underline{Q}} I(*,q)\Phi(q,q') \subset EI'(*,q')$$

the latter juxtaposition being tensor in $X^{\sim}$ of E and $I'(*,q')$, i.e., their concatenation as languages. Hence we require that for all q in $\underline{Q}$, q' in $\underline{Q}'$

$$I(*,q)\Phi(q,q') \subset EI'(*,q').$$

Denoting, for any B, A in $X^{\sim}$,

$$[B,A] = \{z \mid \exists y \in B,\ yz \in A\}$$

(this is one of the internal homs mentioned above), our criterion for existence of E is

(*E) for all q in $\underline{Q}$, q' in $\underline{Q}'$, $I'(*,q') \subset [X, I(*,q)\Phi(q,q')]$.

Exactly analogous considerations show that there is an L provided that

(*L) for all q in $\underline{Q}$, q' in $\underline{Q}'$, $T(q,*) \subset \{\Phi(q,q')T'(q',*), X\}$

(where, for B, A in $X^{\sim}$,

$$\{B,A\} = \{y \mid \exists z \in A,\ yz \in B\}$$

which is the other internal hom). This is the first part of the following.

THEOREM 4. *If* $(\underline{Q},I,T)$ *and* $(\underline{Q}',I',T')$ *are X-automata and* Φ:

$\underline{Q}' \longrightarrow \underline{Q}$, *then there is an* X-*automaton* $(\underline{\Phi}, I_g, T_g)$ *satisfying* $I_g K = I$ *and* $T' = J^* T_g$ *provided that conditions* (*E) *and* (*L) *are satisfied. Moreover, if* E *and* L *satisfy* (*E) *and* (*L), *and* $TL = K^* T_g$ *and* $EI' = I_g J$, *the behaviour of* $(\underline{\Phi}, I_g, T_g)$ *is*

$$\beta(\underline{\Phi}, I_g, T_g) = E\beta(\underline{Q}', I', T) \cup \beta(\underline{Q}, I, T)L.$$

PROOF. By definition

$$\beta(\underline{\Phi}, I_g, T_g) = \sum_{p \in \underline{\Phi}} I_g(*, p) T_g(p, *).$$

Since $\underline{\Phi}$ has objects the disjoint union of those of $\underline{Q}$ and $\underline{Q}'$, we have

$$\beta(\underline{\Phi}, I_g, T_g) = \sum_{q \text{ in } \underline{Q}} I_g(*, q) T_g(q, *) \cup \sum_{q \text{ in } \underline{Q}'} I_g(*, q) T_g(q, *)$$

Now the definitions of I_g and T_g, along with K and J show that for q in $\underline{Q}$, $I_g(*, q) = I(*, q)$ and for q' in $\underline{Q}'$, $T_g(q', *) = T(q', *)$. Similarly for q in $\underline{Q}$,

$$T_g(q, *) = T \circ L(q, *) = T(q, *)L$$

and for q' in $\underline{Q}'$,

$$I_g(*, q') = E \circ I'(*, q') = EI'(*. q').$$

Thus

$$\begin{aligned} \beta(\underline{\Phi}, I_g, T_g) &= \sum_{q \text{ in } \underline{Q}} I(*, q) T(q, *) L \cup \sum_{q' \text{ in } \underline{Q}'} EI'(*, q') T'(q', *) \\ &= \Big(\sum_{q \text{ in } \underline{Q}} I(*, q) T(q, *)\Big) L \cup E\Big(\sum_{q' \text{ in } \underline{Q}'} I'(*, q') T'(q', *)\Big), \\ &= \beta(\underline{Q}, I, T) L \cup E\beta(\underline{Q}', I', T'). \end{aligned}$$

REMARKS. (1) Observe that E (and L) as above provide an extension (and lifting) which need not be Kan, since it lacks the universal property.

(2) Recall that tree automata can be viewed as categories enriched in a bicategory B(T) (see [3, 5]) obtained from a Lawvere theory T (a category whose objects are finite sets $[n] = \{1, 2, \dots, n\}$, $n = 0, 1, 2, \dots$ and which has the category of finite sets as subcategory, such that $[m]$ is the m-fold coproduct of $[1]$) by taking the same objects and as 1-cells subsets of the homsets; 2-cells are just inclusions. B(T)-categories satisfying suitable conditions are then (possibly non-deterministic) T-algebras. If we call $[n]_0$ the trivial one object category over $[n]$, a *tree automaton* is a T-algebra X with an *initial* module i from X to $[0]_0$ and a *final* module t from $[1]_0$ to X (see again [3] or [5] for details). The composite module $i \cdot t$ is the *behaviour* of the automaton, i.e., the set of terms (or trees) which are reco-

gnizable. In this setting there is no longer a description of regular operations, but some of the previous constructions still make sense. In particular, an analogue of Proposition 1 provides the union of behaviours and even the general situation described in diagram (*) can be interpreted if we notice that E, now to and from $[0]_0$, is trivial, while L, to and from $[1]_0$, represents a set of unary operations of the algebra (see [5]).

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