CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 33, n° 2 (1992), p. 135-175

<http://www.numdam.org/item?id=CTGDC_1992__33_2_135_0>

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VOL. XXXIII-2 (1992)

ON THE CATEGORICAL FOUNDATIONS OF HOMOLOGICAL AND HOMOTOPICAL ALGEBRA

by Marco GRANDIS

RÉSUMÉ. On présente ici, de façon synthétique, une étude sur les bases catégoriques de l'algèbre homologique et homotopique, partiellement exposée dans des travaux préliminaires [G3-6]. On veut montrer que ces matières peuvent être fondées sur des notions catégoriques très simples: noyaux et conoyaux d'une part, noyaux et conoyaux homotopiques de l'autre, fournissant respectivement le cadre des catégories semiexactes et homologiques d'une part, semihomotopiques et homotopiques de l'autre.

La structure de base où cela acquiert du sens est donnée, dans le premier cas, par une catégorie avec un *idéal assigné* de morphismes nuls. Dans le deuxième, par une *h-categorie*, c'est-à-dire une catégorie avec *homotopies*, pourvue d'une composition horizontale *réduite* entre morphismes et ces dernières.

L'algèbre homotopique apparaît ainsi -dans ses fondations- comme un enrichissement bidimensionnel de l'algèbre homologique. Toutefois, les développements des deux théories sont tout à fait différents, produisant une distinction formelle entre les deux: une distinction qui n'est pas toujour évidente dans les cas concrets, topologiques ou algébriques, où les deux aspects se trouvent mélangés.

0. Introduction

0.1. Abstract. This is a synthesis of a study on the categorical foundations of homological and homotopical algebra, partially exposed in some preliminary preprints. Our viewpoint is that these topics can be based on very simple categorical notions: kernels and cokernels on one hand, homotopy kernels and homotopy cokernels on the other, respectively yielding the notions of semiexact and homological category on one hand, semihomotopical and homotopical category on the other.

The basic structure where this makes sense is given, in the first case, by a category with an *assigned ideal* of null morphisms: kernels and cokernels are defined by the obvious universal property with respect to this ideal. In the

second case it is given by an *h-category*, i.e. a category with *homotopies* between maps and a *reduced* horizontal composition of homotopies with maps: homotopy kernels and homotopy cokernels have to make a given map *homotopically null*, in a universal way which can be expressed by this reduced composition.

Homotopical algebra appears thus -in its bases- as a sort of two-dimensional enrichment of homological algebra. However, the developments of the two theories are entirely different. This fact has an interest, since it provides a formal distinction between homological and homotopical concepts, without reducing the former to special instances of the latter. On the other hand, this failing parallelism seems to suggest that the real bases are not reached and a unified theory at some deeper level could exist.

Semiexact and homological categories have already been shortly presented in [G3], together with some of their applications in algebraic topology. A more detailed and complete exposition of their theory can be found in a series of two preprints [G4-5], here referred to as Part I, II. A preliminary study of semi-homotopical and homotopical categories is in [G6], here cited as Part III ([1]).

0.2. Categorical settings for homological algebra. Homological algebra can be described as the study of exact sequences and of their preservation properties by functors. It was established in *categories of modules* [CE] and immediately extended to *abelian categories* ([Bu]; [CE] appendix; [Gr]), both with formal advantages (e.g., duality) and with concrete ones (the categories of sheaves of modules are reached, as well as Serre's quotients "modulo C" [Ga]).

Yet, this extension is far from covering all the situations in which exact sequences are considered: the main exception is probably given by homotopy sequences, which are not even confined to the category of groups but degenerate in low degree into pointed sets and actions of groups: therefore, their "exactness" is usually described and studied "step by step", even in complicate situations as homotopy spectral sequences ([BK], IX.4; [Ba], III.2).

Many homological procedures can be freed from additivity and extended to *p-exact* categories (exact in the sense of Puppe [P2; Mt]): see [GV1-2; G1-4]. This extension permits a notion of *distributive homological algebra*, which cannot be formulated in the abelian frame and yields useful universal models for distributive theories, namely for spectral sequences [Ze; G2]. However, there is no substantial progress on the aspect of "usual" categories actually reached.

The frame we propose here includes, besides all p-exact categories, various categories of interest in algebraic topology; from a formal viewpoint, it suffices

([1]) The references I.m, or I.m.n or I.m.n.p apply to [G4], and precisely to its chapter m or its section m.n or to the item (p) of the latter; analogously for Part II [G5] and III [G6].

to define and study the basic homological notions. Our setting has a similarity with the one proposed by Lavendhomme [La; LV], mostly developed for "categories of pairs". Also Ehresmann [Eh] considered kernels with respect to an ideal, in connection with the cohomology of categories. Actually, his basic setting is much more general, being given by: an ideal J of the category **H**, a functor p: **H** → **C**, a subcategory **H'** of **H**, a subcategory **C'** of **C** consisting of monics ([Eh], p. 546); his definition of short exact sequence does not require the first morphism to be a kernel ([Eh], p. 546-547).

0.3. Semiexact and homological categories. A *semiexact* category **A** (1.1), our basic notion for homological algebra, is a category *equipped* with a *closed* ideal of "null morphisms" and provided with kernels and cokernels with respect to this ideal. It is *pointed* (or *p-semiexact*) if it has a zero object *and* its null morphisms are the zero ones. The stronger notion of *homological* category (1.6) makes possible to define subquotients, as the homology of a complex or the terms of a spectral sequence. A (generalized) *exact* category is a semiexact category in which every morphism is exact (1.6); such a category is always homological, while it is exact in the sense of Puppe iff it is pointed (*p-exact*). Some examples are given here below; a longer list is in 1.9.

In a semiexact category, exact sequences and exact functors can be introduced (1.2-3). The normal subobjects of any object form a lattice; their direct and inverse images supply a *transfer* functor Nsb: **A** → **Ltc**, taking values in the homological category of lattices and Galois connections (1.4-5); they provide an essential tool for *elementary* diagram chasing (1.8). Connected sequences of functors, homology theories and satellites extend naturally to this frame (ch. 2) and satellites can be detected by means of effacements (thm. 2.4), as in the abelian case [Gr].

Subquotients in homological categories are introduced in ch. 3, together with their induced morphisms; they are the crucial tool for *non-elementary* diagrammatic lemmas, whenever new arrows are to be established: e.g. in the construction of the connecting morphism (3.4), of the homology sequence for complexes (ch. 4), of the spectral sequence associated to an exact couple (II.7).

If **A** is a homological category, the homology sequence associated to a short exact sequence of complexes over **A** is *of order two*: its exactness is studied in thm. 4.3. Such sequences are always exact iff **A** is *modular* (thm. 4.4), i.e. its transfer functor Nsb: **A** → **Ltc** takes values into the (p-exact) subcategory **Mlc** of modular lattices and modular connections (1.7). However, a very simple condition -the exactness of differentials- supplies a partial exactness property (4.3 b)), sufficient for proving that chain homology is a sequence of satellites for various homological categories (4.5-6); e.g., all the abelian ones, the exact category **Mlc** and *non* modular categories like **Ban** and **Hlb** (Banach and Hilbert spaces). A deeper study of this fact could be of interest.

The importance of distributivity for exact categories was already stressed in

[G2]. The importance of modularity for homological categories appears now, while it cannot be clearly distinguished in a context where it is automatically granted, like the abelian or even the exact one.

0.4. Applications in algebraic topology. Concrete motivations for this setting can be found in various topics of algebraic topology [G3].

Firstly, consider the usual category $\mathbf{Top_2}$ of *pairs* (X, A) of topological spaces (A is a subspace of X): a morphism f: (X, A) $\to$ (Y, B) is a map f: X $\to$ Y which takes A into B; defining such a map to be *null* whenever $f(X) \subset B$, makes $\mathbf{Top_2}$ into a homological category, in a useful way. The short exact sequences correspond to triples of spaces:

$$(1) \qquad (A, B) \rightarrowtail (X, B) \twoheadrightarrow (X, A) \qquad\qquad (X \supset A \supset B),$$

and the pair (X, A) is indeed "X modulo A": the cokernel of the embedding of A into X. Therefore, the first four axioms of Eilenberg-Steenrod [ES] for a homology theory $H = (H_n, \partial_n)$ amount to saying that H is an *exact sequence* of functors, from this homological category $\mathbf{Top_2}$ to a category of modules.

Also a "single space" homology theory for locally compact spaces [ES] can be treated in this way, over the homological category $\mathcal{T}_{LC}$ of locally compact Hausdorff spaces and *partial* proper maps, defined on open subspaces [G3, ch. 2]. Massey relative cohomology of groups forms an exact sequence of functors over the homological category $\mathbf{Grp_2}$ of pairs of groups [G3, ch. 3]. Relative homotopy can be seen as an exact sequence over the homological category of pairs of pointed spaces, with values in the homological category **Act** of *actions* of groups on pointed sets [G3, ch. 4]. A more complex application to the spectral sequence of a tower of fibrations is given in II.8.

0.5. Categorical settings for homotopical algebra. Various settings have been proposed, among which we briefly recall the following ones.

a) Kan [Ka] developed an "abstract homotopy theory" for cubical and simplicial complexes satisfying an extension property, now called Kan complexes. The cubical approach was extended by Kan himself ([Ka], Part II) to categories equipped with a cylinder functor; other extensions of these approaches are in Kamps [K1-2], Huber [Hb1-2], Kleisli [Ks].

b) Homotopy in groupoid-enriched categories is considered in Gabriel-Zisman ([GZ], ch. V); see also Marcum [Mc] and its references.

c) Quillen's setting [Qn] is based on "model categories", in which three sets of distinguished maps are assigned: weak equivalences, fibrations and cofibrations. This does not cover situations which are deficient in path spaces and fibrations, as finite CW-complexes. Non-symmetrical extensions, where weak equivalences and cofibrations (or fibrations) are assigned, were given by K.S. Brown [Br] ("categories of fibrant objects") and Baues [Ba] ("cofibration categories").

d) Heller's "h-c-categories" [H1] are equipped with a homotopy relation and cofibrations. Anderson [An] and Heller's second setting [H2] aim to abstract the features of "homotopy categories" like Ho **Top**.

The present setting is mostly related with the cubical and the 2-categorical ones, in a) and b). I thank K.H. Kamps for providing part of these references.

0.6. Semihomotopical and homotopical categories. Our basic tool is given by the *h-kernel* and *h-cokernel* of a map f: A $\rightarrow$ B. The latter, for instance, is given by the h-pushout (or *standard homotopy pushout*, in Mather's terminology for spaces [Mh]) of the "terminal" map A $\rightarrow$ $\top$ along f, and is determined up to isomorphism.

These notions can be introduced in an *h-category* (5.1), a sort of two-dimensional context abstracting the nearly 2-categorical properties of spaces, maps and homotopies, and coinciding with the notion of "generalized homotopy system" introduced by Kamps ([K1], def. 2.1). Formally, this can be described as a category enriched over graphs with identities (5.2). More concretely, it consists of a category endowed with cells between its maps (*homotopies*) and a *reduced* horizontal composition of cells and maps, sufficient to formulate the universal properties of h-kernels and h-cokernels.

A *right semihomotopical* category **A** (5.3) is an h-category with terminal object $\top$ and homotopy cokernels (with respect to the latter). This produces the *mapping cone* (or h-cokernel) functor C defined on the category $\mathbf{A}^2$ of the maps of **A**, together with the *suspension* endofunctor of **A**, $\Sigma A = C(A \rightarrow \top)$.

Every morphism f: A $\rightarrow$ B has a *cofibration sequence* (5.5), extending the well-known Puppe sequence for topological spaces [P1]:

$$(1)\quad A \xrightarrow{f} B \xrightarrow{x} Cf \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma x} \Sigma Cf \xrightarrow{\Sigma\delta} \Sigma^2 A \ \ldots$$

Dually, one has *left-semihomotopical* categories (5.7), provided with h-kernels (with respect to the initial object $\bot$), hence with a loop endofunctor $\Omega A = K(\bot \rightarrow A)$. A *semihomotopical* category satisfies both conditions, so that every map has a double fibration-cofibration sequence. In a *pointed* semihomotopical category, like $\mathbf{Top}^{\top}$ (pointed spaces), the suspension and loop endofunctors are canonically adjoint: $\Sigma \dashv \Omega$ (5.8).

The definition of *right homotopical* category is just sketched here (5.9): it requires an h4-category (provided with vertical composition and vertical involution for cells, forming a relaxed 2-categorical structure, up to second-order homotopy) and assumes a second-order universal property for h-cokernels. In these hypotheses, the structural maps of h-cokernels are cofibrations, the suspension functor is homotopy invariant, each object ΣA is an h-cogroup and the cofibration sequence (1) is homotopically equivalent to the tower of iterated h-cokernels of f. Analogously for *left homotopical* and *homotopical* categories

(5.9). Some examples are considered in 5.10.

In ch. 6 we introduce the symmetrical notion of *semihomogeneous* theory:

(2) H_n: **A** $\to$ **B**, $\quad h_n: H_n \to H_{n+1}.\Sigma, \quad k_n: H_n.\Omega \to H_{n+1}$,

defined over a *semihomotopical* category **A**, with values into a category **B** equipped with an ideal of *null* morphisms (6.3). Two supplementary conditions, Σ-stability ($h_n: H_n \cong H_{n+1}.\Sigma$) and Σ-exactness, produce an "absolute" homological theory (6.5). The dual axioms, Ω-stability ($H_n.\Omega \cong H_{n+1}$) and Ω-exactness, yield a notion of homotopical theory, containing the usual homotopy theory (6.6). A homogeneous theory is both homological and homotopical: e.g., chain homology.

0.7. Conventions. A universe $\mathcal{U}$ is fixed throughout, whose elements are called *small* sets; a $\mathcal{U}$*-category* has objects and arrows belonging to this universe. The concrete categories we consider are generally large $\mathcal{U}$-categories: e.g. the category **Set** of small sets, or **Grp** of small groups. In a category, an *ideal* is any set of maps stable under composition with any map of the category.

1. Semiexact and homological categories

This chapter contains the basic definitions and properties of semiexact and homological categories. Detailed proofs and a more complete study, including for instance the categories of fractions, can be found in Part I [G4].

1.1. Semiexact categories. A *semiexact* category, or ex1-category, **A** = (**A**, **N**) is a pair satisfying the following two axioms:

(ex0) **A** is a category and **N** is a *closed* ideal of **A** (see below),

(ex1) every morphism f: A $\to$ B of **A** has a kernel and cokernel, with respect to **N**.

The morphisms of **N** are called *null morphisms* of **A**; the objects whose identity is null are called *null objects* of **A**. The closeness of **N** means that every null morphism factors through a null object. Equivalently, one can assign a set of null objects, closed under retracts (I.3.1).

The kernel and cokernel of f: A $\to$ B, written:

(1) ker f: Ker f $\rightarrowtail$ A, $\qquad$ cok f: B $\twoheadrightarrow$ Cok f,

are respectively a (*normal*) monic and a (*normal*) epi. Here, the arrows $\rightarrowtail$, $\twoheadrightarrow$ will always denote normal monics and normal epis, since simple monics and epis have a marginal interest; f is **N**-*monic* if ker f is null, **N**-*epi* if cok f is null.

The morphism f has a unique *normal factorisation* f = mgp (I.3.9) through its *normal coimage* and its *normal image*:

$$(2)\qquad \begin{array}{ccccccc} \mathrm{Ker}\, f & \rightarrowtail & A & \xrightarrow{f} & B & \twoheadrightarrow & \mathrm{Cok}\, f \\ & & {\scriptstyle p}\downarrow & & \uparrow{\scriptstyle m} & & \\ & & \mathrm{Ncm}\, f & \xrightarrow{g} & \mathrm{Nim}\, f & & \end{array} \qquad \begin{array}{l} p = \mathrm{ncm}\, f = \mathrm{cok}\,\mathrm{ker}\, f \\ m = \mathrm{nim}\, f = \mathrm{ker}\,\mathrm{cok}\, f \end{array}$$

which is natural; f is said to be an *exact* morphism if this g is an isomorphism.

The semiexact category **A** is *pointed* (or p-semiexact) if it has a zero object 0 *and* its null morphisms coincide with the zero ones (those factoring through 0): then the ideal **N** is determined by the categorical structure, while kernels and cokernels resume the usual meaning.

1.2. Exact sequences. The sequence (f, g) = (A → B → C) is said to be *of order two* if gf is null (iff nim f ≤ ker g, iff cok f ≥ ncm g). It is *exact* (in B) if nim f = ker g, or equivalently cok f = ncm g. It is *short exact* if f = ker g and g = cok f.

It is easy to prove that the sequence (f, g) is exact iff the following conditions on the diagram (1) hold:

$$(1)\qquad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \nearrow_{u} & & {}_{v}\searrow & \\ \bullet & & & & \bullet \end{array} \qquad\qquad (2)\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ 0 & \xrightarrow{g'} & C \end{array}$$

a) gf is null,

b) whenever gu and vf are null, also vu is so.

Note that the conditions a), b) are meaningful *in any category equipped with an ideal*, even when kernels and cokernels do not exist: such a situation will appear, in a homotopical context, for Puppe sequences (5.10 a)).

If **A** is pointed, the exactness of (1) plainly amounts to saying that the square (2) is *semicartesian* [G1], i.e commutative and verifying: gu = g'u' and vf = v'f' imply vu = v'u'.

1.3. Exact functors. A functor F: **A** → **B** between semiexact categories is said to be *exact* if it preserves kernels and cokernels. Then F preserves null morphisms (f is null iff ker f = 1), null objects, normal factorisations, exact morphisms, exact and short exact sequences.

Actually, the functor F is exact iff it is both *short-exact* (i.e., preserves the short exact sequences) and *long-exact* (i.e., preserves the exact sequences), each of these two conditions being weaker than exactness (I.5.1).

For instance, the singular chain functor C.: $\mathbf{Top}_2$ → **C.Ab** from the (homological) category of pairs of topological spaces to the category of chain complexes of abelian groups is just *short*-exact. The transfer functor (1.5) of a semiexact category is *long*-exact: it is exact iff **A** satisfies the axiom (ex2) in 1.6, which fails in **Grp**.

A *semiexact* (or *ex1*) *subcategory* of the ex1-category **A** is a subcategory **A'** verifying:

a) for every f in **A'** there is some kernel and some cokernel of f in **A** which belong to **A'**,

b) if m is a normal monic of **A** which belongs to **A'**, and mf is in **A'**, so is f; dually, if p is a normal epi of **A** which belongs to **A'**, and fp is in **A'**, so is f.

Then **A'**, equipped with the ideal $\mathbf{N}' = \mathbf{A}' \cap \mathbf{N}$, is an ex1-category and its inclusion in **A** is exact and *conservative* (reflects the isomorphisms).

1.4. Lattices and connections. The category **Ltc** of *lattices and connections* formalizes the structure of normal subobjects in semiexact categories, together with their direct and inverse images.

An object is a small lattice (always assumed to have 0 and 1). A morphism $f = (f_., f^{\cdot})\colon X \to Y$ is a Galois connection between the lattices X and Y, i.e. an adjunction $f_. \dashv f^{\cdot}$:

(1) $f_.\colon X \to Y$ and $f^{\cdot}\colon Y \to X$ are increasing mappings,

(2) $f^{\cdot}f_. \geq \mathrm{id}\,X, \quad f_.f^{\cdot} \leq \mathrm{id}\,Y$ [2],

so that $f_.$ preserves all the existing joins (including $0 = \vee\emptyset$), $f^{\cdot}$ preserves all the existing meets (including $1 = \wedge\emptyset$) and:

(3) $f_.f^{\cdot}f_. = f_.,\qquad f^{\cdot}f_.f^{\cdot} = f^{\cdot}$,

(4) $f^{\cdot}(y) = \max\{x \in X \mid f_.(x) \leq y\}, \qquad f_.(x) = \min\{y \in Y \mid f^{\cdot}(y) \geq x\}$.

The composition is obvious. Isomorphisms can be identified to ordinary lattice-isomorphisms. The category **Ltc** is *selfdual*, under the contravariant endofunctor turning each lattice into the opposite one and reversing a connection.

Ltc has a zero object: the one-point lattice $0 = \{*\}$, and the zero-morphism $f\colon X \to Y$ is described by: $f_.(x) = 0$, $f^{\cdot}(y) = 1$. Kernels and cokernels exist, and the normal factorisation of f is:

(5)
$$\begin{tikzcd} \downarrow f^{\cdot}0 \arrow[r, tail, "m"] & A \arrow[r, "f"] \arrow[d, two heads, "q"] & B \arrow[r, two heads, "p"] & \uparrow f_.1 \\ & \uparrow f^{\cdot}0 \arrow[r, "g"] & \downarrow f_.1 \arrow[u, tail, "n"] & \end{tikzcd}$$

---

[2] These compositions, inequalities and identities belong to the category of ordered sets and increasing mappings.

(6) $m.(x) = x$, $m^{\cdot}(x) = x \wedge f^{\cdot}0$, $p.(y) = y \vee f.1$, $p^{\cdot}(y) = y$,
(7) $q.(x) = x \vee f^{\cdot}0$, $q^{\cdot}(x) = x$, $n.(y) = y$, $n^{\cdot}(y) = y \wedge f.1$,
(8) $g.(x) = f.(x)$, $g^{\cdot}(y) = f^{\cdot}(y)$.

It is easy to show that the morphism f is exact iff:

a) $f^{\cdot}f.(x) = x \vee f^{\cdot}0 \quad (x \in X)$ a*) $f.f^{\cdot}(y) = y \wedge f.1 \quad (y \in Y)$,

while every $a \in X$ determines a generic short exact sequence in **Ltc**:

(9) $\downarrow a \xrightarrow{m} X \xrightarrow{p} \uparrow a \qquad (m.(x) = x,\ m^{\cdot}(x) = x \wedge a;\ p.(x) = x \vee a,\ p^{\cdot}(x) = x)$.

We are also interested in the (p-exact) subcategory **Mlc** of *modular lattices and modular connections* (I.1.3), where a morphism is any exact connection between modular lattices; the latter are characterized by the conditions a), a*) above; but the modularity of the lattices makes these conditions equivalent to the following ones, which are plainly stable under composition:

(10) $f^{\cdot}(f.x \vee y) = x \vee f^{\cdot}y, \qquad f.(f^{\cdot}y \wedge x) = y \wedge f.x \qquad (x \in X, y \in Y)$.

1.5. The transfer functor. In the semiexact category **A**, each object A has a lattice Nsb A of normal subobjects and a lattice Nqt A of normal quotients, anti-isomorphic via kernels and cokernels (I.3.6, I.3.12). Each morphism f: A → B has *direct and inverse images* for normal subobjects:

(1) f_*: Nsb A → Nsb B, $f_*(x) = \text{nim}\, fx = \ker \text{cok}\,(fx)$,

(2) f^*: Nsb B → Nsb A, $f^*(y) = \ker((\text{cok}\, y) \circ f)$ = pullback of y along f,

which form a Galois connection. In particular, if m: M ↣ A and p: A ↠ P (by I.3.12.3-4):

(3) $m_*(m^*(x)) = x \wedge m, \qquad p^*(p_*(x)) = x \vee \ker p \qquad (x \in \text{Nsb}\, A)$.

We say that **A** is a *semiexact* $\mathfrak{U}$*-category* if it is ex1, it is a $\mathfrak{U}$-category (i.e., all its objects and morphisms belong to $\mathfrak{U}$ (0.7)) and moreover all its lattices Nsb A of normal subobjects belong to $\mathfrak{U}$. As a consequence, also all the anti-isomorphic lattices of normal quotients are small.

Since we shall mainly consider such categories, *semiexact* (or *ex1*) category will mean, from now, semiexact $\mathfrak{U}$-category while *unrestricted* semiexact category will refer to the original defintition 1.1. Because of the previous considerations, a semiexact category has a *transfer* functor into the category of small lattices and connections, which generally is just *long-exact* (I.5.7):

(4) $\text{Nsb}_{\mathbf{A}}: \mathbf{A} \to \mathbf{Ltc}, \qquad A \mapsto \text{Nsb}\, A, \qquad f \mapsto (f_*, f^*)$.

1.6. Homological and exact categories. A *homological* category, or ex3-category, is an ex1-category **A** verifying the following axioms:

(ex2) normal monics and normal epis are stable under composition, (ex3) *subquotient axiom*, or *homology axiom*: given a normal monic m: $M \rightarrowtail A$ and a normal epi q: $A \twoheadrightarrow Q$, with $m \geq \ker q$ ($\operatorname{cok} m \leq q$), the morphism qm is exact.

Other equivalent forms are given in I.6.1-3:

(ex2a) the composition of two normal monics or two normal epis is an exact morphism,

(ex2b) for every normal monic m and every normal epi p: $m^*m_* = 1$, $p_*p^* = 1$ (or equivalently, because of 1.5.3: (m_*, m^*) and (p_*, p^*) are exact connections),

(ex2c) the transfer functor Nsb: $\mathbf{A} \to \mathbf{Ltc}$ is exact,

(ex3a) *pullback axiom*: the pullback of a pair $\cdot \twoheadrightarrow \cdot \leftarrowtail \cdot$ is of type $\cdot \leftarrowtail \cdot \twoheadrightarrow \cdot$

(ex3a*) *pushout axiom*: the pushout of a pair $\cdot \leftarrowtail \cdot \twoheadrightarrow \cdot$ is of type $\cdot \twoheadrightarrow \cdot \leftarrowtail \cdot$.

A semiexact category **A** is (generalized) *exact* (or ex4) if all its morphisms are exact; it is *strictly exact* (or ex5), if moreover two parallel null morphisms always coincide (I.7.4): this last notion is equivalent to saying that every connected component of **A** is pointed exact (*p-exact*), i.e. exact in the sense of Puppe-Mitchell [Mt]. Every exact category is homological, as it follows trivially from the form (ex2a). The category **Ltc** is pointed homological, not exact; its subcategory **Mlc** is p-exact.

A conservative ex1-functor F: $\mathbf{A} \to \mathbf{B}$ reflects the properties ex2, ex3 and ex4. In particular, every semiexact subcategory of an homological (or exact) category is so.

1.7. Modular semiexact categories. An object A of the semiexact category **A** will be said to be *modular* if its lattice Nsb A of normal subobjects is so. A morphism f: $A \to B$ is *left-modular*, or *right-modular*, or *modular* if its associated connection Nsb f: Nsb A $\to$ Nsb B is *left-exact*, *right-exact* or exact (1.4), i.e. satisfies (1), or (2), or both:

(1) $f^*f_*x = x \vee f^*0 \quad (x \in \mathrm{Nsb}\,A)$, $\qquad$ (2) $f_*f^*y = y \wedge f_*1 \quad (y \in \mathrm{Nsb}\,B)$.

More particularly, we say that f is *left-modular over* x (resp. *right-modular over* y) when the above property holds for this particular normal subobject x of A (resp. y of B).

A semiexact category **A** is said to be *modular* if all its objects and morphisms are so, or equivalently if its transfer functor Nsb: $\mathbf{A} \to \mathbf{Ltc}$ factors through the (p-exact) subcategory **Mlc** of modular lattices and modular connections (1.4). Then **A** is necessarily ex2 (use the condition (ex2b) in 1.6) and the transfer functor Nsb: $\mathbf{A} \to \mathbf{Mlc}$ is exact.

Every exact category is modular. Indeed, the exact functor $\mathrm{Nsb}_{\mathbf{A}}$: $\mathbf{A} \to \mathbf{Ltc}$ preserves the exact morphisms: therefore all the maps of **A** are modular, and it is not difficult to see that this implies that also the lattices Nsb A are so (I.7.6).

The pointed homological category K-**Tvs** of topological vector spaces is modular, non-exact: its normal subobjects are the linear subspaces. Instead, the homological categories **K-Hvs** (Hausdorff vector spaces), **Ban** (Banach spaces) and **Hlb** (Hilbert spaces) are not modular: in these cases the normal subobjects can be identified to the *closed* linear subspaces, which generally do not form a modular lattice (e.g. for the classic Hilbert space L^2). Other examples are considered in I.8.5.

1.8. Elementary diagram chasing. Diagram lemmas in abelian categories are a well known tool for homological algebra. Loosely speaking, and in order to extend them to the present setting, let us distinguish between *elementary* lemmas, whose thesis just involves the morphisms assigned in the hypothesis, and *non-elementary* ones, which state the existence of some new arrow. In this sense, the Five Lemma and the 3×3-Lemma are elementary, while the Snake Lemma -stating the existence and properties of the connecting morphism- is not.

As a general fact, an elementary lemma can be proved by a sort of abstract diagram chasing, using direct and inverse images of normal subobjects, as we show below for the Five Lemma (see also [Ma], XII.3); the previous modular properties of direct and inverse images (1.7) are often to be used: for instance, the use of the modular property $f^*f_*(x) = x \vee f^*0$ ($x \in \mathrm{Nsb}A$) substitutes the following standard argument of diagram chasing in *concrete* categories: knowing that $f(a') = f(a'')$, with $a', a'' \in X$, consider $a = a' - a'' \in \mathrm{Ker}\, f$. Therefore, these lemmas can be extended to *modular semiexact* categories, and even to the *semiexact* ones if suitable hypotheses of exactness or modularity on specific morphisms are assumed. Instead the construction of new arrows, as the connecting morphism, requires induction on subquotients in a *homological* category and is deferred to ch. 3.

Five Lemma. Given a commutative diagram with exact rows, in the modular semiexact category **A**:

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "a"] & B' \arrow[r, "g"] \arrow[d, "b"] & C' \arrow[r, "h"] \arrow[d, "c"] & D' \arrow[r, "k"] \arrow[d, "d"] & E' \arrow[d, "e"] \\ A'' \arrow[r, "u"] & B'' \arrow[r, "v"] & C'' \arrow[r, "w"] & D'' \arrow[r, "x"] & E'' \end{tikzcd} \tag{1}$$

a) if a is N-epi, b and d are N-monic, then c is N-monic,

a*) if e is N-monic, b and d are N-epi, then c is N-epi,

b) if **A** is exact, a is epi, e is monic, b and d are isomorphisms, then c is iso.

It suffices to prove a). Since $c^*0 \leq c^*w^*0 = h^*d^*0 = h^*0 = g_*1$:

(2) $c^*0 = g_*g^*c^*0 = g_*b^*v^*0 = g_*b^*u_*1 = g_*b^*u_*a_*1 = g_*b^*b_*f_*1 = g_*f_*1 = 0.$

1.9. Examples. The categories **Grp**, $\mathbf{Rng}_*$ of groups and *general* rings (without unit assumption) are pointed semiexact, non homological: their normal monics are not stable under composition.

The following categories are p-homological, not exact; the proof of the axioms is trivial or easy (and is sketched in I.4, I.6.6-7):

- **Abm**: abelian monoids; **Ltc**: lattices and Galois connections (1.4)
- **K-Tvs, K-Hvs**: topological or Hausdorff vector spaces, on the topological or Hausdorff field K
- **Ban, Hlb**: Banach or Hilbert spaces and continuous linear mapping
- $\mathbf{Set}^{\mathsf{T}}$ (pointed sets), $\mathbf{Top}^{\mathsf{T}}$ (pointed spaces) and $\mathbf{Cph}^{\mathsf{T}}$ (pointed compact Hausdorff spaces)
- $\mathcal{S}$: sets and partial mappings (equivalent to $\mathbf{Set}^{\mathsf{T}}$); $\mathcal{C}$: spaces and partial maps
- $\mathcal{T}$: spaces and partial maps, defined on open subspaces
- $\mathcal{T}_{LC}$: locally compact Hausdorff spaces and partial proper maps, defined on open subspaces (equivalent to $\mathbf{Cph}^{\mathsf{T}}$ via one-point compactification).

The following categories are homological with respect to a given set of null objects (i.e., to the ideal of morphisms which factor through them); a reference is given when the proof is not elementary:

- **Mod**: modules (M, R) over arbitrary rings, w.r.t. the *null* modules (0, R)
- $\mathbf{Set}_2$ (pairs of sets), $\mathbf{Top}_2$ (pairs of spaces), w.r.t. the *null* pairs (X, X)
- general "categories of pairs", with respect to identities [G3, 4.9-11]
- **Cov**: *coverings* p: X $\rightarrow$ X', with respect to homeomorphisms [G3, 2.7]
- **Act**: actions of groups on pointed sets, w.r.t. actions on $\{*\}$ [G3, ch. 4]
- the category $\mathbf{A}^{\mathbf{I}}$ of functors $\mathbf{I} \rightarrow \mathbf{A}$, where **I** is a small category and **A** is a homological category, with respect to the functors turning all the objects of **I** into null objects of **A**
- the category **C.A** of chain complexes over the homological category **A**.

Further, the category Gr **Ab** of graded abelian groups, with morphisms of *any* degree, is (generalized) exact with respect to the objects whose components are zero, and *not* pointed.

Finally, the category $\mathbf{EX}_4$ of exact $\mathcal{U}$-categories and exact functors is homological (in the *unrestricted* sense, 1.5), with respect to the exact categories equivalent to **1**; the normal subobjects coincide with the thick subcategories, the normal quotients with the exact categories of fractions. The same holds for the categories of p-exact or abelian $\mathcal{U}$-categories (I.9.9).

2. Connected sequences and satellites in semiexact categories

The classical terminology for abelian categories can be easily extended to the semiexact ones. In this general setting, satellites can be identified by means of "effacements", as in the abelian case [Gr]. **A** and **B** are always semiexact categories.

2.1. Connected sequences. A *connected* (resp. *exact*) *d-sequence* (F^n, d^n) of functors $\mathbf{A} \to \mathbf{B}$ $(n \geq 0)$ between semiexact categories is given by the following data:

a) a sequence of functors $F^n\colon \mathbf{A} \to \mathbf{B}$, which generally are not exact,

b) for every short exact sequence $a = (A' \rightarrowtail A \twoheadrightarrow A'')$ of **A**, a sequence of maps $d^n = d^n_a : F^nA'' \to F^{n+1}A'$ $(n \geq 0)$, natural for morphisms of short exact sequences, such that the following sequence be of order two (resp. exact) in **B**:

$$(1) \qquad \dots \quad F^n(A') \longrightarrow F^n(A) \longrightarrow F^n(A'') \xrightarrow{d^n} F^{n+1}(A') \quad \dots$$

For instance, if **A** is homological, the cochain homology functors and their differentials form a connected d-sequence (H^n, d^n) from $C^{\cdot}\mathbf{A}$ to **A**, whereas the chain homology functors give a connected ∂-sequence (H_n, ∂_n) from $C.\mathbf{A}$ to **A**; these sequences are exact iff **A** is modular (4.4).

Connected sequences of functors $\mathbf{X} \to \mathbf{A}$, where **X** is semiexact and **A** is homological, are often obtained by a composition:

$$(2) \qquad \mathbf{X} \xrightarrow{C} C.\mathbf{A} \xrightarrow{H_n} \mathbf{A},$$

of a *short-exact* functor C (1.3) with the connected sequence (H_n, ∂_n) of "algebraic" homology functors over the complexes of **A**. If **A** is modular (a fortiori if it is exact, or abelian), the sequence is exact. For instance, both singular homology and the relative homology of groups are of this type.

Let **X** be a semiexact category, provided with a relation R between parallel maps (the *homotopy* relation), a set Σ of morphisms (the *excision* maps) and an object P (the *standard point*). A *homology theory* $H = (H_n, \partial_n)$ on the data (X, R, Σ, P) with values in the semiexact category **A** can be defined by rephrasing Eilenberg-Steenrod's axioms [ES]. Various examples are treated in [G3].

2.2. Right satellites. The connected d-sequence (F^n, d^n) of functors from **A** to **B** $(n \geq 0)$ is *right-universal* if, for every connected sequence (G^n, δ^n) and every natural transformation $\varphi^0\colon F^0 \to G^0$ there exists precisely one *natural transformation* of connected sequences (φ^n) extending φ^0.

Plainly, this naturality means that each $\varphi^n: F^n \to G^n$ is natural and, for any short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of **A**, the obvious square commutes in **B**: $\delta^n . \varphi^n A'' = = \varphi^{n+1}A' . d^n$.

The solution, if it exists, is determined by F^0 up to a unique isomorphism of connected sequences; the functors F^n are called *right satellites* of F^0: $S^nF^0 = F^n$.

Let **A** and **B** be p-semiexact categories, which are also preadditive (i.e., enriched over **Ab**), as **K-Tvs**, **K-Hvs**, **Ban**, **Hlb** (1.9) or any abelian category. If **A** has sufficient normally injective (resp. projective) objects, every additive functor F: **A** $\to$ **B** has all right (resp. left) satellites, which can be constructed by the iterating procedure exposed in [CE], ch. III. If **A** is also additive (has finite biproducts) and homological (as all the examples above), the global construction of right (resp. left) derived functors is also available.

A general explicit construction for left satellites, without any additivity condition, can also be given if **A** is a category of pairs (II.6.5-7). The more general problem of identifying satellites is now dealt with.

2.3. Effacements. Consider a *connected* d-sequence $F = (F^n, d^n)_{n \in \mathbb{N}}$ between *semiexact* categories **A** and **B**.

An *n-effacement* of an object A of **A**, with respect to F, will be a normal monic $m: A \rightarrowtail Q$ producing a short exact sequence of **A**:

(1) $(m, p) = (A \rightarrowtail Q \twoheadrightarrow A')$, $\qquad p = \mathrm{cok}\, m$,

such that, in the associated order two F-sequence, we have:

(2) $\ldots \quad F^{n-1}Q \xrightarrow{F^{n-1}p} F^{n-1}A' \xrightarrow{d^{n-1}} F^nA \xrightarrow{F^n m} F^nQ \quad \ldots$

(3) $d^{n-1} = \mathrm{cok}\, F^{n-1}(p)$.

If the connected sequence F is *exact*, this condition is equivalent to:

(3') $F^n m$ is null in **B** and $d^{n-1}: F^{n-1}A' \to F^nA$ is an exact morphism,

so that, if also the category **B** is *exact*, we come back to the classic formulation: $F^n m$ is null.

The naturality problems one meets in working with such a tool are expressed in the diagrams below:

$$(4) \quad \begin{array}{ccc} A & \xrightarrow{m} & Q \\ \| & & \\ A & \xrightarrow{m'} & Q' \end{array} \qquad\qquad (5) \quad \begin{array}{ccc} A & \xrightarrow{m} & Q \\ \downarrow f & & \downarrow f' \\ B & \xrightarrow{m'} & R \end{array}$$

given two n-effacements of the object A, as in (4), or more generally, two n-effacements of A and B and a map $f: A \to B$ as in (5), how *to connect* them.

The simplest way (yet sufficient also in various non-abelian cases: see 4.6) is to assign, for all $n > 0$, a *functorial effacement*: $m^n_A : A \rightarrowtail Q^n(A)$, i.e. a functor Q^n: **A** $\to$ **A** and a natural transformation $m^n: 1_{\mathbf{A}} \to Q^n$ whose components are n-effacements, satisfying the *diagonal condition*:

a) for every normal monic $f: A \rightarrowtail B$ in **A**, the diagonal $m^n_B f = Q^n(f).m$ of (5) is again an n-effacement of A.

Note that, if **A** is ex2, while **B** and F are exact, this condition a) is automatically satisfied, because of the previous remarks.

By means of functorial effacements, the problem (4) is bypassed while (5) is solved in a natural way. Actually, weaker conditions are sufficient to prove the existence of satellites: F has *sufficient*, or *connected*, or *normally injective* effacements (II, ch. 5). If all the functors F^n (for $n > 0$) annihilate on the normally injective objects of **A**, the last conditions just means that every object A has a normal embedding into a normally injective object.

2.4. Functorial effacement theorem. Let be given a *connected* sequence $F = (F^n, d^n)$: **A** $\to$ **B** between semiexact categories. If the sequence F has a functorial effacement (2.3), then it is right-universal and $S^nF^0 = F^n$ $(n \geq 0)$.

Proof. Choose a functorial effacement $m^n: 1_{\mathbf{A}} \to Q^n$ $(n > 0)$ of F. Take a connected sequence (G^n, δ^n): **A** $\to$ **B** and a natural transformation $\varphi^0: F^0 \to G^0$; assume we have already built $\varphi^p: F^p \to G^p$, for $0 \leq p < n$, satisfying the required naturality conditions and let us prove that there exists a unique natural transformation φ^n which takes on the process.

a) *Uniqueness and construction.* The effacement $m^n_A : A \rightarrowtail Q^n(A)$ produces a short exact sequence:

(1) $(m, p) = (A \rightarrowtail Q \twoheadrightarrow A')$, $\quad m = m^n_A$, $\quad p = \mathrm{cok}\, m$, $\quad F^n m$ null,

together with a commutative diagram with exact rows, in **B**:

$$\begin{tikzcd}
F^{n-1}Q \arrow[r, "F^{n-1}p"] \arrow[d, "\varphi^{n-1}Q"] & F^{n-1}A' \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}A'"] & F^nA \arrow[r, "\text{null}"] \arrow[d, dashed, "\varphi^nA"] & F^nQ \\
G^{n-1}Q \arrow[r, "G^{n-1}p"] & G^{n-1}A' \arrow[r, "\delta^{n-1}"] & G^nA &
\end{tikzcd} \tag{2}$$

where, by hypothesis, d^{n-1} is a cokernel of $F^{n-1}p$; since $\delta^{n-1}.\varphi^{n-1}A'.F^{n-1}p = (\delta^{n-1}.G^{n-1}p).\varphi^{n-1}Q$ is null, there exists precisely one map φ^nA: $F^nA \to G^nA$ which makes the diagram commutative. This argument defines φ^nA and proves at the same time that it is uniquely determined by φ^{n-1}.

b) *Naturality on morphisms*, i.e. naturality of $\varphi^n: F^n \to G^n$. Every morphism $f: A \to B$ in **A** embeds in the commutative diagram (3):

$$(3)\quad \begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"] & Q \arrow[r, two heads] \arrow[d, "g"] & A' \arrow[d, "g'"] \\ B \arrow[r, tail, "m'"'] & R \arrow[r, two heads] & B' \end{tikzcd} \qquad (4)\quad \begin{tikzcd}[column sep=small] & F^{n-1}A' \arrow[rr, "d"] \arrow[dl] \arrow[dd, dashed] & & F^nA \arrow[dl] \arrow[dd, "u"] \\ F^{n-1}B' \arrow[rr, crossing over] \arrow[dd] & & F^nB & \\ & G^{n-1}A' \arrow[rr, dashed] \arrow[dl, dashed] & & G^nA \arrow[dl] \\ G^{n-1}B' \arrow[rr] & & G^nB \arrow[from=uu, crossing over, "v"'] & \end{tikzcd}$$

where the horizontal short exact rows are given by the effacements $m = m^n_A$, $m' = m^n_B$ of A and B and $g = Q^n(f)$. This produces the cube (6) in **B**, whose morphism d^{n-1}: $F^{n-1}A' \to F^nA$ is epi. All its squares, except the right one, are already known to commute: from the naturality of φ^{n-1} (left square), of d^{n-1} (upper square) and δ^{n-1} (lower square) or from the definition of $u = \varphi^nA$ and $v = \varphi^nB$ (back and front square); therefore, also the right square commutes, because of an obvious "sixth face lemma" which we state below (2.5).

c) *Non-standard effacements.* The diagrams above prove also a fact that we need below: any n-effacement m': $A \rightarrowtail R$ of A, which factors as $m' = gm$ through the standard one ($m = m^n_A$) by some morphism g, would produce the same morphism φ^nA, by the procedure exposed in a). Indeed, form the diagram (3) with $B = A$, $f = 1_A$, $m' = gm$; afterwards form (4), where $u = \varphi^nA$ (produced by the standard effacement m) and v is the morphism produced by m'. As before, the right square of the cube is commutative; since its slanting arrows, F^nf and G^nf, are now identities, it follows that $u = v$.

d) *Naturality on short exact sequences.* Given a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in **A**, the normal monic f: $A \rightarrowtail B$ embeds in a diagram (3) as above, where the composition $m'' = m'f = gm$: $A \to R$ is an n-effacement of A (diagonal condition in 2.3); this can be used for the calculus of φ^nA, because of c). In this way, we form the commutative diagram (5), with short exact rows:

$$(5)\quad \begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail, "m'"] & C \arrow[d] \\ A \arrow[r, tail, "m''"'] & R \arrow[r, two heads] & X \end{tikzcd} \qquad (6)\quad \begin{tikzcd}[column sep=small] & F^{n-1}C \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & F^nA \arrow[dl] \arrow[dd] \\ F^{n-1}X \arrow[rr, crossing over] \arrow[dd] & & F^nA & \\ & G^{n-1}C \arrow[rr, dashed] \arrow[dl, dashed] & & G^nA \arrow[dl, "1"] \\ G^{n-1}X \arrow[rr] & & G^nA \arrow[from=uu, crossing over] & \end{tikzcd}$$

and deduce the cube (6) of **B**: we want to prove that its back square is commutative. Again, this follows from the lemma here below, since $G^nA \to G^nA$ is an identity, whence monic; the commutativity of the other five squares follows from the naturality of φ^{n-1} (left square), of φ^n (right square), from the connected sequences F and G (upper and lower square), or from the definition of $\varphi^n A$ and the previous remarks on m" (front square).

2.5. The sixth face lemma. Given a cubic diagram in any category:

(1)
$$\begin{tikzcd}
& \bullet \arrow[rr] \arrow[dl] \arrow[dd, dashed, "a"] & & \bullet \arrow[dl] \arrow[dd] \\
\bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd, "b"'] & \\
& \bullet \arrow[rr, dashed] \arrow[dl, dashed] & & \bullet \arrow[dl] \\
\bullet \arrow[rr] & & \bullet &
\end{tikzcd}$$

- if b is monic and all the squares except possibly the upper one commute, also the latter does,

- if a is epi and all the squares except possibly the lower one commute, also the latter does.

3. Subquotients and regular induction in homological categories

Various diagrammatic constructions concerning chain homology or spectral sequences depend on the study of subquotients, of their induced morphisms and of the calculus of direct and inverse images along induced morphisms: in particular the Snake Lemma (3.4-5); all this requires at least the subquotient axiom (ex3), introduced in 1.6. A is always a homological category.

3.1. Subquotients. The axiom (ex3) can also be expressed in the following form (I.6.5): given an object A and two subobjects m: $M \rightarrowtail A$ (the *numerator*), n: $N \rightarrowtail A$ (the *denominator*), with $m \geq n$, there is a bicartesian square (1), determined up to isomorphism:

(1)
$$\begin{tikzcd}
M \arrow[r, tail, "m"] \arrow[d, two heads, "q'"] & A \arrow[d, two heads, "q"] \\
S \arrow[r, tail, "m'"] & Q
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
N \arrow[r, equal] \arrow[d, tail] & N \arrow[d, tail, "n"] & \\
M \arrow[r, tail, "m"] \arrow[d, two heads, "q'"] & A \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] & P \arrow[d, equal] \\
S \arrow[r, tail, "m'"] & Q \arrow[r, two heads] & P
\end{tikzcd}$$

which can be embedded into the commutative diagram (2), with short exact rows and columns. Note that, since (1) is bicartesian:

(3) $m = q^*m'$, $\qquad q = m_\circ(q')$, $\qquad n = \ker q = m_*(\ker q')$.

The object $S \cong \mathrm{Ncm}\ qm \cong \mathrm{Nim}\ qm$, determined up to isomorphism by $m \geq n$ in Nsb A, will be called a *subquotient* of A and written M/N (3). It is null iff M = N.

More precisely, a (formal) *subquotient* $s: S \twoheadrightarrow A$ will be such a bicartesian square (1) of normal monics and normal epis, up to isomorphism in M and Q. Or also a diagram $S \twoheadleftarrow \bullet \rightarrowtail A$, up to a central isomorphism, determining the square (1) by pushout (1.6); or a diagram $S \rightarrowtail \bullet \twoheadleftarrow A$, determining (1) by pullback. Even without disposing of a category of relations over A (which exists for exact categories), we shall write:

(4) $s = m \cdot \tilde{q}' = \tilde{q} \cdot m'$, $\quad \mathrm{num}\, s = m_*(1) = q^*(m')$, $\quad \mathrm{den}\, s = m_*q'^*(0) = q^*(0)$,

meaning that s *is* the diagram (1): the expressions $m \cdot \tilde{q}'$ and $\tilde{q} \cdot m'$ can be justified, as vertical compositions in a double category Ind A (II.2.8).

3.2. Regular induction. Let be given a morphism $f: A \to B$ and two subquotients $s: M/N \twoheadrightarrow A$, $t: H/K \twoheadrightarrow B$. We say that f *induces (regularly)* from M/N to H/K whenever:

(1) $f_*(M) \leq H$ and $f_*(N) \leq K$,

in which case one can prove in the usual way that f extends uniquely to a translation (2) of the bicartesian square of s (3.1.1) to the analogous one for t:

(2)
$$\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dd, two heads, dashed, "q"] & & B \arrow[dd, two heads, "v"] \\
M \arrow[ur, tail, "m"] \arrow[rr] \arrow[dd, two heads, "q'"'] & & H \arrow[ur, tail, "h"] \arrow[dd, two heads, "v'"'] & \\
& A/N \arrow[rr, dashed] & & B/K \\
M/N \arrow[ur, tail, "m'"'] \arrow[rr, "g"'] & & H/K \arrow[ur, tail, "h'"'] &
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
M/N \arrow[u, two heads, "s"] \arrow[r, "g"'] & H/K \arrow[u, two heads, "t"']
\end{tikzcd}$$

determining the (regularly) *induced* morphism $g: M/N \to H/K$. The commutative cube (2) can be contracted into the *inductive square* (4) (a cell of the double category Ind A (II.2.8)). Regular induction is consistent with composition and identities.

In particular, if A = B and f is the identity, there are *canonical* morphisms, induced by 1_A:

(3) Without the axiom (ex3), we should distinguish between a "left subquotient" (Ncm h = Cok u) and a "right" one (Nim h = Ker v), the first being a quotient of a subobject (M), the second a subobject of a quotient (Q).

(4) $g\colon M/N \to M'/N'$, for $M \le M'$, $N \le N'$ in Nsb A.

3.3. Direct and inverse images along an induced map. After the construction of induced morphisms, this is the second basic tool for non-elementary diagram chasing (1.8).

With the previous notations (3.2), the direct image of $x \in$ Nsb M/N and the inverse image of $y \in$ Nsb H/K along the induced morphism g can be calculated by using each of the two "factorisations" (3.1.4) of the two subquotients s, t. This produces four formulas, for both cases (as proved in II.1.5-8):

$$(1)\quad g_*(x) = v'_*h^*f_*m_*q'^*(x) = v'_*h^*f_*q^*m'_*(x) = h'^*v_*f_*m_*q'^*(x) = h'^*v_*f_*q^*m'_*(x),$$

$$(2)\quad g^*(y) = q'_*m^*f^*h_*v'^*(y) = q'_*m^*f^*v^*h'_*(y) = m'^*q_*f^*h_*v'^*(y) = m'^*q_*f^*v^*h'_*(y).$$

3.4. Snake Lemma (the connecting morphism). In the homological category **A** let be given the two middle squares of the following diagram, commutative and with exact rows (g = cok f, h = ker f):

$$(1)\quad \begin{tikzcd}
\mathrm{Ker}\,a \arrow[r, "f"] \arrow[d, tail, "a'"] & \mathrm{Ker}\,b \arrow[r, "g'"] \arrow[d, tail, "b'"] & \mathrm{Ker}\,c \arrow[d, tail, "c'"] \arrow[ddd, dashed, bend left=60, "d", to path={-- ([xshift=8ex]\tikztostart.east) |- ([xshift=8ex, yshift=-12ex]\tikztostart.east) -| ([xshift=-22ex]\tikztotarget.west) |- (\tikztotarget)}] \\
A' \arrow[r, "f"] \arrow[d, "a"] & B' \arrow[r, two heads, "g"] \arrow[d, "b"] & C' \arrow[d, "c"] \\
A'' \arrow[r, tail, "h"] \arrow[d, two heads, "a''"] & B'' \arrow[r, "k"] \arrow[d, two heads, "b''"] & C'' \arrow[d, two heads, "c''"] \\
\mathrm{Cok}\,a \arrow[r, "h'"] & \mathrm{Cok}\,b \arrow[r, "k'"] & \mathrm{Cok}\,c
\end{tikzcd}$$

and complete the diagram by kernels, cokernels and the induced rows.

Then there is a *connecting morphism* $d\colon \mathrm{Ker}\,c \to \mathrm{Cok}\,a$, regularly induced by $b\colon B' \to B''$ on the subquotients Ker c (of B') and Cok a (of B"):

$$(2)\quad s = (\mathrm{Ker}\,c \rightarrowtail C' \twoheadleftarrow B'), \qquad t = (\mathrm{Cok}\,a \twoheadleftarrow A'' \rightarrowtail B''),$$

and forming a sequence *of order two*, natural for translations of the middle squares:

$$(3)\quad \mathrm{Ker}\,a \xrightarrow{f} \mathrm{Ker}\,b \xrightarrow{g'} \mathrm{Ker}\,c \xrightarrow{d} \mathrm{Cok}\,a \xrightarrow{h'} \mathrm{Cok}\,b \xrightarrow{k'} \mathrm{Cok}\,c.$$

Proof. This regular induction holds:

(4) b_* (num s) $= b_*g^*(c') = b_*g^*c^*(0) = b_*b^*k^*(0) \leq k^*(0) = h =$ num t,

(5) b_* (den s) $= b_*(\ker g) = b_*f_*(1) = h_*a_*(1) = h_*(\ker a'') =$ den t,

so that the sequence (3) is established, together with its naturality. Now g'f' is null, because it is annihilated by c', an N-monic, and we just need to show that dg' is null, as the rest will follow by duality. We show that $d_*g'_*(1) = 0$, where d_* is computed "winding along the diagram", by means of the calculus of direct images along induced morphisms (3.3.1) and of (ex2b):

$$\begin{aligned}(6)\quad d_*g'_*(1) &= a''_*h^*b_*g^*c'_*g'_*(1) = a''_*h^*b_*g^*g_*b'_*(1) = \\ &= a''_*h^*b_*(b' \vee \ker g) = a''_*h^*b_*(\ker b \vee f_*(1)) = \\ &= a''_*h^*b_*f_*(1) = a''_*h^*h_*a_*(1) = a''_*a_*(1) = 0.\end{aligned}$$

3.5. Exactness properties. Various properties of the connecting morphism sequence (3.4.3) are proved in II.3.4, essentially through the previous diagram-chasing tools: direct and inverse images along induced morphisms. We state here, without proof, the main ones:

a) if f is right-modular over ker b (i.e. $f_*f^*(b^*0) = b^*0 \wedge f_*1$) then nim f' = ker g',

b) if b is left-modular over nim f (i.e. $b^*b_*(f_*1) = f_*1 \vee b^*0$) then nim g' = ker d,

b*) if b is right-modular over ker k (i.e. $b_*b^*(k^*0) = k^*0 \wedge b_*1$) then nim d = ker h',

a*) if k is left-modular over nim b (i.e. $k^*k_*(b_*1) = b_*1 \vee k^*0$) then nim h' = ker k',

d) if b is exact, then so is d and the sequence 3.4.3 is exact in the "central objects", Ker c and Cok a,

f) if f is right-modular, b is modular and k is left-modular, then the sequence 3.4.3 is exact,

g) if f is a normal monic, b is modular and k is a normal epi, then the sequence 3.4.3 is exact, begins with a normal monic and ends with a normal epi.

4. Complexes and homology

The homology sequence theorem can be extended to homological categories: as for the Snake Lemma (3.4-5), on which it depends, we get an *order two* sequence, which is *exact* under some modularity conditions (4.3) and always exact iff **A** is modular (4.4). As an application of the effacement theorem (2.4), we prove that the chain homology functors are a sequence of satellites for various homological categories (4.6). **A** is always a homological category.

4.1. Cochain complexes. We generally treat *cochain* complexes $A^\cdot = A = (A^n, d^n)$ and occasionally *chain* complexes $A. = (A_n, \partial_n)$, both indexed over $\mathbb{N}$.

In both cases, for the sake of simplicity, we always speak of cycles (respectively written Z^n or Z_n), boundaries (B^n or B_n) and homology (H^n or H_n); when useful, we specify: *cochain* homology or *chain* homology, according to the case.

A *morphism* of complexes $f^{\bullet}\colon A^{\bullet} \to C^{\bullet}$ is a morphism of diagrams, i.e. a sequence of morphisms $f^n\colon A^n \to C^n$ such that $d^n f^n = f^{n+1} d^n$ $(n \geq 0)$.

This forms the category $C^{\bullet}\mathbf{A}$ of cochain complexes of $\mathbf{A}$; it has a natural structure of homological category, created by the conservative, faithful functor:

(1) $U\colon C^{\bullet}\mathbf{A} \to \mathbf{A}^{\mathbb{N}}, \qquad A^{\bullet} \mapsto (A^n)$,

thus, a morphism $f^{\bullet}$ is null (or a normal monic, or an exact morphism) iff all its components are so, and $\mathrm{Ker}\, f^{\bullet} = (\mathrm{Ker}\, f^n, d^n)$, with differentials induced by the ones of $A^{\bullet}$ (and null).

4.2. Homology. The complex $A^{\bullet} = (A^n, d^n)$ determines, in each component A^n, the subobjects of cycles and boundaries, their quotients and the homology subquotient:

(1) $Z^n = Z^nA^{\bullet} = \mathrm{Ker}\, d^n, \qquad B^n = B^nA^{\bullet} = \mathrm{Nim}\, d^{n-1}, \qquad (Z^n \geq B^n \text{ in } \mathrm{Nsb}\, A^n)$,

(2) $\bar{Z}^n = \bar{Z}^nA^{\bullet} = \mathrm{Ncm}\, d^n = A^n / Z^n, \qquad \bar{B}^n = \bar{B}^nA^{\bullet} = \mathrm{Cok}\, d^{n-1} = A^n / B^n$,

(3) $H^n = H^nA^{\bullet} = Z^n / B^n$,

producing functors $Z^n, B^n, \bar{Z}^n, \bar{B}^n, H^n\colon C^{\bullet}\mathbf{A} \to \mathbf{A}$.

These objects form a commutative diagram, with a bicartesian subquotient square (3.1):

(4)
$$\begin{tikzcd}
A^{n-1} \arrow[rr, "d^{n-1}"] \arrow[d, two heads] & & A^n \arrow[rr, "d^n"] \arrow[dr, two heads] & & A^{n+1} \\
\bar{B}^{n-1} \arrow[r, "\delta^{n-1}"] & Z^n \arrow[ur, tail] \arrow[dr, two heads] & & \bar{B}^n \arrow[r, "\delta^n"] & Z^{n+1} \arrow[u, tail] \\
& & H^n \arrow[ur, tail] & &
\end{tikzcd}$$

where $\delta^n\colon \bar{B}^n \to Z^{n+1}$ is induced by the differential d^n, so that $\mathrm{nim}\, \delta^{n-1} = (B^n \rightarrowtail Z^n)$ and:

(5) $H^nA^{\bullet} \cong \mathrm{Cok}\,(B^n \rightarrowtail Z^n) = \mathrm{Cok}\, \delta^{n-1}$,

(6) $H^nA^{\bullet} \cong \mathrm{Ker}\,(\bar{B}^n \twoheadrightarrow \bar{Z}^n) = \mathrm{Ker}\, \delta^n$.

4.3. Theorem: the homology sequence (II.3.5). Let be given a short exact sequence of cochain complexes, in $C^{\bullet}\mathbf{A}$:

$$(1) \qquad U \overset{m}{\rightarrowtail} V \overset{p}{\twoheadrightarrow} W, \qquad\qquad m = \ker p, \qquad p = \mathrm{cok}\, m.$$

a) There is a homology sequence *of order two*, natural for translations of (1):

$$(2) \qquad \ldots H^n(V) \xrightarrow{\bar{p}n} H^n(W) \xrightarrow{d^n} H^{n+1}(U) \xrightarrow{\bar{m}^{n+1}} H^{n+1}(V) \ \ldots$$

where $\bar{m}^n = H^n(m)$, $\bar{p}^n = H^n(p)$ and d^n is induced by the differential d^n_V of the complex V.

b) *Central exactness* (4): if the differential d^n_V of the central complex V is an exact morphism, so is the differential d^n of the homology sequence; the sequence itself is exact *in the domain* of d^n (i.e., $H^n(W)$) as well as *in its codomain* ($H^{n+1}(U)$).

c) If the following conditions hold for every $n \geq 0$, the homology sequence is *exact*:

(3) $(B^nV \vee U^n) \wedge Z^nV = B^nV \vee (U^n \wedge Z^nV)$,

(4) $d^*d_*(U^n) = U^n \vee Z^nV, \qquad d_*d^*(U^{n+1}) = U^{n+1} \wedge B^{n+1}V$ (5).

d) These conditions are automatically satisfied whenever **A** is modular.

Proof. See II.3.5. The proof follows from an iterated application of the Snake Lemma and of its exactness properties (3.4-5).

4.4. Theorem: homology and modularity (II.3.6). The following conditions on the homological category **A** are equivalent:

a) **A** is modular,

b) the sequence $(H^n, d^n)\colon C^{\cdot}\mathbf{A} \to \mathbf{A}$ of cochain homology functors of **A** is exact (i.e., the homology sequence 4.3.2 is always exact),

b') the sequence $(H_n, \partial_n)\colon C_{\cdot}\mathbf{A} \to \mathbf{A}$ is exact,

c) the connecting-morphism sequence 3.4.3 is always exact, for every (commutative) diagram 3.4.1 whose central rows are exact,

d) idem, for every diagram 3.4.1 whose central rows are *short exact*.

Proof. See II.3.6.

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(4) This partial exactness result is the key for proving the universality of chain homology in non-exact cases (4.5).

(5) I.e., the differential $d = d^n_V$ is left-modular over U^n and right-modular over U^{n+1} (1.7).

4.5. Theorem: homology and graph-factorisation. Let A be a homological category. Then the *order two* homology d-sequence (H^n, d^n): $C^{\cdot}A \rightarrow A$ for cochain complexes is right universal, provided that A has a *functorial g-factorisation* (or graph-factorisation, the term being motivated by some examples below, 4.6 a) - c)).

The latter consists in assigning:

i) for every map f of A, a *g-factorisation* $f = f''.f'$, where f' is a normal monic and f'' a normal epi,

ii) for every commutative diagram of A, where the rows are g-factorisations:

$$(1) \qquad \begin{tikzcd} A \arrow[r, tail, "f'"] \arrow[rr, bend left] \arrow[d, "a"'] & \bullet \arrow[r, two heads, "f''"] \arrow[d, dashed, "c"] & B \arrow[d, "b"] \\ A' \arrow[r, tail, "g'"] \arrow[rr, bend right] & \bullet \arrow[r, two heads, "g''"] & B' \end{tikzcd} \qquad .$$

a morphism c making the whole diagram commutative, *consistently with vertical composition* (and vertical identities [6]) for such diagrams.

Since this notion is self-dual, the existence of a functorial g-factorisation in A yields also the left universality of chain homology (H_n, ∂_n): $C.A \rightarrow A$.

Proof. Given the cochain complex A in $C^{\cdot}A$, and $n > 0$, we are going to construct a functorial embedding $m = m^n_A : A \rightarrowtail Q$ such that:

(2) $H^n(Q)$ is null and d^{n-1}_Q is an exact morphism.

This is necessarily an n-effacement of A. Indeed, the normal monic m produces a short exact sequence $(m, p) = (A \rightarrowtail Q \twoheadrightarrow C)$ and a homology sequence:

$$(3) \quad \ldots \; H^{n-1}Q \xrightarrow{H^{n-1}p} H^{n-1}C \xrightarrow{d^{n-1}} H^nA \xrightarrow{H^n m} H^nQ \; \ldots$$

Since d^{n-1}_Q is an exact map, by the "central exactness" part of the homology sequence theorem (4.3 b)), this sequence is exact in $H^{n-1}C$ and in H^nA and its differential d^{n-1} is exact too. Last, since $H^n(Q)$ is null, d^{n-1} is the cokernel of $H^{n-1}(p)$.

Let $d^{n-1}_A = z\delta$ be the factorisation of the differential of A through the normal subobject $z: Z^n \rightarrow A^n$ of n-cycles of A and $\delta = \delta''\delta'$ the (chosen) g-factorisation of δ. We form the complex Q (with $Q^r = A^r$ for $r \neq n\text{-}1$) and the effacement m as

[6] That is, if $f = g$, $a = 1$ and $b = 1$, the assigned morphism c is the identity.

in the following commutative diagram:

$$(4)\qquad \begin{tikzcd}
\dots & A^{n-2} \arrow[r, "d^{n-2}"] \arrow[dd, equal] & A^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr, "\delta"] \arrow[dd, tail, "\delta'"'] & & A^n \arrow[r, "d^n"] \arrow[dd, equal] & A^{n+1} \arrow[dd, equal] & \dots & A \arrow[dd, tail, "m"] \\
& & & Z^n \arrow[ur, tail, "z"] & & & & \\
\dots & Q^{n-2} \arrow[r, "d^{n-2}"'] & Q^{n-1} \arrow[rr, "d^{n-1}"'] \arrow[ur, "\delta''"'] & & Q^n \arrow[r, "d^n"'] & Q^{n+1} & \dots & Q
\end{tikzcd}$$

It is easy to check that Q is indeed a complex; m is a normal monic as all its components are so. The condition (2) is satisfied, as $d_Q^{n-1} = z\delta''$ is trivially exact and $H^n(Q)$ is null:

(5) $\mathrm{nim}\, d_Q^{n-1} = \mathrm{nim}\, z\delta'' = z_*(\delta''_*(1)) = \mathrm{nim}\, z = Z = \ker d_A^n = \ker d_Q^n$.

Therefore, the functoriality of g-factorisations produces a functorial effacement $m_A^n : A \rightarrowtail Q(A)$ of the d-sequence H^n, provided we verify the diagonal condition of 2.3. A normal monomorphism of complexes $f: A \rightarrowtail B$ produces a commutative diagram:

$$(6)\qquad \begin{tikzcd}
A \arrow[r, tail, "m"] \arrow[d, tail, "f"] & Q(A) \arrow[d, dashed, "Qf"] \\
B \arrow[r, tail, "m'"] & Q(B)
\end{tikzcd}$$

and we have to show that the normal monic $m'' = m'f: A \rightarrowtail Q(B)$ is an n-effacement of A. This proceeds as before, since the complex Q(B) satisfies the hypotheses (2): it is n-acyclic and its differential of degree n-1 is exact.

4.6. Categories with functorial g-factorisation. Various homological categories, including all the abelian ones, have functorial g-factorisation, whence their sequence of chain or cochain homology functors are universal.

a) Abelian categories. There is a functorial g-factorisation (1) through the biproduct $A \oplus B$:

$$(1)\qquad \begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail, "g"'] & & B \\
& A\oplus B \arrow[ur, two heads, "q"'] &
\end{tikzcd}
\qquad\qquad
(2)\qquad \begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & B \\
& A\oplus B \arrow[ur, two heads, "h"'] &
\end{tikzcd}$$

q is the cartesian projection and $g = (1_A, f)$ is the *graph* of f. Dually, there is also a g-factorisation (2), via the *co-graph* h of co-components f and 1_B ($h(a, b) = f(a) + b$, in a category of modules).

b) The modular homological category **K-Tvs** of topological vector spaces (I.4.2, I.6.6; 1.7). Here the normal subobjects are the linear subspaces, the normal quotients are the quotients modulo the former. There is a functorial g-factorisation of type (1): the cartesian projection q is an open surjection, hence - also topologically- a quotient; the injective morphism g is a topological embedding: if U is open in A, $U \times B$ is open in $A \oplus B$ and $g^{-1}(U \times B) = \{a \mid (a, fa) \in U \times B\} = U$. It is also possible to give a factorisation of type (2).

c) The homological categories **K-Hvs**, **Ban** and **Hlb** of Hausdorff vector spaces, Banach and Hilbert spaces. Here the normal subobjects are the *closed* linear subspaces, the normal quotients are the quotients modulo the former, with the obvious structure; these categories are *not* modular (1.7). We get again a functorial g-factorisation, either by construction (1) (the image of g is the graph of f, hence it is closed) or by construction (2) ($i(A) = A \times \{0\}$ is closed).

d) The category $\mathfrak{I}$ of sets and partial bijections. Also this non-abelian, p-exact category has a functorial g-factorisation: a partial bijection $f: A \to B$ determines an amalgamated sum $A +_f B = (A+B)/R$ where the equivalence relation $R \subset (A+B)\times(A+B)$ is generated by $f \subset A \times B$ (identifying $x \in \mathrm{Def}\, f$ with $fx \in \mathrm{Val}\, f$). The factorisation $A \rightarrowtail A +_f B \twoheadrightarrow B$, consisting of an embedding and a restriction, is now obvious.

e) The p-exact category **Mlc** of modular lattices and modular connections (1.4). A functorial g-factorisation can be obtained by an *amalgam* of a connection, similar to the construction above (II.6.4).

f) Plainly, a non-exact homological category **A** in which the exact morphisms are stable under composition can*not* have g-factorisations: this is the case of $\mathbf{Set}^{\mathsf{T}}$, $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{C}$ (I.4.3-4). More generally, the same negative conclusion holds whenever the exact morphisms of **A** generate a proper subcategory; this happens, for instance, for $\mathbf{Set}_2$ and $\mathbf{Top}_2$, since their exact morphisms are injective mappings (I.3.4).

5. Semihomotopical and homotopical categories

A semihomotopical category, our basic notion for homotopical algebra, is produced by the existence of h-kernels and h-cokernels in a very simple 2-dimensional structure, called h-category.

A cell in a 2-graph will be written as $\alpha: a' \to a'': A \to B$ if we want to express all of its domains and codomains: the vertical ones (a' and a'') as well as the horizontal ones (A and B); it will be written as α: a' $\to$ a" or α: A $\twoheadrightarrow$ B (with a bold arrow) when we just need to express the former or the latter.

5.1. Definition. An h-*category* **A** is a category and a 2-graph, having the same underlying 1-graph and provided with:

a) a *reduced horizontal composition* law $y \circ \alpha \circ x$ (also written $y\alpha x$):

$$(1) \quad y \circ \alpha \circ x \colon ya'x \to ya''x \colon X \to Y \qquad\qquad X \xrightarrow{x} A \underset{a''}{\overset{a'}{\underset{\longrightarrow}{\overset{\longrightarrow}{\scriptstyle\alpha\downarrow}}}} B \xrightarrow{y} Y$$

whose *horizontal identities* are, by definition, the identical morphisms 1_A of the objects,

b) a *vertical structure* consisting of a cell 1_a: a $\to$ a (the *vertical identity* of a), for every map a,

so that this axiom is satisfied:

(hc) horizontal identities are neutral for the horizontal composition, which is associative and distributes with respect to vertical identities:

$$(2) \quad 1_B \circ \alpha \circ 1_A = \alpha \qquad\qquad A \overset{\alpha}{\twoheadrightarrow} B$$

$$(3) \quad y' \circ (y \circ \alpha \circ x) \circ x' = (y'y) \circ \alpha \circ (xx') \qquad\qquad X' \xrightarrow{x'} X \xrightarrow{x} A \overset{\alpha}{\twoheadrightarrow} B \xrightarrow{y} Y \xrightarrow{y'} Y'$$

$$(4) \quad y \circ 1_a \circ x = 1_{yax} \qquad\qquad X \xrightarrow{x} A \xrightarrow{a} B \xrightarrow{y} Y.$$

We write αx for $1_B\ \alpha\ x$ and $y\alpha$ for $y\ \alpha\ 1_A$. Plainly, one can also assign two *binary* reduced horizontal compositions αx and $y\alpha$, under the axiom:

$$\text{(hc')} \quad \alpha \circ 1_A = \alpha = 1_B \circ \alpha, \qquad 1_a \circ x = 1_{ax}, \qquad y \circ 1_a = 1_{ya},$$

$$(\alpha \circ x) \circ x' = \alpha \circ (xx'), \qquad y' \circ (y \circ \alpha) = (y'y) \circ \alpha, \qquad y \circ (\alpha \circ x) = (y \circ \alpha) \circ x.$$

A category can always be thought of as a *trivial* h-category, whose only cells are vertical identities. Every 2-category has an underlying h-category.

An *h-functor* F: **A** $\to$ **B** is a functor between h-categories together with a transformation of cells which preserves the additional structure: vertical domains and codomains, reduced horizontal composition and vertical identities. If **B** is a category (with its trivial h-structure), this just means that F is a functor and turns the cells of **A** into equalities of maps of **B**. A weaker notion of homotopy invariant functor is probably more important (III.2.5).

Richer structures are considered in III.2.2: for instance an *h4-category* is provided with vertical composition and vertical involution for homotopies, together with a second-order homotopy relation for cells $\sim$, so that the whole complex is a sort of relaxed 2-category (up to $\sim$). In this case, the (first-order) homotopy relation for maps, $f \simeq g$ if there exists a cell $\alpha: f \to g$, is a congruence.

5.2. Remarks. Formally, a (locally small) h-category **A** is the same as a category enriched over the monoidal category of 1-truncated cubical sets (or 1-truncated simplicial sets, or also directed graphs with identities), as it appears if we take as hom-objects the diagrams:

(1) $\mathbf{A}(A, B) = (\partial_\varepsilon: \mathbf{A}_1(A, B) \rightleftarrows \mathbf{A}_0(A, B): s), \qquad \partial_\varepsilon s = 1 \qquad (\varepsilon = 0, 1)$

where the components $\mathbf{A}_0(A, B)$ and $\mathbf{A}_1(A, B)$ are respectively given by the arrows and by the cells of **A**; ∂_0, ∂_1 are the vertical domain and vertical codomain (of a cell); s is the vertical identity (of an arrow). The monoidal (closed) structure of 1-truncated cubical sets we use is given by:

(2) $(X \otimes Y)_0 = X_0 \times Y_0, \qquad (X \otimes Y)_1 = (X_0 \times Y_1) \dot{\amalg} (X_1 \times Y_0)$ (7)

and supplies a category structure in degree 0 (for the arrows), together with the composition of a cell with an arrow on both sides, in degree 1, so that our axiom (hc') be satisfied (5.1).

In other words, an h-category is the same as a *generalized homotopy system* in the sense of Kamps [K1]: a category **A** equipped with a functor $\mathbf{A}_1$: $\mathbf{A}^{op} \times \mathbf{A} \to$ **Set** and with natural transformations $\partial_\varepsilon: \mathbf{A}_1 \to \mathbf{A}_0$, $s: \mathbf{A}_0 \to \mathbf{A}_1$ such that $\partial_\varepsilon s = 1$ (where $\mathbf{A}_0$ is the hom-functor of **A**).

In an h-category, the *terminal* object $\top$ will be defined by the following equivalent *two-dimensional* properties:

(3) for every object X, there is a unique cell $X \twoheadrightarrow \top$,

(3') for every object X, there is a unique morphism $\top_X: X \to \top$ and a unique cell $X \twoheadrightarrow \top$ (the vertical identity of the latter).

5.3. Definition. A *right-semihomotopical* category will be an h-category **A** satisfying the following axioms:

(rh.0) **A** has a terminal object $\top$ (5.2),

(rh.1) every arrow $f: A \to B$ of **A** has an *h-cokernel*, with respect to the ideal of $\top$-null morphisms: this is the *h-pushout* hck $f = (x, x''; \xi)$ of $\top_A: A \to \top$ along f, i.e. a cell $\xi: xf \to x''\top_A: A \to X$ such that:

(7) Here the symbol $\dot{\amalg}$ means pushout over $X_0 \times Y_0$. More precisely, $(X \otimes Y)_1$ is the pushout of the mappings $X_0 \times s: X_0 \times Y_0 \to X_0 \times Y_1$ and $s \times Y_0: X_0 \times Y_0 \to X_1 \times Y_0$.

$$(1)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\tau_A"'] & B \arrow[d, "x"] \arrow[dl, phantom, "\xi\swarrow"] \\ \top \arrow[r, "x''"'] & X \end{tikzcd}$$

for each η: $yf \to y''\tau_A$: $A \to Y$, there is exactly one a: $X \to Y$ such that: $y = ax$, $y'' = ax''$, $\eta = a\xi$.

It should be noted that, according to Mather's original terminology for topological spaces [Mh], this h-pushout is the *standard homotopy pushout*; it is determined up to isomorphism, while a general homotopy pushout is just determined up to homotopy equivalence.

We write $Cf = X$ (*mapping cone* of f) the h-cokernel object of f and $c(f) = x$ the main h-cokernel map; the latter will be called a *principal injection* of X (and a principal cofibration when it is so, see 5.9). The triple (x, x'', ξ) is *jointly epi*: if a and b are parallel morphisms, the relations $ax = bx$, $ax'' = bx''$ and $a\xi = b\xi$ imply $a = b$. In a right semihomotopical category, principal injections are stable under strict pushouts (III.3.2).

5.4. The h-cokernel functor. Let **A** be a right-semihomotopical category. The h-cokernel defines a *mapping cone* (or h-cokernel) functor:

(1) C: $\mathbf{A}^2 \to \mathbf{A}$,

since, given a morphism (a, b): $f \to g$ in $\mathbf{A}^2$, as in (2):

$$(2)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "x"] \arrow[d, "b"] & Cf \arrow[d, dashed, "h"] \\ C \arrow[r, "g"'] & D \arrow[r, "y"'] & Cg \end{tikzcd} \qquad \begin{array}{l} \mathrm{hck}\, f = (x, x''; \xi) \\ \\ \mathrm{hck}\, g = (y, y''; \eta) \end{array}$$

the cell ηa: $yga = ybf \to y''\tau_C a = y''\tau_A$: $A \to Cg$ determines one morphism $h = C(a, b)$: $Cf \to Cg$, such that: $hx = yb$, $hx'' = y''$ and $h\xi = \eta a$.

In particular, we have two endofunctors of **A**, *suspension* and *cone*:

(3) $\Sigma A = C(\tau_A$: $A \to \top)$, $\qquad\qquad CA = C(1_A$: $A \to A)$.

The object ΣA comes equipped with a cell, the *suspension evaluation of* A:

(4) ev^A: $a'\tau_A \to a''\tau_A$: $A \to \Sigma A$,

universal among the cells from A, whose vertical domain and codomain are null. Therefore, given a map f: $A \to B$, Σf: $\Sigma A \to \Sigma B$ is the unique morphism which satisfies the conditions:

(5)
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & \Sigma A \arrow[d, dashed, "\Sigma f"] \\ B \arrow[r, "\beta"'] & \Sigma B \end{tikzcd} \qquad \begin{array}{l} \Sigma f.a' = b', \quad \Sigma f.a'' = b'', \\ \\ (\Sigma f)\, ev^A = ev^B f. \end{array}$$

Note that the triple (a', a'', ev^A) is jointly epi, i.e. cancellable with respect to maps into A; therefore, if A is pointed, the cell ev^A itself is cancellable in this sense: $ev^A \circ u = ev^A \circ v$ implies $u = v$.

If **A** is h4, but actually as soon as it has a vertical involution of cells, the functor Σ gets an involutive automorphism i: $\Sigma \to \Sigma$ which will be called *inversion* (III.4.6) and used as an *extension of the sign-changing procedure* to non-additive situations: if **A** is the category of chain complexes over some additive category, i_A is the *sign-change* automorphism (of the suspended complex ΣA).

5.5. The cofibration sequence of a map. The morphism f: A $\to$ B produces a *differential* δ: Cf $\to$ ΣA, determined by the conditions (2):

(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\xi\swarrow"] & B \arrow[r] \arrow[d, "x"] & T \arrow[dd, "a'"] \\ T \arrow[r, "x"'] \arrow[d, equal] & Cf \arrow[dr, "\delta"] & \\ T \arrow[rr, "a''"'] & & \Sigma A \end{tikzcd}$$

(2)
$$\delta x = a' T_B \qquad \delta x'' = a'' \qquad \delta\xi = ev^A$$

where the upper square formed by δ is a (strict) pushout (III.3.2).

The differential δ can also be obtained by means of the h-cokernel functor C: $\mathbf{A}^2 \to \mathbf{A}$ (5.4):

(3)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "x"] \arrow[d] & Cf \arrow[d, dashed, "\delta"] \\ A \arrow[r] & T \arrow[r, "a'"'] & \Sigma A = C(T_A) \end{tikzcd}$$

(4) $\delta = C(1_A, T_B)$,

which implies that δ is natural for morphisms (a, b): f $\to$ g in $\mathbf{A}^2$.

This differential produces the *cofibration sequence*, or Puppe sequence [P1], of the morphism f in the right-semihomotopical category **A**:

(5) $\quad A \xrightarrow{f} B \xrightarrow{x} Cf \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma x} \Sigma Cf \xrightarrow{\Sigma\delta} \Sigma^2 A \ \ldots$

We show now that the cofibration sequence is linked to a tower of principal injections, in which each morphism is the h-cokernel of the previous one (5.6; 5.9).

5.6. The cofibration diagram. For the map f: $A \to B$, let $B_0 = A$, $B_1 = B$, $x_0 = f\colon B_0 \to B_1$, and define inductively $x_n = c(x_{n-1})$ by means of the h-cokernel of $x_{n-1}\colon B_{n-1} \to B_n$:

(1) $\quad x_n\colon B_n \to B_{n+1}, \qquad \xi_n\colon x_n . x_{n-1} \to x''_n . \top_{B_{n-1}}\colon B_{n-1} \to B_{n+1} \qquad (n \geq 1)$

so that the cofibration sequences of the morphisms x_n form the rows of a *cofibration diagram* for the morphism f, *commutative except for the squares marked with* #:

(2)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "x_1"] \arrow[d, equal] & Cf \arrow[r, "\delta"] \arrow[d, equal] & \Sigma A \arrow[r, "\Sigma f"] \arrow[dr, phantom, "\#"] & \Sigma B \arrow[r, "\Sigma x_1"] \arrow[d, equal] & \Sigma Cf \arrow[r, "\Sigma\delta"] \arrow[d, equal] & \Sigma^2 A \arrow[r] \arrow[dr, phantom, "\#"] & \ldots \\
B_0 \arrow[r, "f"] \arrow[d, equal] & B_1 \arrow[r, "x_1"] \arrow[d, equal] & B_2 \arrow[r, "x_2"] \arrow[d, equal] & B_3 \arrow[r, "\delta_1"] \arrow[u, "s_0"] \arrow[d, equal] & \Sigma B_1 \arrow[r] \arrow[dr, phantom, "\#"] & \Sigma B_2 \arrow[r] \arrow[d, equal] & \Sigma B_3 \arrow[r] \arrow[u, "\Sigma s_0"] \arrow[d, equal] & \ldots \\
B_0 \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d, equal] & B_2 \arrow[r] \arrow[d, equal] & B_3 \arrow[r, "x_3"] \arrow[d, equal] & B_4 \arrow[r, "\delta_2"] \arrow[u, "s_1"] \arrow[d, equal] & \Sigma B_2 \arrow[r] \arrow[dr, phantom, "\#"] & \Sigma B_3 \arrow[r] \arrow[d, equal] & \ldots \\
B_0 \arrow[r, "f"] & B_1 \arrow[r, "x_1"] & B_2 \arrow[r, "x_2"] & B_3 \arrow[r, "x_3"] & B_4 \arrow[r, "x_4"] & B_5 \arrow[r, "\delta_3"] \arrow[u, "s_2"] & \Sigma B_3 \arrow[r] & \ldots \\
\ldots & \ldots & \ldots & & & & &
\end{tikzcd}$$

but admitting a commutative rearrangement, under stronger hypotheses (5.9). Here $\delta_n\colon B_{n+2} \to \Sigma B_n$ $(n \geq 0)$ is defined as in 5.5.2:

(3) $\quad \delta_n x_{n+1} = b'_n \top, \qquad \delta_n x''_{n+1} = b''_n, \qquad \delta_n \xi_{n+1} = ev^{B_n}\colon b'_n \to b''_n\colon B_n \to \Sigma B_n,$

while $s_n\colon Cx_{n+2} = B_{n+3} \to \Sigma B_n$ is defined through the universal property of Cx_{n+2}:

(4) $\quad s_n x_{n+2} = \delta_n, \qquad s_n x''_{n+2} = b'_n, \qquad s_n \xi_{n+2} = 1.$

5.7. Semihomotopical categories. Dually, a *left semihomotopical* category is an h-category **A** with initial object $\perp$, where every morphism f: $A \to B$ of **A** has an *h-kernel* with respect to the ideal of $\perp$-null morphisms: this is the h-pullback hkr f = $(x', x; \xi)$ of $\perp_B\colon \perp \to B$ along f.

The triple (x', x, ξ) is jointly monic. We write Kf the h-kernel object and kf = x: $Kf \to A$ the main structural map: this is called a principal projection (and a principal fibration if it is so); such maps are stable under pullbacks (III.3.2).

A has a *loop-endofunctor*, dual to suspension:

(1) $\Omega A = K(\perp_A\colon \perp \to A)$,

and the object ΩA comes equipped with a cell $ev_A\colon \perp_A a' \to \perp_A a''\colon \Omega A \to A$ (*loop evaluation of* A), universal among the cells *to* A, whose vertical domain and codomain are null. Each map determines a *fibration* sequence:

$$(2) \quad \dots\ \Omega^2 B \xrightarrow{\Omega\partial} \Omega Kf \xrightarrow{\Omega x} \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\partial} Kf \xrightarrow{x} A \xrightarrow{f} B.$$

A *semihomotopical* category A is an h-category which is both left and right semihomotopical. Then it has an initial object $\perp$, a terminal object $\top$, h-kernels with respect to the ideal of $\perp$-null morphisms ($A \to \perp \to B$) and h-cokernels with respect to the ideal of $\top$-null morphisms. Each map has a *fibration-cofibration* sequence, formed by the glueing of the two sequences.

In particular, a pointed semihomotopical category with trivial h-structure (5.1) is the same as a p-semiexact category. Then, the fibration-cofibration sequence of a map $f\colon A \to B$ reduces to:

(3) $0 \to \mathrm{Ker}\, f \to A \to B \to \mathrm{Cok}\, f \to 0.$

5.8. The suspension-loop adjunction. Assume now that the semihomotopical category A is *pointed* ($\perp = 0 = \top$), so that the $\perp$-null and $\top$-null maps coincide.

Conversely, it is easy to see that if these ideals coincide, then A is pointed: the unique map $\perp \to \top$ is an isomorphism, because both $1_\perp$ and $1_\top$ have to factor through it.

We show now that, as in the case of pointed spaces, there is a canonical adjunction $\Sigma \dashv \Omega$. This fails in the non-pointed case, e.g. in **Top** which has a trivial left-homotopical structure (5.10).

For every object A, the homotopy $ev^A\colon A \to \Sigma A$ has null domain and codomain, hence it factors uniquely through the universal cell $ev_{\Sigma A}\colon \Omega\Sigma A \to \Sigma A$, yielding the unit $y\colon 1 \to \Omega\Sigma$ of the adjunction; the counit $x\colon \Sigma\Omega \to 1$ is obtained similarly:

(1) $y_A\colon A \to \Omega\Sigma A, \qquad ev_{\Sigma A} \circ y_A = ev^A,$

(2) $x_A\colon \Sigma\Omega A \to A, \qquad x_A \circ ev^{\Omega A} = ev_A\,,$

$$(3) \quad \begin{tikzcd} A \arrow[r, "ev^A"] \arrow[dr, dashed, "y_A"'] & \Sigma A \\ & \Omega\Sigma A \arrow[u, "ev_{\Sigma A}"'] \end{tikzcd} \qquad\qquad (4) \quad \begin{tikzcd} \Omega A \arrow[r, "ev_A"] \arrow[d, "ev^{\Omega A}"'] & A \\ \Sigma\Omega A \arrow[ur, dashed, "x_A"'] & \end{tikzcd}$$

The naturality of the transformation y over a morphism f: $A \to B$ follows from the diagram (5):

$$
(5)\quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"] & \Omega\Sigma A \arrow[r, "ev_{\Sigma A}"] \arrow[d, "\Omega\Sigma f"] & \Sigma A \arrow[d, "\Sigma f"] \\
B \arrow[r, "y_B"'] & \Omega\Sigma B \arrow[r, "ev_{\Sigma B}"'] & \Sigma B
\end{tikzcd}
\qquad
(6)\quad
\begin{tikzcd}
\Sigma A \arrow[r, "\Sigma y_A"] & \Sigma\Omega\Sigma A \arrow[r, "x_{\Sigma A}"] & \Sigma A \\
A \arrow[r, "y_A"'] \arrow[u, "ev^A"] & \Omega\Sigma A \arrow[r, "ev_{\Sigma A}"'] \arrow[u, "ev^{\Omega\Sigma A}"] & \Sigma A \arrow[u, equal]
\end{tikzcd}
$$

where the outer rectangle commutes by definition of y and of Σf, the right-hand square commutes by definition of $\Omega(\Sigma f)$, and $ev_{\Sigma B}$ is monic on morphisms (5.4). Analogously, x is natural.

Last, we check one of the coherence conditions: $x_{\Sigma A} . \Sigma y_A = 1_A$. The diagram (6) above is commutative by definition of Σy_A (left square) and of $x_{\Sigma A}$ (right square), whence:

(7) $(x_{\Sigma A} . \Sigma y_A) \circ ev^A = ev_{\Sigma A} \circ y_A = ev^A = 1_A \circ ev^A$,

and the thesis follows from the cancellation property of ev^A.

5.9. Homotopical categories. A *right homotopical* category is an h4-category (5.1) which is right semihomotopical, with h-cokernels verifying a higher-order universal property (III.3.11).

In these hypotheses, the structural maps of h-cokernels are cofibrations (III.3.9), the suspension endofunctor is homotopy invariant (III.4.5), every suspension ΣA has a canonical h-cogroup structure (III.4.7) and the cofibration sequence of a map f is equivalent to a tower of principal cofibrations, its iterated h-cokernels.

This last result comes from the fact that the squares marked with # in the cofibration diagram of f (5.6.2) are now *homotopically anticommutative* (III.5.4), i.e. commute homotopically up to inversion (5.4) in the last vertex of the square. Therefore, by *reversing* some arrows (all δ_n and s_n, for n odd) and composing the columns, we can form the *contracted cofibration diagram* of f:

$$
(1)\quad
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "x_1"] & Cf \arrow[r, "\delta"] & \Sigma A \arrow[r, "\Sigma f"] & \Sigma B \arrow[r, "\Sigma x_1"] & \Sigma Cf \arrow[r, "\Sigma\delta"] & \Sigma^2 A \ \ldots \\
B_0 \arrow[r, "f"'] \arrow[u, equal] & B_1 \arrow[r, "x_1"'] \arrow[u, equal] & B_2 \arrow[r, "x_2"'] \arrow[u, equal] & B_3 \arrow[r, "x_3"'] \arrow[u, "s_0"] \arrow[ur, phantom, "\simeq"] & B_4 \arrow[r, "x_4"'] \arrow[u, "s'_1"] \arrow[ur, phantom, "\simeq"] & B_5 \arrow[r, "x_5"'] \arrow[u, "s'_2"] \arrow[ur, phantom, "\simeq"] & B_6 \ \ldots \arrow[u, "s'_3"]
\end{tikzcd}
$$

(2) $s'_1 = i_{B_1} s_1, \quad s'_2 = s_2, \quad s'_3 = (\Sigma s_0).i_{B_3} s_3, \ \ldots$

connecting the cofibration sequence of f (the upper row) to the *tower of* *cofibrations* of f (the lower row), where each map x_n is the h-cokernel of the previous one.

This diagram is homotopically commutative, all its vertical arrows are homotopy equivalences and all x_n are principal cofibrations (for $n \geq 1$). Further, the two rows are *exact* with respect to the ideal of homotopically $\top$-null morphisms (in the sense of the last remark in 1.2).

If **A** is *homotopical*, i.e., right and left homotopical, there is a similar *contracted fibration-cofibration* diagram connecting the fibration-cofibration sequence of f to its double tower of fibrations and cofibrations.

5.10. Some examples. a) The category **Top** of topological spaces, with usual homotopies, is homotopical. The initial object is empty and the whole left-homotopical stucture is trivial: all h-kernels (and loop-objects) are empty. Instead, the terminal object $\top = \{*\}$ produces a well-known right-homotopical structure: the h-cokernel of the map $f\colon X \to Y$ is its mapping cone, the suspension is the classical one. The cofibration sequence of f is exact with respect to the ideal of homotopically constant maps (5.9); it can be noted that strict kernels, with respect to this ideal, need not exist (for instance, for the identity of the sphere S^1).

b) The category $\mathbf{Top}^\top$ of pointed topological spaces, equipped with pointed homotopies, is pointed homotopical. The h-kernel of $f\colon X \to Y$ is the *homotopy fibre*:

(1) $Kf = \{(x, \eta) \in X \times PY \mid \eta(0) = 0_Y, \eta(1) = f(x)\}$,

where PY is the space of paths of Y, with the compact-open topology, pointed at the constant path in 0_Y; the space of loops is the obvious one. The fibration sequence of f produces the exact homotopy sequence of the map f. The h-cokernel of f and the suspension of X are obtained from the non-pointed ones by collapsing the subspace $\{0_X\} \times I$; the cofibration sequence of f is the original Puppe sequence.

c) The h-category of pointed finite CW-complexes, pointed cellular maps and pointed cellular homotopies is right-homotopical; as well known, it lacks path-spaces and homotopy kernels.

d) If **A** is a *preadditive* category (enriched over **Ab**), the category of unbounded chain complexes $C_*\mathbf{A}$ is h4, with respect to the usual homotopies. If **A** is *additive*, $C_*\mathbf{A}$ is homotopical: the h-kernel of the morphism $f\colon A \to B$ and the loop-object are as follows (a morphism between biproducts is written in matrix notation):

(1) $(Kf)_n = A_n \oplus B_{n+1}$, $\qquad \partial_n = \begin{pmatrix} \partial & 0 \\ f & -\partial \end{pmatrix} : (Kf)_n \to (Kf)_{n-1}$,

(2) $\Omega A = K(0 \rightarrow A)$, $\quad (\Omega A)_n = A_{n+1}$, $\quad \partial_n^{\Omega A} = -\partial_{n+1}$,

(3) $ev_A: 0 \rightarrow 0: \Omega A \rightarrow A$, $\quad (ev_A)_n = \mathrm{id}\, A_{n+1}$,

while the h-cokernel and the suspension endofunctor derive by duality: $(C_*\mathbf{A})^{op} \cong C_*(\mathbf{A})^{op}$. Loop and suspension are reciprocal shifts: $C_*\mathbf{A}$ is stable (6.2).

Note that $C_*\mathbf{A}$ has kernels and cokernels if and *only* if $\mathbf{A}$ does; but of course the pullbacks of principal fibrations and pushouts of principal cofibrations always exist, as it happens in every semihomotopical category (III.3.2).

e) Let $\mathbf{A}$ be a category with zero object. The category $G_*\mathbf{A}$ of $\mathbb{Z}$-graded objects over $\mathbf{A}$ and maps of degree 0 has an h-structure which is not trivial, even if the associated homotopy relation between parallel morphisms is so: an *endocell* $\alpha: a \rightarrow a: A \rightarrow B$ is given by any pair (a, α), where α is a morphism from A to B of degree 1; there are no other cells. The suspension and loop endofunctors always exist, and are reciprocal shifts. The h-category $G_*\mathbf{A}$ is not semihomotopical, in general: the existence of the h-kernel of an arbitrary map $f: A \rightarrow B$ in $\mathbf{A}$ depends on the existence of strict kernels and binary products; dually for h-cokernels.

f) The category R-**Dga** of differential graded unital R-algebras has initial object given by R itself and terminal object equal to the null R-algebra; usual homotopies yield an h-structure, *without* vertical composition. This h-category is semihomotopical (with trivial right structure).

g) The 2-category $\mathbf{Cat}_i$ of small categories, functors and functorial isomorphisms is *strictly* h4 (with identical cell-homotopy) and homotopical. Analogously its 2-full sub-2-category **Gpd** of small groupoids. In both cases, h-pullbacks are comma squares.

6. Homological theories on semihomotopical categories

We introduce here a symmetrical notion of semihomogeneous theory over a pointed semihomotopical category $\mathbf{A}$, with values into a category $\mathbf{B}$ equipped with an ideal of *null* morphisms. The usual conditions of Σ-stability and Σ-exactness produce an homological theory; the dual axioms, Ω-stability and Ω-exactness, concern homotopical theories.

6.1. Absolute theories. Absolute homological theories for topological spaces, defined by *single-space* axioms, where introduced by Dold-Thom [DT] (credited to Puppe) and G.M. Kelly [Ke], and shown to be equivalent to the relative formulation of Eilenberg-Steenrod.

Actually, the term "absolute" is somewhat misleading: the crucial distinction between the two presentations is not the fact of being relative or absolute, but the fact of being based on short exact sequences (in semiexact categories, 0.4) or on the suspension functor (in semihomotopical categories); in other words, to pertain to homological or homotopical algebra. And indeed, single-space theories for locally compact spaces plainly belong to the first type (0.4).

Analogously, the theory of chain complexes over abelian categories splits in the present setting in two parts: complexes over homological categories (ch. 4) and complexes over additive categories (5.10); in the latter case, homology is not defined by a subquotient, but by taking $H_n(A) = [S, \Omega^n A]$, for a fixed complex S.

6.2. The adjunction diagram. In the pointed semihomotopical category **A**, every morphism f: A → B produces a commutative *adjunction fibration-cofibration diagram* (just the solid arrows):

$$(1)\qquad \begin{tikzcd}
\cdots & \Sigma\Omega Kf \arrow[r, "\Sigma\Omega kf"] \arrow[d, "x_{Kf}"] & \Sigma\Omega A \arrow[r, "\Sigma\Omega f"] \arrow[d, "x_A"] & \Sigma\Omega B \arrow[r, "\Sigma\partial"] \arrow[d, "x_B"] & \Sigma Kf \arrow[r, "\Sigma kf"] \arrow[d, dotted, "u"] & \Sigma A \cdots \arrow[d, equal] \\
\cdots \Omega B \arrow[r, "\partial"] \arrow[d, equal] & Kf \arrow[r, "kf"] \arrow[d, dotted, "v"] & A \arrow[r, "f"] \arrow[d, "y_A"] & B \arrow[r, "cf"] \arrow[d, "y_B"] & Cf \arrow[r, "\delta"] \arrow[d, "y_{Cf}"] & \Sigma A \cdots \\
\cdots \Omega B \arrow[r, "\Omega cf"'] & \Omega Cf \arrow[r, "\Omega\delta"'] & \Omega\Sigma A \arrow[r, "\Omega\Sigma f"'] & \Omega\Sigma B \arrow[r, "\Omega\Sigma cf"'] & \Omega\Sigma Cf \cdots &
\end{tikzcd}$$

where the middle row is the fibration-cofibration sequence of f, the upper row is the Σ-image of the latter, shifted forward of three steps, and dually for the lower row. The commutative squares come from the naturality of counit x and unit y; the rectangles commute because their rows are zero: $kf.\partial = 0$, $\delta.cf = 0$ (5.5.2).

Further, *if* **A** *is homotopical*, it is possible to fill-in the dotted arrows, by means of two adjoint morphisms, the *counit* and *unit* of f:

(2) $u_f\colon \Sigma Kf \to Cf$, $\qquad\qquad v_f\colon Kf \to \Omega Cf$,

producing new squares, which are commutative *up to homotopy* (III.7.4). It is easy to see that $x_A \simeq u_f$ for f: A → 0, while $y_A \simeq v_g$, for g: 0 → A.

A homotopical category **A** will be said to be *h-stable* if it is pointed and all the maps $u_f\colon \Sigma Kf \to Cf$, $v_f\colon Kf \to \Omega Cf$ are homotopy equivalences (whence all $x_A\colon \Sigma\Omega A \to A$, $y_A\colon A \to \Omega\Sigma A$ are also so). The homotopical category $C_*\mathbf{A}$ of chain complexes over an additive category **A** is stable in a stricter sense: all the above maps are isomorphisms.

In an h-stable homotopical category, all the vertical arrows in the adjunction diagram of a map are homotopy equivalences: the fibration-cofibration sequence of f is turned into itself, up to homotopy, by the functors Σ and Ω, with a three-place shift forwards or backwards, respectively.

6.3. Semihomogeneous theories. A *semihomogeneous theory* (or *pseudoconnected sequence of h-functors*) on the pointed semihomotopical category **A**, with values into the category **B**, is a family $H = ((H_n), (h_n))$ indexed on integral numbers, where:

(ht.0) *Naturality and homotopical invariance*: for every $n \in \mathbb{Z}$,

(1) $H_n\colon \mathbf{A} \to \mathbf{B}, \qquad h_n\colon H_n \to H_{n+1}.\Sigma\colon \mathbf{A} \to \mathbf{B},$

are respectively an h-functor (turning homotopical arrows into equal ones) and a natural transformation. This condition is selfdual, since one can equivalently assign a natural transformation:

(2) $k_n\colon H_n.\Omega \to H_{n+1}\colon \mathbf{A} \to \mathbf{B},$

connected to the former by means of the following conversion formulae:

(3) $k_nA = H_{n+1}x_A \circ h_n\Omega A = (H_n\Omega A \to H_{n+1}(\Sigma\Omega A) \to H_{n+1}A),$

(4) $h_nA = k_n\Sigma A \circ H_ny_A = (H_nA \to H_n(\Omega\Sigma A) \to H_{n+1}\Sigma A).$

6.4. Two pseudosequences. Given a semihomogeneous theory H, on the pointed semihomotopical category **A**, a morphism $f\colon A \to B$ generates an unbounded (solid) commutative diagram in **B** (where $n' = n-1$, $n'' = n+1$), the *double pseudosequence* of f:

(1)
$$\begin{tikzcd}
 & & & & & & H_{n-2}A \arrow[d] \arrow[r, dashed] & {} \\
 & & & H_{n'}\Omega A \arrow[r, "H_{n'}\Omega f"] \arrow[d, "k_{n'}A"] & H_{n'}\Omega B \arrow[r, "H_{n'}\partial"] \arrow[d, "k_{n'}B"] & H_{n'}Kf \arrow[r, "H_{n'}kf"] \arrow[d, dashed, "h_{n'}f"] & H_{n'}A \arrow[r, dashed, "H_{n'}f"] \arrow[d, "h_{n'}A"] & {} \\
 {} \arrow[r, dashed] & H_n\Omega B \arrow[r] \arrow[d, "k_nB"] & H_nKf \arrow[r] \arrow[d, dashed, "h_nf"] & H_nA \arrow[r] \arrow[d, "h_nA"] & H_nB \arrow[r] \arrow[d, "h_nB"] & H_nCf \arrow[r] & H_n\Sigma A \arrow[r, dashed] & {} \\
 {} \arrow[r, dashed, "H_{n''}f"'] & H_{n''}B \arrow[r, "H_{n''}cf"'] \arrow[d] & H_{n''}Cf \arrow[r, "H_{n''}\delta"'] & H_{n''}\Sigma A \arrow[r, "H_{n''}\Sigma f"'] & H_{n''}\Sigma B & & & \\
 {} \arrow[r, dashed] & H_{n+2}B & & & & & &
\end{tikzcd}$$

since the squares are commutative by naturality of h or k, while the rows of the rectangles are zero, as in the adjunction diagram 6.2.1. If **A** is homotopical, there are morphisms $h_nf\colon H_nKf \to H_{n+1}Cf$ (deriving from the adjoint morphisms $u_f\colon \Sigma Kf \to Cf$, $v_f\colon Kf \to \Omega Cf$ of 6.2.2) which complete commutatively the diagram (III.8.3).

From this diagram (1) we extract the *Σ-pseudosequence* of f ($n'' = n+1$):

(2)
$$\begin{tikzcd}
 & & & \mathbf{H_nA} \arrow[r, "H_nf"] \arrow[d, "h_nA"] & \mathbf{H_nB} \arrow[r, "H_ncf"] \arrow[d, "h_nB"] & \mathbf{H_nCf} \\
\mathbf{H_{n''}A} \arrow[r, "H_{n+1}f"'] \arrow[d, dashed] & \mathbf{H_{n''}B} \arrow[r, "H_{n+1}cf"'] \arrow[d, dashed] & \mathbf{H_{n''}Cf} \arrow[r, "H_{n+1}\delta"'] \arrow[ur, dashed, "\partial"] & H_{n''}\Sigma A \arrow[r, "H_{n+1}\Sigma f"'] & H_{n''}\Sigma B & \\
 {} & {} & & & &
\end{tikzcd}$$

and the *Ω-pseudosequence* of f (n' = n-1):

(3)
$$\begin{tikzcd}
 & & & & {} \arrow[d, dashed] & {} \arrow[d, dashed] \\
 & H_{n'}\Omega A \arrow[r] \arrow[d, "k_{n-1}A"'] & H_{n'}\Omega B \arrow[r, "H_{n-1}\partial"] \arrow[d, "k_{n-1}B"'] & \mathbf{H_{n'}Kf} \arrow[r, "H_{n-1}kf"] & \mathbf{H_{n'}A} \arrow[r, "H_{n-1}f"] & \mathbf{H_{n'}B} \\
\mathbf{H_nKf} \arrow[r, "H_nkf"'] & \mathbf{H_nA} \arrow[r, "H_nf"'] & \mathbf{H_nB} \arrow[ur, dashed, "\partial"'] & & &
\end{tikzcd}$$

highlighting in bold characters the *main part* of these diagrams: if h (resp.: k) is a functorial *iso*morphism, the main part of the former (resp.: the latter) will supply the homological (resp.: homotopical) sequence of f, with the dotted differential, while the remaining auxiliary part is to supply the exactness of the sequence, around its "glueing points".

6.5. Homological theories. A semihomogeneous theory H will be said to be a *homological theory* if:

(ht.1) *Σ-stability*: all transformations $h_n\colon H_n \to H_{n+1}.\Sigma$ are isomorphisms,

(ht.2) *Σ-exactness*: for every map $f\colon A \to B$ in **A** and every n, the sequence (1) is exact in **B** (1.2):

$$(1) \qquad H_nA \xrightarrow{H_nf} H_nB \xrightarrow{H_ncf} H_nCf \xrightarrow{H_n\delta} H_n\Sigma A \xrightarrow{H_n\Sigma f} H_n\Sigma B.$$

Then, because of the stability condition, each functor H_k determines H_n for $n \leq k$: $H_n \cong H_k.\Sigma^{k-n}$. Because of both conditions, the Σ-pseudosequence just considered (6.4.2) becomes the exact *homology sequence* of f in **B**, with differential ∂ as specified below:

$$(2) \qquad \dots\ H_nA \xrightarrow{H_nf} H_nB \xrightarrow{H_ncf} H_nCf \xrightarrow{\partial_n} H_{n-1}A\ \dots$$

$$(3) \qquad \partial_n = (h_{n-1}A\,)^{-1}.H_n\delta\colon H_nCf \to H_{n-1}A \quad (\delta\colon Cf \to \Sigma A,\ h_{n-1}A\colon H_{n-1}A \to H_n\Sigma A).$$

If the category **A** is homotopical, it suffices in (ht.2) to require the exactness of the sequence $H_nA \to H_nB \to H_nCf$, since in the (homotopically commutative) contracted cofibration diagram of f (5.9.1), the vertical morphisms s_0 and s'_1 are homotopy equivalences, so that the exactness of (1) in H_nCf or in $H_n\Sigma A$ follows from (4) applied, respectively, to the morphism $x_1 = cf\colon B \to Cf$ or to $x_2 = c(x_1)\colon B_2 \to B_3$.

6.6. Homotopical theories. By duality in the domain, a semihomogeneous theory π will be said to be a *homotopical theory* if:

(ht.1*) *Ω-stability*: all transformations $k_n\colon \pi_n.\Omega \to \pi_{n+1}$ are isomorphisms,

(ht.2*) *Ω-exactness*: for every map $f\colon A \to B$ in **A** and every n, the sequence (1) is exact in **B** (1.2):

$$(1) \qquad H_n\Omega A \xrightarrow{\pi_n\Omega f} \pi_n\Omega B \xrightarrow{\pi_n\delta} \pi_nKf \xrightarrow{\pi_nkf} \pi_nA \xrightarrow{\pi_nf} \pi_nB.$$

Now the morphism f produces the exact *homotopy* sequence:

$$(2) \qquad \ldots\ \pi_nKf \xrightarrow{\pi_nkf} \pi_nA \xrightarrow{\pi_nf} \pi_nB \xrightarrow{\partial_n} \pi_{n-1}Kf\ \ldots$$

(3) $\partial_n = \pi_{n-1}\partial.(k_{n-1}B)^{-1}\colon \pi_nB \to \pi_{n-1}Kf.$

Again, if the category **A** is homotopical, it suffices in (ht.2*) to require the exactness of the sequence in π_nA.

In a homotopical theory π, each functor π_k determines π_n for $n \geq k$: $\pi_n \cong \pi_k.\Omega^{n-k}$. For a *positive* homotopical theory these axioms, of course, are just assumed for $n \geq 0$.

6.7. Homogeneous theories. Let **A** be pointed homotopical. A *homogeneous* theory, from **A** to **B**, will be a semihomogeneous theory H which is both homological and homotopical and verifies also:

(ht.3) *central isomorphism axiom*: for every map $f\colon A \to B$, the morphism $h_nf\colon H_nKf \to H_{n+1}Cf$ recalled in 6.4 is an isomorphism.

In a homogeneous theory, every H_n determines all the other:

(1) $H_n A \cong H_0(\Omega^nA), \qquad H_{-n} A \cong H_0(\Sigma^nA) \qquad (n \geq 0).$

Further, the double pseudosequence 6.4.1 collapses, since all its vertical arrows are isomorphisms.

6.8. Theorem: characterization of homogeneous theories. Let **A** be a pointed homotopical category and $H = (H_n, h_n, k_n)$ a *semihomogeneous* theory over **A**.

a) In the presence of the axioms (ht. 1, 1*, 3), the homology and homotopy sequences (8.5-6) of H coincide up to isomorphism:

$$
(1)\qquad
\begin{array}{ccccccccc}
\ldots & H_{n+1}B & \xrightarrow{\partial'_{n+1}} & H_nKf & \xrightarrow{H_nkf} & H_nA & \xrightarrow{H_nf} & H_nB & \ldots \\
 & \| & & \downarrow{\scriptstyle h_nf} & & \| & & \| & \\
\ldots & H_{n+1}B & \xrightarrow[H_{n+1}cf]{} & H_{n+1}Cf & \xrightarrow[\partial''_{n+1}]{} & H_nA & \xrightarrow[H_nf]{} & H_nB & \ldots
\end{array}
$$

so that Σ-exactness (ht.2) is equivalent to Ω-exactness (ht.2*); if they hold, H is homogeneous and (1) is its exact sequence.

b) If the category **B** is exact and H is both Σ-stable and Ω-stable (ht. 1, 1*), then H is homogeneous iff it verifies two conditions out of the remaining three: Σ-exactness (ht.2), Ω-exactness (ht.2*) or central isomorphism (ht.3).

c) If **A** is h-stable (7.8), H is Σ-stable iff it is Ω-stable, and in this case necessarily satisfies the central isomorphism axiom (ht.3). In particular, H is homological iff it is homotopical, iff it is homogeneous: such theories are characterized by the following axioms: (ht. 0, 1, 2) or equivalently (ht. 0, 1*, 2*).

Proof. a) and b) are proved in III.8.9; c) is an easy consequence.

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