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A NEW PROOF OF REITERMAN'S THEOREM

by Grzegorz JARZEMBSKI

In [3] J.Reiterman introduced a concept of an implicit operation in order to get a very elegant finite analog of the Birkhoff variety theorem. In this note we analyse implicit operations from a categorical point of view. As a result we obtain a simple, purely categorical proof of Reiterman theorem.

1 Basic concepts

For all unexplained notions of category theory we refer the reader to [2].

We shall identify classes of objects of a category considered with full subcategories they generate. For every natural number n, by $\tilde{n}$ we denote the set $\{1, 2, \ldots, n\}$.

Let $(\mathcal{A}, U : \mathcal{A} \longrightarrow Set)$ be an arbitrary concrete category over sets. For any natural number n, by an *n-ary implicit operation* in $\mathcal{A}$ we mean any natural transformation $\phi : U^n \longrightarrow U$ [3].

Roughly speaking, an implicit operation is a family of functions $(\phi_A : U^n A \longrightarrow UA : A \in Ob\mathcal{A})$ "compatible" with all $\mathcal{A}$-morphisms.

By $IO(\mathcal{A})_n$ we shall denote the class of all n-ary implicit operations in $\mathcal{A}$.

Definition 1 *Any pair (ϕ, ψ) of n-ary implicit operations in $\mathcal{A}$ is called here an "equation"; (notation $(\phi = \psi)$).*

We say that an equation $(\phi = \psi)$ is satisfied in $B \in Ob\mathcal{A}$ iff $\phi_B = \psi_B$.

For a given set of equations E, by Mod$E$ we denote the full subcategory of $\mathcal{A}$ consisting of $\mathcal{A}$-objects satisfying all equations in E.

We call a full subcategory $\mathcal{D} \subset \mathcal{A}$ *equationally definable* iff $\mathcal{D} =$ Mod$E$ for some set of equations E.

The following observation needs only a routine verification:

Lemma 1 *Every equationally definable full subcategory $\mathcal{D} \subset \mathcal{A}$ satisfies the following:*

1. *$\mathcal{D}$ is closed under formation of all exisiting concrete limits,*

2. *whenever $A \in \mathcal{D}$, $h : A \longrightarrow B$ and Uh is a surjection, then $B \in \mathcal{D}$ ($\mathcal{D}$ is closed under formation of homomorphic images),*

3. *whenever $A \in \mathcal{D}$, $m : B \longrightarrow A$ and Um is a monomorphism, then $B \in \mathcal{D}$ ($\mathcal{D}$ is closed under formation of subobjects).*

J. Reiterman has proved the converse of Lemma 1 for equationally definable classes of finite algebras of an arbitrary finite type Ω ([3]). His result has been next generalized by Banaschewski ([1]) for arbitrary finitary types. In this paper we present a new proof of Reiterman theorem based on a categorical analysis of the concept of an implicit operation.

2 Two restrictions

Definition 2 *A concrete category $(\mathcal{A}, U)$ is said to be small-based provided for every natural number n $IO(\mathcal{A})_n$ is a set.*

Let $(\tilde{n} \downarrow U)$ denote the category of all U-morphisms with a domain $\tilde{n}$. Let

$$U_n : (\tilde{n} \downarrow U) \longrightarrow Set$$

be the "forgetful functor", i.e. $U_n(h : \tilde{n} \longrightarrow UA, A) = UA$, for every (h, A) in $(\tilde{n} \downarrow U)$.

Lemma 2 *A concrete category $(\mathcal{A}, U)$ is small-based if and only if for every natural number n there exists a limit of the functor U_n and then*

$$IO(\mathcal{A})_n \cong limU_n$$

Proof. A limit of U_n is formed by the set $IO(\mathcal{A})_n$ together with limit projections $\pi_{(h,A)} : IO(\mathcal{A})_n \longrightarrow U_n(h, A) = UA$ such that

$$\pi_{(h,A)}(\phi) = \phi_A(h)$$

for every $(h, A) \in (\tilde{n} \downarrow U)$ and $\phi \in IO(\mathcal{A})_n$.

Definition 3 *We say that a concrete category* (A, U) *has enough implicit operations iff for every* $A, B \in \mathcal{A}$ *and* $h : UA \longrightarrow UB$, $h = Ug$ *for some* $\mathcal{A}$*-morphism* $g : A \longrightarrow B$ *provided* h *preserves all implicit operations* ($h \cdot \phi_A = \phi_B \cdot h^n$ *for every* $n \in N$ *and* $\phi \in IO(\mathcal{A})_n$*).*

Lemma 3 *Assume that a* $(\mathcal{A}, U)$ *is a concrete, small-based with enough implicit operations. Then there exists a (finitary) monad* T *over* Set *such that the following conditions are satisfied:*

1. $(\mathcal{A}, U)$ *is concretely isomorphic to a full subcategory of the Eilenberg-Moore category* Set^T ,
2. *for every* $n \in N$, $IO(\mathcal{A})_n \cong limU_n$ *is a carrier of a finitely generated monadic* T*-algebra such that every limit projection is a* T*-morphism.*

Proof. Since $(\mathcal{A}, U)$ is small-based, the family of all implicit operations form a finitary type: $IO = (IO(\mathcal{A})_n : n \in N)$.

Let T be a monad over Set such that its Eilenberg-Moore category Set^T is the category of all IO-algebras. Since the concrete category considered has enough implicit operations, it is concretely isomorphic to a full subcategory of T-algebras.

In the considered case the functor U_n factorizes through the forgetful functor $U^T : Set^T \longrightarrow Set$ in an obvious way. U^T creates limits, hence, by Lemma 2, every set $IO(\mathcal{A})_n$ carries a structure of a monadic T-algebra such that every limit-projection $\pi_{(h,A)} : IO(\mathcal{A})_n \longrightarrow A$ is a homomorphism.

Obviously, every $IO(\mathcal{A})_n$ is then a finitely generated algebra of the type IO - it is generated by the set of natural projections $\{\pi_i^n : U^n \longrightarrow U : i = 1, 2, \ldots, n\}$.

It must be stressed however that a monad T satisfying the conditions of Lemma 3 is not uniquely determined. It need not to be finitary too. An embedding $\mathcal{A} \longrightarrow Set^T$ will be used only in order to support an investigation on equationally definable subcategories of $\mathcal{A}$. As we will show later, in particular cases a "good" choice of a monad T is crucial.

3 Characterization of equationally definable subcategories

Assume that a concrete category $(\mathcal{A}, U)$ is a small-based category with enough implicit operations. Let $T = (T, \mu, \eta)$ be an arbitrary but fixed monad over Set such that the conditions stated in Lemma 3 are satisfied. We shall assume that $\mathcal{A}$ is a full subcategory of the Eilenberg-Moore category Set^T. The T-algebra of n-ary implicit operations will be denoted by $IO(\mathcal{A})_n$.

Let $\pi^n : \tilde{n} \longrightarrow IO(\mathcal{A})_n$ be a function such that for every $i \in \tilde{n}$, $\pi^n(i) = \pi_i^n : U^n \longrightarrow U =$ the i-th natural projection.

We assume also that the extension of π^n to the T-morphism $(T\tilde{n}, \mu_{\tilde{n}}) \longrightarrow IO(\mathcal{A})_n$ is surjective for every natural number n.

Definition 4 *We call a T-algebra D a $T_{\mathcal{A}}$-algebra iff for every $n \in N$ and a function $f : \tilde{n} \longrightarrow U^T D$ there exists a (unique) T-morphism $f^* : IO(\mathcal{A})_n \longrightarrow D$ such that*

$$f^* \cdot \pi^n = f$$

We shall call f^* an extension of f.

Clearly, $\mathcal{A}$ is a subclass of $T_{\mathcal{A}}$-algebras: for every $B \in Ob\mathcal{A}$ and $h : \tilde{n} \longrightarrow UB$ its extension is the limit-projection $\pi_{(h,A)} : IO(\mathcal{A})_n \longrightarrow B$ (Lemma 2).

Notice also that every $IO(\mathcal{A})_n$ is a $T_{\mathcal{A}}$-algebra. This is a consequence of the fact that the class of $T_{\mathcal{A}}$-algebras is closed under formation of limits. It is easily checked that the class of $T_{\mathcal{A}}$-algebras is closed within Set^T under formation of homomorphic images and subalgebras, i.e. the class of $T_{\mathcal{A}}$-algebras is the "Birkhoff subcategory generated by $\mathcal{A}$" within Set^T.

Definition 5 *For each $\phi \in IO(\mathcal{A})_n$ and a $T_{\mathcal{A}}$-algebra D we define an n-ary operation in D, $\phi_D : (U^T D)^n \longrightarrow U^T D$ as follows:*

$$\phi_D(h) = h^*(\phi)$$

for every $h : \tilde{n} \longrightarrow U^T D$.

Thus h^* gives "values of all n-ary implicit operations in D at the valuation h".

It is easily checked that T-morphisms between $T_{\mathcal{A}}$-algebras preserve these operations.

Having all implicit operations extended to all $T_{\mathcal{A}}$-algebras we extend also a notion of satisfaction for equations of implicit operations:

Definition 6 *Let $\phi, \psi \in IO(\mathcal{A})_n$. A $T_{\mathcal{A}}$-algebra A satisfies the equation $(\phi = \psi)$ iff for every $h : \tilde{n} \longrightarrow U^T A$,*

$$h^*(\phi) = h^*(\psi)$$

If $A \in Ob\mathcal{A}$ and $h : \tilde{n} \longrightarrow UA$ then $h^*(\phi) = \pi_{(h,A)}(\phi) = \phi_A(h)$, i.e. the newly defined concept of satisfaction and that of Definition 1 coincide for $\mathcal{A}$-objects.

For any class $\mathcal{B}$ of $T_{\mathcal{A}}$-algebras we write $H(\mathcal{B})$, $S(\mathcal{B})$, $P(\mathcal{B})$ in order to denote the class of all homomorphic images, subalgebras and products of the algebras from $\mathcal{B}$, resp.

We call a class $\mathcal{B}$ an *HSP-class* iff $\mathcal{B} = HSP(\mathcal{B})$.
Since we deal with finitary implicit operations only, we shall use one more closure operator. In what follows by $C(\mathcal{B})$ we shall denote the class of all $T_{\mathcal{A}}$-algebras with all finitely generated subalgebras in $\mathcal{B}$ (a monadic T-algebra A is said to be *finitely generated* provided it is isomorphic to a quotient of a free monadic T-algebra $(T\tilde{n}, \mu_{\tilde{n}})$ for some natural number n).

Theorem 4 *For an arbitrary class $\mathcal{B}$ of $T_{\mathcal{A}}$-algebras the following conditions are equivalent:*

1. *$\mathcal{B}$ is an HSP-class and $\mathcal{B} = C(\mathcal{B})$*

2. *There exists a set E of equations (of implicit operations) such that $\mathcal{B}$ consists of all $T_{\mathcal{A}}$-algebras satisfying all equations in E.*

Proof. 2. $\Rightarrow$ 1. The equations $\mathcal{B} = P(\mathcal{B}) = H(\mathcal{B}) = S(\mathcal{B})$ need only a routine verification.

For every $T_{\mathcal{A}}$-algebra B and a function $h : \tilde{n} \longrightarrow U^T B$ its extension h^* factorizes as a surjective T-morphism followed by a monomorphism:

$$h^* = m \cdot e : IO(\mathcal{A}_n) \longrightarrow D_h \longrightarrow B$$

Clearly, D_h is a finitely generated $T_{\mathcal{A}}$-algebra. Hence a $T_{\mathcal{A}}$-algebra B satisfies an equation $(\phi = \psi)$ iff every finitely generated subalgebra of B satisfies it.

This proves $C(\mathcal{B}) \subset \mathcal{B}$.

1. $\Rightarrow$ 2. Since $\mathcal{B}$ is closed under products and subalgebras, for every natural number n there exists a "reflection map" - a surjective T-morphism $e_n : IO(\mathcal{A})_n \longrightarrow B_n$ such that $B_n \in \mathcal{B}$ and every morphism of T-algebras $h : IO(\mathcal{A})_n \longrightarrow A$ with $A \in \mathcal{B}$ factorizes through e_n. Let

$$E_n = ker(e_n) = \{(\phi, \psi) : \phi, \psi \in IO(\mathcal{A})_n, e_n(\phi) = e_n(\psi)\}$$

and

$$E = \bigcup(E_n : n \in N)$$

We prove that E is a set of equations we are looking for.

Clearly, each $T_{\mathcal{A}}$-algebra in $\mathcal{B}$ satisfies all equations in E.

Assume that A is a finitely generated $T_{\mathcal{A}}$-algebra satisfying all equations in E. A is finitely generated hence there exists a surjective T-morphism $f : IO(\mathcal{A})_n \longrightarrow A$ for some $n \in N$. Since A satisfies all equations in E, we obtain $E_n \subset ker f$. Hence $f = f^o \cdot e_n$ for some T-morphism f^o. f is surjective, hence f^o is a surjective morphism, too.

Thus $A \in H(\mathcal{B}) = \mathcal{B}$.

An arbitrary $T_{\mathcal{A}}$-algebra B satisfies all equations form E, iff all its finitely generated subalgebras satisfy all these equations, i.e. if all those subalgebras are in $\mathcal{B}$. This means $B \in C(\mathcal{B}) = \mathcal{B}$.

The proof is complete.

Corollary 5 . *For any class $\mathcal{B}$ of $T_{\mathcal{A}}$-algebras, the class $CHSP(\mathcal{B})$ is the smallest equationally definable class of $T_{\mathcal{A}}$-algebras containing $\mathcal{B}$.*

Proof. Observe that $HC(\mathcal{B}) \subset CH(\mathcal{B})$ and $SPC(\mathcal{B}) \subset CSP(\mathcal{B})$ for any class of $T_{\mathcal{A}}$-algebras $\mathcal{D}$. We omit a routine calculation.

The next Corollary summarizes investigation of this section.

Corollary 6 *For any class $\mathcal{D} \subset \mathcal{A}$, the following conditions are equivalent*

1. *$\mathcal{D}$ is an equationally definable class,*

2. $\mathcal{D} = CHSP(\mathcal{D}) \cap \mathcal{A}$

4 A new proof of Reiterman theorem

Consider now equationally definable classes in the concrete category $Alg_{fin}\Omega$ of all finite algebras of a given finitary type Ω. We are going to give a new proof of Reiterman characterization theorem based on the results of the previous section. Our proof also covers the generalization of Reiterman theorem given by Banaschewski ([1]).

Since $Alg_{fin}\Omega$ is a small category, it is small-based. It follows from Lemma 2 that in this case the set of n-ary implicit operations is represented as a limit of the poset of all finite quotients of a free Ω-algebra generated by the set $\tilde{n} = \{1, 2, \ldots, n\}$. From this it easily follows that every implicit operation ϕ is "locally explicit" - i.e., for every finite algebra A there is a term t with $t_A = \phi_A$ ([3]).

Obviously, the concrete category $Alg_{fin}\Omega$ has enough implicit operations.

For the category $Alg_{fin}\Omega$ we may consider three (at least) concrete full embeddings of it into the following Eilenberg-Moore categories over sets:

- the category $Alg\Omega$ of all Ω-algebras,

- the category $AlgIO$, where IO is the type built of all implicit operations in $Alg_{fin}\Omega$,

- the category of all compact Hausdorff IO-algebras and continuous homomorphisms (by endowing each finite algebra with the discrete topology).

In our proof of Reiterman theorem we shall use the third embedding.

Recall that by a *pseudovariety* of finite Ω-algebras we mean each class of finite algebras closed under formation of finite products, subalgebras and homomorphic images.

Theorem 7 *(Reiterman theorem)*
For any class $\mathcal{D} \subset Alg_{fin}\Omega$ the following conditions are equivalent:

1. *$\mathcal{D}$ is a pseudovariety ,*

2. *$\mathcal{D} =$ $ModE$ for some set of equations of implicit operations of finite algebras.*

Proof. Each Mod$E$ is clearly a pseudovariety.

Conversely, let D be a pseudovariety. To find E with $D =$ Mod$E$ consider the monad T over Set whose Eilenberg-Moore category is the category of all compact Hausdorff IO-algebras (and continuous homomorphisms).

Clearly, $Alg_{fin}\Omega$ is a full concrete subcategory of Set^T and for every natural n, the set of n-ary implicit operations in $Alg_{fin}\Omega$ is a carrier of a finitely generated monadic T-algebra (Lemma 3).

By Corollary 6, it is enough to prove that $\mathcal{D}$ contains every finite algebra C lying in HSP($\mathcal{D}$) $\subset Set^T$. That is, there is a continuous and surjective homomorphism $e : B \longrightarrow C$, where B is a compact Hausdorff IO-algebra, and B is a closed subalgebra of $\prod_{i\in I} A_i$ for a collection $(A_i : i \in I) \subset \mathcal{D}$.

We may assume that every A_i has the form $A_i = B/\rho_i$ for some congruence ρ_i on B. Then

$$\bigcap(\rho_i : i \in I) = \nabla_B$$

where ∇_B denotes the diagonal in $B \times B$.

Every congruence ρ_i has a finite index and it is a a kernel of a continuous projection $pr_i : B \longrightarrow A_i$, hence it is a closed subset of the compact space $B \times B$.

The epimorphism $e : B \longrightarrow C$ is continuous, the diagonal ∇_C is open in the compact space $C \times C$, hence the inverse image $ker(e) = (e \times e)^{-1}(\nabla_C)$ is an open subset of $B \times B$ containig ∇_B. Hence

$$\bigcap(\rho_i : i \in I) = \nabla_B \subset ker(e)$$

Since $B \times B$ is compact, there exists a finite subset $I_1 \subset I$ such that

$$\bigcap(\rho_i : i \in I_1) \subset ker(e)$$

Clearly, the congruence $\bigcap(\rho_i : i \in I_1)$ has a finite index. Thus $B_0 = B/\bigcap(\rho_i : i \in I_1)$ is a finite algebra and C is a homomorphic image of B_0.
Moreover, B_0 is a subalgebra of the product of the family of finite algebras $\{A_i : i \in I_1\}$.

Hence $C \in \mathcal{D}$.

The proof is complete.

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