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RATIONAL NILPOTENT GROUPS AS SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES

by Salvina PICCARRETA

RESUME. Soit X un CW-complexe. On considère le groupe $\mathcal{E}(X)$ dont les éléments sont les classes d'homotopie des équivalences de self-homotopie de X, et ses sous-groupes $\mathcal{E}_\sharp(X)$ et $\mathcal{E}_*(X)$ dont les éléments induisent respectivement l'identité en homotopie et en homologie. Dans cet article, les groupes rationnels de nilpotence 1, de nilpotence 2 et de rang inférieur ou égal à 6, dont le sous-groupe commutateur a un rang égal à 1, sont réalisés comme $\mathcal{E}_\sharp(X)$ et $\mathcal{E}_*(X)$ lorsque X est la rationalisation d'un CW-complexe fini.

1 Introduction

Let X be a CW complex. We denote by $\mathcal{E}(X)$ the set of homotopy classes of self homotopy equivalences of X. It is well known that $\mathcal{E}(X)$ is a group with respect to composition of homotopy classes. In this paper we consider the subgroups $\mathcal{E}_\sharp(X) = \mathcal{E}_{\sharp N}(X)$ of maps inducing the identity on the homotopy groups of X up to dimension N (homological if X is a finite complex, homotopical if X has a finite number of non-trivial, finitely generated homotopy groups) and $\mathcal{E}_*(X)$ of the maps inducing the identity on the homology groups.

Dror and Zabrodsky (see [9]) proved that when X has the homotopy type of a finite complex, $\mathcal{E}_\sharp(X)$ and $\mathcal{E}_*(X)$ are nilpotent and Maruyama (see [19], [20]) showed that for any set of primes P the homomorphisms:

$$\mathcal{E}_\sharp(X) \to \mathcal{E}_\sharp(X_P)$$

$$\mathcal{E}_*(X) \to \mathcal{E}_*(X_P)$$

induced by the P-localizations of X are the P-localizations of nilpotent groups. Hence $\mathcal{E}_\sharp(X_\mathbb{Q})$ and $\mathcal{E}_*(X_\mathbb{Q})$ are rational nilpotent groups.
In [2] and [3] Arkowitz and Curjel prove that $\mathcal{E}_\sharp(X)$, $\mathcal{E}_*(X)$ and all their subgroups are finitely generated if X is a 1-connected finite complex.
A group G is said to be *nilpotent* of nilpotency n, if $n+1$ is the maximum integer, such that the n-th commutator, G^n, defined recursively by $G^n = [G, G^{n-1}]$ is not trivial.
A nilpotent group G has a cyclic series, and it is said to be of *finite rank,* written $\rho(G) < \infty$, if the number of infinite cyclic factors is finite, this number, $\rho(G)$ is is called the *Hirsch rank* of G.
Any finitely generated nilpotent group is of finite rank (see Remark 2.7 in [2]). This means in particular that $\mathcal{E}_\sharp(X)$ has finite rank if X is a 1-connected finite complex. Furthermore, $\rho(\mathcal{E}_\sharp(X)) = \rho(\mathcal{E}_\sharp(X)/T)$, where T is the torsion subgroup, because $T \lhd \mathcal{E}_\sharp(X)$ implies $\rho(\mathcal{E}_\sharp(X)) = \rho(\mathcal{E}_\sharp(X)/T) + \rho(T)$ and the rank of any finite rank group is equal to zero if and only if the group is periodic.
The Mal'cev completion is a functor from the category of finitely generated torsion free nilpotent groups (in what follows *f.t.n.* for short) to the category of unipotent algebraic $\mathbb{Q}$-groups (i.e. the category of groups G of matrices with elements in $\mathbb{Q}$, such that for every $x \in G$ $(x - I)^n = 0$ for some positive integer n, where I is the identity matrix) and it is equal to the rationalization on this category [18] (the details of the construction of the functor are in [7]).
Let T be the torsion subgroup of $\mathcal{E}_\sharp(X)$, using the properties of Mal'cev completion we can deduce that for any 1-connected finite complex:

- $\dim_\mathbb{Q}(\mathcal{E}_\sharp(X))_\mathbb{Q} = \rho(\mathcal{E}_\sharp(X)/T) = \rho(\mathcal{E}_\sharp(X))$
- $nil(\mathcal{E}_\sharp(X)_\mathbb{Q}) = nil(\mathcal{E}_\sharp(X)/T)$.

The same properties hold for $\mathcal{E}_*(X)$.
A basic problem about self-equivalences is the realizability of $\mathcal{E}(X)$, i.e., when for a given group Π there exists a space X, such that $\mathcal{E}(X) \cong \Pi$, and in particular when there exists a finite CW complex X, such that $\mathcal{E}(X) \cong \Pi$. Actually very little is known about the problem (see [1]).

In this paper we give examples of unipotent algebraic $\mathbb{Q}$-groups that can be realized as $\mathcal{E}_\sharp(X_\mathbb{Q})$ (see Theorems 3.1, 3.3, 3.6, 3.7) and $\mathcal{E}_*(X_\mathbb{Q})$ (see Theorems 4.1, 4.3, 4.2, 4.4) for some CW complex X of dimension N. We will obtain these results working on the Sullivan or Quillen models of the space considered. Let $\mathcal{M}$ (respectively L) be the Sullivan (resp. the Quillen) model of X. It is well known ([11], Chap. XIV) that there is a contravariant equivalence between the homotopy category of rational spaces of finite type and the homotopy category of finite type minimal algebras; thus, there is an anti-isomorphism between $\mathcal{E}(X_\mathbb{Q})$ and $\mathcal{E}(\mathcal{M})$. Now the i-th degree indecomposables, $Q^i(\mathcal{M})$, of $\mathcal{M}$ correspond to the i-th rational homotopy group of X ([11], p.136) and so $\mathcal{E}_\sharp(X_\mathbb{Q})$ is anti-isomorphic to $\mathcal{E}_{\sharp N}(\mathcal{M})$, group of homotopy classes of self-equivalences which induce the identity on $Q^i(\mathcal{M})$, for $i \leq N$ (see [5], Remark 2.3). Furthermore there exists a covariant equivalence between the homotopy category of rational spaces of finite type and the homotopy category of finite type minimal Lie algebras ([23], Chap. III). This gives an isomorphism between $\mathcal{E}(X_\mathbb{Q})$ and $\mathcal{E}(\mathrm{L})$. The i-th degree indecomposables, $Q^i(\mathrm{L})$, of L correspond to the i-th rational homology group of X. So there is an isomorphism between $\mathcal{E}_*(X_\mathbb{Q})$ and $\mathcal{E}_\sharp(\mathrm{L})$ group of homotopy classes of self-equivalences which induce the identity on $Q^i(\mathrm{L})$.

2 Minimal models and self equivalences

In what follows we will use the conventions of [8], [11] and [16]. The direct sum of homogeneous vector spaces V^k, $k \in \mathbb{Z}, k \geq 0\}$, over the rationals $\mathbb{Q}$, is called a graded vector space. An element $v \in V$ is homogeneous if $v \in V^k$ for some k and for a homogeneous element $v \in V$ denote by $|v|$ the degree k. Usually we will assume that V is finite dimensional and, if $v_1, \ldots, v_r$ is a base for V, we will write $V = < v_1, \ldots, v_r >$.

Let $\Lambda(V)$ be the exterior algebra of the vector space V. We use the notion of homotopy for maps $\varphi, \psi : \mathcal{M} \longrightarrow \mathcal{N}$ of minimal algebras given in [16], p.240. Given an algebra $\mathcal{M} = \Lambda(V)$ with differential d, let us define a differential graded (DG for short) algebra $\mathcal{M}^I = \Lambda(V \oplus \overline{V} \oplus \widehat{V})$, with differential also denoted by d, as follows: $\widehat{V}$ is an isomorphic copy of V and $\overline{V}$ is the desuspension of V (i.e., $\overline{V}^p = V^{p+1}$); moreover the differential d of $\mathcal{M}^I$ agrees with the differential on $\mathcal{M}$, $d(\overline{v}) = \hat{v}$ and $d(\hat{v}) = 0$, for $\overline{v} \in \overline{V}$ and $\hat{v} \in \widehat{V}$. Furthermore there is a degree -1 derivation $i : \mathcal{M}^I \longrightarrow \mathcal{M}^I$ defined on generators by $i(v) = \overline{v}$, $i(\overline{v}) = 0$ and $i(\hat{v}) = 0$. This allows us to construct a degree 0 derivation $\gamma : \mathcal{M}^I \longrightarrow \mathcal{M}^I$ by setting $\gamma = di + id$ and the map $\alpha : \mathcal{M}^I \longrightarrow \mathcal{M}^I$:

$$\alpha = \sum_{n=0}^{\infty} \frac{1}{n!}\gamma^n.$$

We say that there exists a homotopy beginning at φ and ending at ψ, if there is a DG algebra morphism $H : \mathcal{M}^I \longrightarrow \mathcal{N}$, such that $H|_{\mathcal{M}} = \varphi$ and $H\alpha|_{\mathcal{M}} = \psi$.

The analogous definition of homotopy for maps $\varphi, \psi : \mathrm{L} \longrightarrow \mathrm{L}'$ of minimal Lie algebras is given in [23], p. 49.

Let $\mathcal{M}$ be a minimal algebra. We can define the group of homotopy equivalences on $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, the subgroups $\mathrm{Aut}_\sharp(\mathcal{M})$ and $\mathrm{Aut}_{\sharp N}(\mathcal{M})$ of homotopy equivalences which induce respectively the identity on all the indecomposable elements and on the indecomposable elements with dimension less than or equal to N. We can also define the group of homotopy classes of elements in $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}_\sharp(\mathcal{M})$, which we will denote respectively by $\mathcal{E}(\mathcal{M})$, $\mathcal{E}_\sharp(\mathcal{M})$ and $\mathcal{E}_{\sharp N}(\mathcal{M})$. Analogously for a minimal Lie algebra L we can define $\mathrm{Aut}(\mathrm{L})$, $\mathrm{Aut}_\sharp(\mathrm{L})$, $\mathrm{Aut}_{\sharp N}(\mathrm{L})$, $\mathcal{E}(L)$, $\mathcal{E}_\sharp(\mathrm{L})$ and $\mathcal{E}_{\sharp N}(\mathrm{L})$.

In what follows will be very important to decide whether two self-maps on a minimal model are homotopic. As a particular case of Theorem 2.5 in [5] we obtain the following sufficient conditions for the existence of a homotopy between two maps of DG algebras, which can be easily dualized for DG Lie algebras.

Proposition 2.1 *Let $f, g : (\Lambda(V), d) \longrightarrow (A, d')$ be morphisms of DG algebras, with $\Lambda(V)$ free and 1-connected. Let $K \subseteq V$ be such that $d(V) \subseteq \Lambda(K)$; if $f|_K = g|_K$ and $[f(v) - g(v)] = 0$ for every $v \in V$, then there exists a homotopy between f and g.*

Let $f, g : (\mathbb{L}(V), d) \longrightarrow (L', d')$ be morphisms of DG Lie algebras, with $\mathbb{L}(V)$ free and 1-reduced. Let $K \subseteq V$ be such that $d(V) \subseteq \mathbb{L}(K)$; if

$f|_K = g|_K$ and $[f(v) - g(v)] = 0$ for every $v \in V$, then there exists a homotopy between f and g.

Proof. Let $f, g : (\Lambda(V), d) \longrightarrow (A, d')$ be as assumed. For every $v \in V$ there exists χ_v, such that

$$f(v) - g(v) = d(\chi_v),$$

χ_v being zero for every $v \in K$, thus we can construct the DG map $H : \Lambda(V)^I \longrightarrow A$ defined uniquely by the conditions:

$$\begin{array}{rcl} H(v) = & f(v) & \text{if } v \in V \\ H(\overline{v}) = & -\chi_v & \text{if } \overline{v} \in \overline{V} \\ H(\hat{v}) = & -d(\chi_v) & \text{if } \hat{v} \in \widehat{V} \end{array}$$

H is a DG map, in fact

$$\begin{array}{rcl} H(d(v)) - d(H(v)) & = & f(d(v)) - d(f(v)) = 0 \\ H(d(\overline{v})) - d(H(\overline{v})) & = & H(\hat{v}) - d(-\chi_v) = \\ & = & -d(\chi_v) + d(\chi_v) = 0 \\ H(d(\hat{v})) - d(H(\hat{v})) & = & 0 - d^2(\chi_v) = 0 \end{array}$$

Note that $\alpha(v) = v + \hat{v} + \xi$, where $\xi = \sum \frac{(id)^n(v)}{n!}$, is in the ideal generated in $\Lambda(V)^I$ by the elements $\overline{v}$ for $v \in K$. In fact, for $v \in V$, $d(v) \in \Lambda(K)$ is in the algebra and $i(v)$ is in the ideal generated in $\Lambda(V)^I$, by the elements $\overline{w}$, for $w \in \Lambda(V)$. Thus, $H(\xi) = 0$ and

$$H\alpha(v) = H(v) + H(\hat{v}) = g(v)$$

i.e. H is a homotopy between f and g.
The proof for DG Lie algebras is analogous. □

Next result gives a necessary condition for two maps to be homotopic:

Proposition 2.2 *Let $f, g : (\Lambda(V), d) \longrightarrow (A, d')$ be morphisms of DG algebras with $\Lambda(V)$ free and 1-connected. Given a cocycle $v \in V$, if f is homotopic to g then $f(v) - g(v)$ is a coboundary.*
Let $f, g : (\mathbb{L}(V), d) \longrightarrow (L', d')$ be morphisms of DG Lie algebras with $\mathbb{L}(V)$ free and 1-reduced. Given a cycle $v \in V$, if f is homotopic to g then $f(v) - g(v)$ is a boundary.

Proof. As $dv = 0$, $\alpha(v) = v + \hat{v}$, if there exists a homotopy H between f and g then:

$$\begin{aligned} f(v) - g(v) &= H(v) - H\alpha(v) = H(v) - H(v) - H(\hat{v}) = -H(\hat{v}) = Hd(\overline{v}) \\ &= dH(\overline{v}). \end{aligned}$$

□

3 Unipotent algebraic $\mathbb{Q}$-groups as $\mathcal{E}_\sharp(X_\mathbb{Q})$

Note that the N-dimensional CW complexes considered in this section have minimal models without indecomposables elements in dimension greater than N.
Suppose that $\mathcal{E}_\sharp(X)$ is abelian; then its rationalization is simply the tensor product with $\mathbb{Q}$ and so $\mathcal{E}_\sharp(X_\mathbb{Q})$ is isomorphic to the direct sum of n copies of $\mathbb{Q}$.

Theorem 3.1 *For any positive integer n there is a product of spheres X such that $\mathcal{E}_\sharp(X_\mathbb{Q}) \cong \mathbb{Q}^n$.*

Proof. Given m, k, odd integers $m, k \geq 3$, let $X = (S^m)^k \times (S^{mk})^n$. The Sullivan model of X is

$$\mathcal{M} = \Lambda((x_1, \ldots, x_k, y_1, \ldots, y_n), 0),$$

where

$$|x_i| = m \qquad |y_j| = mk.$$

As $d = 0$ we get $Aut_\sharp(\mathcal{M}) = \mathcal{E}_\sharp(\mathcal{M})$.
In dimension m there are no decomposables and in dimension mk the only decomposable element is $\chi = \prod_{i=1}^k x_i$. It follows that the map:

$$\begin{aligned} F : \mathbb{Q}^n &\longrightarrow \mathcal{E}_\sharp(\mathcal{M}) \\ (p_1, \ldots, p_n) &\longmapsto [\psi] \end{aligned}$$

where

$$\psi(x_i) = x_i$$
$$\psi(y_j) = y_j + p_j\chi$$

is an isomorphism. □

Recall that the Prüfer rank of a group G is the least integer r such that every finitely generated subgroup of G is generated by r elements. Denote by $Tr_1(n, \mathbb{Q})$ the group of all lower triangular square matrices of order n over $\mathbb{Q}$ whose diagonal elements are 1. The famous Theorem of Mal'cev characterizes torsion free nilpotent groups of finite Prüfer rank:

Theorem 3.2 *(see [15]) Let G be a torsion free nilpotent group with Prüfer rank r. Then, for some integer n, the group G is isomorphic to a subgroup of $Tr_1(n, \mathbb{Q})$.*

Denote by $U(n) \subset GL(n, \mathbb{C})$ the group of matrices, U, such that $U\overline{U}^T = I$.

Theorem 3.3 *Let $X_{r,q}$ be the homogeneous space $X_{r,q} = \frac{U(r+1+q)}{U(1)\times U(q)}$ with $q \geq r-1$.*
Then for any positive integer r, $\mathcal{E}_\sharp(X_{r,q\mathbb{Q}}) \cong Tr_1(r, \mathbb{Q})$.

Proof. Let $\mathcal{M}$ be the minimal model of X. As $\mathcal{E}_\sharp(\mathcal{M})$ and $\mathcal{E}_\sharp(X_\mathbb{Q})$ are anti-isomorphic, to prove the proposition it is sufficient to prove that $\mathcal{E}_\sharp(\mathcal{M})$ and $Tr_1(r, \mathbb{Q})$ are anti-isomorphic.
The Sullivan model of the homogeneous space $\frac{U(r+1+q)}{U(1)\times U(q)}$, is described in [6] and is given as follows: $\mathcal{M} = \Lambda(V_0 \oplus V_1)$, where
$V_0 =< u_1, w_1, \ldots, w_r >$, $V_1 =< v_1 >$, and

$$|u_1| = 2 \qquad |w_k| = 2(q+k)+1 \qquad |v_1| = 2q+1$$

$$d|_{V_0} = 0 \qquad d(v_1) = (u_1)^{q+1}.$$

By Proposition 8.4 in [6], for any $\varphi \in \mathrm{Aut}_\sharp(\mathcal{M})$, there exists ψ in $\mathrm{Aut}_\sharp(\mathcal{M})$ homotopic to φ and such that $\psi(u_1) = u_1$, $\psi(v_1) = v_1$ and $\psi(w_k) = w_k + \chi_k$, with $\chi_k \in \Lambda(V_0)$.

If $|\chi_k| = |w_k|$, for $\chi_k \in \Lambda(V_0)$, then $\chi_k \in \Lambda^+(w_1, \ldots, w_r) \otimes \Lambda(u_1)$ for degree reasons, but if $q \geq r-1$:

$$|w_{i_1} w_{i_2} w_{i_3}| \geq |w_1 w_2 w_3| = 2q+3+2q+5+2q+7 > |w_r|,$$

then χ_k is a sum of monomials, involving one and only one w_l.
We can construct the surjective anti-homomorphism:

$$F : Tr_1(r, \mathbb{Q}) \longrightarrow \mathcal{E}_\sharp(\mathcal{M})$$
$$[\alpha_{s,t}] \longmapsto [\psi],$$

where:

$$\psi(u_i) = u_i \quad \psi(v_1) = v_1$$
$$\psi(w_k) = w_k + \sum_{l=1}^{k-1} \alpha_{k,l} w_l (u_1)^{k-l}.$$

By Proposition 2.2 F is also injective. In fact the coboundary in $\Lambda^+(w_1, \ldots, w_r) \otimes \Lambda(u_1)$ with minimum degree is $w_1(u_1)^{q+1}$ and, as $q \geq r-1$:

$$|w_1(u_1)^{q+1}| = 2+2q+1+2q+2 \geq 2+2q+2r+1 > |w_r|.$$

□

In [5] it is proved that $\frac{U(n)}{U(n_1)\times\ldots\times U(n_k)}$ and $\frac{U(m)}{U(n_1)\times\ldots\times U(n_k)} \times S^{2(m+1)-1} \times \ldots S^{2n-1}$, where $m = n_1 + \ldots + n_k$, have the same rational homotopy type. Moreover, $\frac{U(n)}{U(1)\times U(n-1)} \cong \mathbb{CP}^{n-1}$, so that:

Corollary 3.4 *For any positive r and for any $q \geq r-1$ there is a product of spheres, $X_{r,q} = S^{2(2+q)-1} \times \ldots \times S^{2(1+q+r)-1}$, such that $\mathcal{E}_\sharp((X_{r,q} \times \mathbb{CP}^q)_\mathbb{Q}) \cong Tr_1(r, \mathbb{Q})$. As a consequence every f.t.n group of Prüfer rank r is a subgroup of $\mathcal{E}_\sharp(X_{r,q} \times \mathbb{CP}^q)_\mathbb{Q}$.*

There are some more results about rational nilpotent groups of nilpotency 2. In fact Grunewald, Scharlau and Segal give a classification of f.t.n. groups of nilpotency 2 and Hirsch rank less than or equal to 6 (see [13] and [14]). As an application, Grunewald and O'Halloran classify their Mal'cev completion and the Mal'cev completion of the f.t.n. groups with commutator subgroup $[G, G]$ with Hirsch rank equal to 1, (see [12]). More precisely, the Mal'cev completion of these groups is:

1. if $\rho([G,G]) = 1$: $K_r \times (\mathbb{Q}_a)^s$,

where r is a suitable positive integer, s a suitable non-negative integer, $\mathbb{Q}_a$ is the additive group of rationals $(\mathbb{Q}, +)$ and

$$K_r = \left\{ \begin{bmatrix} 1 & & & & & \\ a_{2,1} & \cdot & & & & \\ \cdot & & \cdot & & 0 & \\ \cdot & 0 & & \cdot & & \\ \cdot & & & & \cdot & \\ a_{r+2,1} & \cdots & \cdots & \cdots & a_{r+2,r+1} & 1 \end{bmatrix}_{(r+2)\times(r+2)} \right\}$$

$a_{i,j} \in \mathbb{Q}$; $a_{i,j} = 0$ for $i \neq r+2$ and $j \neq 1$

2. if G has nilpotency 2

- **a.** and $\rho(G) = 1$: none
- **b.** and $\rho(G) = 2$: none
- **c.** and $\rho(G) = 3$: $Tr_1(3, \mathbb{Q})$
- **d.** and $\rho(G) = 4$: $Tr_1(3, \mathbb{Q}) \times \mathbb{Q}_a$
- **e.** and $\rho(G) = 5$
  - **e.1** $Tr_1(3, \mathbb{Q}) \times \mathbb{Q}_a \times \mathbb{Q}_a$
  - **e.2** K_2
  - **e.3** D,

$$D = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ b & x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & c & d & 1 \end{bmatrix} \right\} \quad x, a, b, c, d \in \mathbb{Q}$$

- **f.** and $\rho(G) = 6$
  - **f.1** $Tr_1(3, \mathbb{Q}) \times \mathbb{Q}_a \times \mathbb{Q}_a \times \mathbb{Q}_a$
  - **f.2** $K_2 \times \mathbb{Q}_a$
  - **f.3** $D \times \mathbb{Q}_a$
  - **f.4** $Tr_1(3, \mathbb{Q}(\sqrt{d}))$ over $\mathbb{Q}$, with d square-free integer,

f.5 R,

$$R = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 & x_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y_3 & x_3 & 1 \end{bmatrix} \right\} \quad x_i, y_i \in \mathbb{Q}$$

It is possible to realize some of these groups as $\mathcal{E}_\sharp(X_\mathbb{Q})$ for a finite CW complex X.

We begin with a technical lemma, which will be frequently used in the sequel:

Lemma 3.5 *For any rationalizable elliptic space X (i.e. a space having finite rational cohomology and rational homotopy) and for any finitely generated abelian group H, there is a product of spheres Y such that:*

$$\mathcal{E}_\sharp((X \times Y)_\mathbb{Q}) = \mathcal{E}_\sharp(X_\mathbb{Q}) \times H_\mathbb{Q}.$$

Proof. Let $H_\mathbb{Q} \cong \mathbb{Q}^n$ and N be a positive integer, bigger than the cohomological and homotopical dimension of $X_\mathbb{Q}$; we choose $Y = (S^m)^k \times (S^{mk})^n$, with m, k odd integers, $k \geq 3$ and $m > \max(N, 3)$.

Let $\mathcal{M}_X = \Lambda(V)$ (respectively $\mathcal{M}_Y = \Lambda(W)$) be the Sullivan model of X (resp. Y).

We shall prove that the natural inclusion:

$$i : \mathcal{E}_\sharp(\mathcal{M}_X) \times \mathcal{E}_\sharp(\mathcal{M}_Y) \hookrightarrow \mathcal{E}_\sharp(\mathcal{M}_X \otimes \mathcal{M}_Y)$$

is surjective.

Let $\varphi \in Aut_\sharp(\mathcal{M}_X \otimes \mathcal{M}_Y)$, $v \in V$, $w \in W$ be homogeneous elements. As $|v|$ is less than N, $\varphi(v) \in \mathcal{M}_X$. On the other hand, w is a cocycle, so $\varphi(w)$ is a cocycle, but the cocycles in $(\mathcal{M}_X)^+ \otimes \mathcal{M}_Y$ with degree m or mk are coboundaries. If p is the projection of $\mathcal{M}_X \otimes \mathcal{M}_Y$ on $\mathcal{M}_Y$, we define $\psi \in Aut_\sharp(\mathcal{M}_X) \times Aut_\sharp(\mathcal{M}_Y)$ as

$$\psi(v) = \varphi(v) \quad \psi(w) = p\varphi(w).$$

Let $K = V$, $d(V \oplus W) \subseteq \Lambda(V)$; moreover we have seen that $[\psi(w) - p\varphi(w)] = 0$, hence by Proposition the maps 2.1 ψ and φ are homotopic; then the inclusion i is surjective.
We know from Theorem 3.1 that $\mathcal{E}_\sharp(Y_\mathbb{Q}) \cong H$, so we can conclude:

$$\mathcal{E}_\sharp((X \times Y)_\mathbb{Q}) = \mathcal{E}_\sharp(X_\mathbb{Q}) \times \mathcal{E}_\sharp(Y_\mathbb{Q}) = \mathcal{E}_\sharp(X_\mathbb{Q}) \times H_\mathbb{Q}.$$

□

Theorem 3.6 *For any f.t.n. group G such that $\rho([G,G]) = 1$, there is a product of spheres X, such that $\mathcal{E}_\sharp(X_\mathbb{Q}) \cong G_\mathbb{Q}$.*

Proof. We know that $G_\mathbb{Q} \cong K_r \times (\mathbb{Q}_a)^s$, for some non negative integer s. First we shall prove that there exists an isomorphism

$$F : K_r \longrightarrow \mathcal{E}_\sharp(X_\mathbb{Q})$$

for $X = (S^2)^r \times S^{2l+1} \times S^{2l+3} \times S^{2l+2r+1}$, with $l > 2r$. A Sullivan model for X is $\mathcal{M} = (\Lambda(u_1, \ldots, u_r, w_1, w_2, w_{r+1}, v_1, \ldots, v_r), d)$, with with degrees and differential

$$|u_i| = 2, \; |v_i| = 3, \; |w_j| = 2l + 2j - 1$$

and

$$d(u_i) = d(w_j) = 0, \; d(v_i) = (u_i)^2.$$

We can construct the map:

$$F : K_r \longrightarrow \mathcal{E}_\sharp(\mathcal{M})$$

$$[\alpha_{s,t}] \longmapsto [\psi],$$

where

$$\psi(u_i) = u_i \quad \psi(v_i) = v_i \quad \psi(w_1) = w_1$$

$$\psi(w_2) = w_2 + \sum_{i=1}^{r} \alpha_{i+1,1} w_1 u_i$$

$$\psi(w_{r+1}) = w_{r+1} + \alpha_{r+2,1} w_1 u_1 \cdots u_r + \sum_{i=1}^{r} \alpha_{r+2,i+1} w_2 u_1 \cdots \widehat{u_i} \cdots u_r$$

As w_i is a cocycle, its image under a DG algebras map is a cocycle. The cocycles in $\Lambda(u_1, \ldots, u_r, v_1, \ldots, v_r)$, which have $(u_i)^2$ or v_i as a factor are coboundaries (see [10], Chapter II) then, up to coboundaries, the decomposables cocycles in dimension $2l+3$ have the form $\sum_{i=1}^{r} p_i w_1 u_i$ and in dimension $2l+2r+1$ have the form $q w_1 u_1 \cdots u_r + \sum_{i=1}^{r} s_i w_2 u_1 \cdots \widehat{u_i} \cdots u_r$, with $p_i, q, s_i \in \mathbb{Q}$. Besides there are not decomposables with degree equal to 2, 3 or $2l+1$. Thus Proposition 2.1 implies that F is surjective.
F is injective as a consequence of Proposition 2.2.
Letting $K = \langle u_1, \ldots, u_r \rangle$, Proposition 2.1 allows us to deduce:

$$F([\alpha_{s,t}])F([\beta_{s,t}]) \sim F([\beta_{s,t}][\alpha_{s,t}]),$$

in fact:

$$[(F([\beta_{s,t}])F([\alpha_{s,t}]))(w_i) - F([\alpha_{s,t}][\beta_{s,t}])(w_i)] = 0.$$

We can conclude that F is an anti-isomorphism.
For any $s \geq 1$, $(\mathbb{Q}_a)^s$ is an abelian group; hence, Lemma 3.5 implies the existence of a product of spheres Y such that $\mathcal{E}_\sharp((X \times Y)_\mathbb{Q}) = K_r \times (\mathbb{Q}_a)^s$. □

Theorem 3.7 *For any f.t.n. group G there is a product of spheres and homogeneous spaces X, such that $\mathcal{E}_\sharp(X_\mathbb{Q}) \cong G_\mathbb{Q}$ if G satisfies one of the following conditions:*

1. *G has nilpotency equal to 2 and $\rho(G) \leq 5$*

2. *$G_\mathbb{Q} = G'_\mathbb{Q} \times (\mathbb{Q}_a)^s$, where G' is as in the Case 1.*

Proof. We distinguish the proofs according to the previous classification of f.t.n. groups with nilpotency 2 and Hirsch rank less than or equal to 6.
Case c. $Tr_1(3,\mathbb{Q}) = \mathcal{E}_\sharp((\frac{U(4+q)}{U(1)\times U(q)})_\mathbb{Q})$ for $q \geq 2$ as a consequence of Theorem 3.3 .
Cases d, e.1, f.1. For any $s \geq 1$, $Tr_1(3,\mathbb{Q}) \times (\mathbb{Q}_a)^s \cong \mathcal{E}_\sharp((\frac{U(4+q)}{U(1)\times U(q)} \times Y)_\mathbb{Q})$ for a suitable product of spheres Y, as a consequence of Lemma 3.5.

Note that, as $G = Tr_1(3,\mathbb{Q}) \times (\mathbb{Q}_a)^s \cong K_1 \times (\mathbb{Q}_a)^s$, it is possible to realize it even as $\mathcal{E}_\sharp(X_\mathbb{Q})$, where X is a product of spheres (see Theorem 3.6).
Cases e.2, f.2. K_2 and $K_2 \times (\mathbb{Q}_a)^s$ are realized in the previous theorem.
Case e.3. $D \cong \mathcal{E}_\sharp(X_\mathbb{Q})$, where $X = \frac{U(7)}{U(1)\times U(2)}$.
Let $V = (u_1, v_1, w_1, w_2, w_3, w_4)$, $\mathcal{M}_X = \Lambda(V)$ is the Sullivan model for X.
We can construct the map:

$$F : D \longrightarrow \mathcal{E}_\sharp(\mathcal{M})$$

$$[\alpha_{s,t}] \longmapsto [\psi],$$

where

$$\begin{aligned}
&\psi(u_1) = u_1 \quad \psi(v_1) = v_1 \quad \psi(w_1) = w_1\\
&\psi(w_2) = w_2 + \alpha_{2,1}w_1u_1\\
&\psi(w_3) = w_3 + \alpha_{3,2}(= \alpha_{5,4})w_2u_1 + \alpha_{3,1}w_1(u_1)^2\\
&\psi(w_4) = w_4 + \alpha_{6,5}w_3u_1 + \alpha_{6,4}w_2(u_1)^2.
\end{aligned}$$

As $(u_1)^3$ is a coboundary, letting $K = u_1$ and $\phi \in Aut_\sharp(\mathcal{M})$ be such that ϕ and ψ coincide on all generators of V besides w_4 and $\phi(w_4) = w_4 + \alpha_{6,5}w_3u_1 + \alpha_{6,4}w_2(u_1)^2 + cw_1(u_1)^2$, Proposition 2.1 implies that ϕ is homotopic to ψ, thus, F is surjective and an anti-homomorphism, Proposition 2.2 implies that F is injective.
Case f.3. For any $s \geq 1$, $D \times (\mathbb{Q}_a)^s \cong \mathcal{E}_\sharp((\frac{U(7)}{U(1)\times U(2)} \times Y)_\mathbb{Q})$, where Y is a suitable product of spheres as a consequence of Lemma 3.5. □

4 Unipotent algebraic $\mathbb{Q}$-groups as $\mathcal{E}_*(X_\mathbb{Q})$

A Hall base for $\mathbb{L}(V)$ is a totally ordered, graded vector space base for the graded vector space L=$\mathbb{L}(V)$. The construction of this base is inductive on the bracket length, in [17] or [22] rules are given to determine the elements of length $s+1$, after having chosen and ordered the elements of length s. We observe that in the graded case, a Hall

base for $\mathbb{L}(V)$ includes also the squares of base products of odd degree (see [4], [21]).
We can dualize the above results for the group $\mathcal{E}_*(X_{\mathbb{Q}})$. We shall omit proofs, when they are dualizations of proofs in the previous section, choosing the correct Hall basis for the Quillen model of the considered spaces.

In what follows we shall denote $S^{m(1)} \vee S^{m(2)} \vee \ldots \vee S^{m(k)}$ by $\bigvee_{i=1}^{k} S^{m(i)}$.

Theorem 4.1 *For any n, $\mathcal{E}_*(\bigvee_{i=1}^{2} S^m \vee \bigvee_{j=1}^{n} S_{\mathbb{Q}}^{2m-1}) \cong \mathbb{Q}^n$.*

Theorem 4.2 *For any positive integer r, $\mathcal{E}_*(X_{\mathbb{Q}}) \cong Tr_1(r, \mathbb{Q})$, where $X = S^3 \vee S^{2n+3} \vee S^{2n+5} \ldots \vee S^{2n+2r+1}$ with $n > \max(1, r-2)$.*

In what follows we shall refer to the previous classification of f.t.n. groups G with either $\rho([G,G]) = 1$ or nilpotency 2 and $\rho(G) \leq 6$

Theorem 4.3 *For any f.t.n. group G such that $\rho([G,G]) = 1$ there is a wedge of spheres X, such that $\mathcal{E}_*(X_{\mathbb{Q}}) \cong G_{\mathbb{Q}}$.*

Proof. We must show that for any abelian group H there exists an isomorphism

$$F : \mathcal{E}_*(X_{\mathbb{Q}}) \longrightarrow K_r \times H$$

for $X = S^{1+2q} \vee S^{2q+2} \vee \bigvee_{k=1}^{r} S^{4q+2} \vee S^{6q+2} \vee \bigvee_{i=1}^{2} S^{3q+3} \vee \bigvee_{j=1}^{n} S^{6q+5}$, $q > 2$ being an even integer and $H \cong \mathbb{Q}^n$.
A Quillen model for X is $\mathcal{L} = (\mathbb{L}(V), 0)$, where
$V = < u, v_1, w_1, \ldots, w_r, v_2, x_1, x_2, y_1, \ldots, y_n >$, and degrees are as follows

$$|u| = 2q,\ |v_1| = 2q{+}1,\ |w_k| = 4q{+}1,\ |v_2| = 6q{+}1,\ |x_i| = 3q{+}2,\ |y_j| = 6q{+}4.$$

In particular this implies that for any $s \geq 1$ there exists a wedge of spheres X, such that $\mathcal{E}_\sharp(X_{\mathbb{Q}}) = K_r \times (\mathbb{Q}_a)^s$. □

Theorem 4.4 *For any f.t.n. group G there is a wedge of spheres or a cofibration of wedge of spheres X, such that $\mathcal{E}_*(X_{\mathbb{Q}}) \cong G_{\mathbb{Q}}$ if one of the following alternative holds:*

1. *G has nilpotency equal to 2 and $\rho(G) \leq 5$*

2. *$G_{\mathbb{Q}} = G'_{\mathbb{Q}} \times (\mathbb{Q}_a)^s$ where G' is as in Case 1.*

Proof. Cases c, d, e.1, e.2. As $Tr_1(r,\mathbb{Q}) \equiv K_1$, $Tr_1(r,\mathbb{Q}) \times (\mathbb{Q}_a)^s$, $K_2 \times (\mathbb{Q}_a)^s$ for $s \geq 0$ are realized in the previous Theorem.
Case e.3. Let H be an abelian group, $H \cong \mathbb{Q}^n$; we shall prove that $D \times H \cong \mathcal{E}_*(X_{\mathbb{Q}})$, where X is a suitable cofibration of wedge of spheres. Given $q > 2$ integer, in [23] it is proved that it is always possible to construct a space X

$$S^{8q+1} \longrightarrow S^{1+2q} \vee S^{2q+2} \vee S^{4q+2} \vee S^{6q+2} \vee S^{8q+2} \vee \bigvee_{i=1}^{2} S^{4q+1} \vee \bigvee_{j=1}^{n-1} S^{8q+1} \longrightarrow X$$

such that the Quillen model of X is $\mathcal{L} = (\mathbb{L}(V), d)$,
$V =< u, w_1, \ldots, w_4, z, x_1, x_2, y_1, \ldots, y_{n-1} >$, where

$$|u| = 2q, \ |w_k| = 2qk + 1, \ |z| = 8q, \ |x_i| = 4q, \ |y_j| = 8q.$$

and

$$d(z) = [u, [u, [u, w_1]]], \quad d_{|\mathbb{L}(V \setminus z)} = 0$$

□

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