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# A NOTE ON NUCLEI OF QUANTALE MODULES

*by J. PASEKA*

**Résumé.** Le but de cet article est d'établir des théorèmes utiles de factorisation pour des $Q$-modules de la même manière que pour des locales. On montre qu'un noyau d'un module associé à un prénoyau d'un module est l'intersection de noyaux de modules d'une forme particulière.

Our motivating source was the paper [12] where the investigation of I-simple involutive quantales $\mathcal{Q}(M)$ and representations $Q \to \mathcal{Q}(M)$ given by a left action is developed. Such involutive quantales could play a similar role as do points in topological spaces or irreducible representations in C*-algebras (see [9]). Since the representations $Q \to \mathcal{Q}(M)$ coincide with quantale modules we will be interested in studying factorization on quantale modules. This note is closely related to the papers [10] and [11] where the interested reader can find unexplained terms and notation concerning the subject. For facts concerning quantales and quantale modules in general we refer to [14] and [15]. For motivating examples concerning quantale modules we recommend the paper [13].

The paper is organized as follows. Section 1 introduces the notion of a left $Q$-module and the motivation for studying such a structure is given. Section 2 contains the basic theorems for factorization on $Q$-modules and the structure of the complete lattice of all module nuclei is investigated. It is shown that any module nucleus associated to a module prenucleus is a meet of module nuclei of a special form.

A *quantale* is a complete lattice $Q$ with an associative binary multiplication satisfying

$$x \cdot \bigvee_{i\in I} x_i = \bigvee_{i\in I} x \cdot x_i \quad \text{and} \quad (\bigvee_{i\in I} x_i) \cdot x = \bigvee_{i\in I} x_i \cdot x$$

for all $x$, $x_i \in Q$, $i \in I$ ($I$ is a set). 1 denotes the greatest element of $Q$, 0 is the smallest element of $Q$. A quantale $Q$ is said to be *unital* if there is an element $e \in Q$ such that $e \cdot a = a = a \cdot e$ for all $a \in Q$. A *subquantale* $Q'$ of a quantale $Q$ is a subset of $Q$ closed under $\bigvee$ and $\cdot$.

By a *morphism of quantales* will be meant a $\bigvee$- and $\cdot$-preserving mapping $f : Q \to Q'$. If a morphism preserves the top element we say that it is *strong.*

A non-trivial ($0 \neq 1$ and $1 \cdot 1 = 1$) quantale $Q$ is said to be *simple* if any surjective homomorphism $Q \to Q'$ is either an isomorphism or a constant morphism.

By the *quantale $\mathcal{Q}(M)$ of endomorphisms of the sup-lattice $M$* will be meant the simple unital quantale of sup-preserving mappings from $M$ to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication corresponding to the composition of mappings, and with the unit given by the identity mapping.

# 1 Quantale modules

**Definition 1.1** [1], [15] *Let $Q$ be a quantale. A* left module over $Q$ *(shortly a left $Q$-module) is a sup-lattice $M$, together with a* module action $\_\bullet\_ : Q \times M \to M$ *satisfying*

$$
\begin{aligned}
(a \cdot b) \bullet m &= a \bullet (b \bullet m) && \text{(M1)}\\
(\bigvee S) \bullet m &= \bigvee\{s \bullet m : s \in S\} && \text{(M2)}\\
a \bullet \bigvee X &= \bigvee\{a \bullet x : x \in X\} && \text{(M3)}
\end{aligned}
$$

*for all $a, b \in Q$, $m \in M$, $S \subseteq Q$, $X \subseteq M$. So we have two maps $- \twoheadrightarrow_l - : M \times M \to Q$, $- \twoheadrightarrow_r - : Q \times M \to M$ such that, for all $a \in Q$, $m, n \in M$, $a \bullet m \leq n$ iff $a \leq m \twoheadrightarrow_l n$ iff $m \leq a \twoheadrightarrow_r n$. $M$ is called a* unital *$Q$-module if $Q$ is a unital quantale with the unit $e$ and $e \bullet m = m$ for all $m \in M$. $M$ is said to be a* strong *$Q$-module if $1 \bullet m = 1$ for all $m \in M, m \neq 0$. Let $a \not\leq b$. Then $M$* separates $a$ from $b$ *if there is an element $m \in M$ such that $a \bullet m \neq b \bullet m$.*

*Let $M$ and $N$ be modules over $Q$ and let $f : M \to N$ be a sup-lattice homomorphism. $f$ is a* module homomorphism *if $f(a \bullet m) = a \bullet f(m)$ for all $a \in Q, m \in M$.*

Note first that if $M$ is a sup-lattice then $M$ is a left $\mathcal{Q}(M)$-module such that $f \bullet m = f(m)$ for all $f \in \mathcal{Q}(M)$ and all $m \in M$. Secondly, we may dually define the notion of a (strong, separating, unital) right $Q$-module. Moreover, all propositions stated for left $Q$-modules are valid in a dualized form for right $Q$-modules.

We shall denote by $Q$-**Mod** (resp. **Mod**-$Q$) the category of left $Q$-modules (resp. right $Q$-modules).

Let us make the following elementary observation. Evidently, for a quantale $Q$, any left $Q$-module $M$ is a unital left $Q[e]$-module with $Q[e]$ defined as in [11] by

$$Q[e] = \{a \vee \varepsilon : a \in Q, \varepsilon \in \{0, e\}\},$$

$e$ arbitrary such that $e \notin Q$, and the multiplication $\bullet_e$ defined by

$$(a \vee \varepsilon) \bullet_e m = \begin{cases} a \bullet m & \text{if } \varepsilon = 0 \\ a \bullet m \vee m & \text{if } \varepsilon = e. \end{cases}$$

So we may always assume that any left quantale module is unital over a unital quantale.

Note that, for any quantale $Q$, any left $Q$-module M and all $n \in M$, the antitone maps $-\twoheadrightarrow_l n : M \to Q$ and $-\twoheadrightarrow_r n : Q \to M$ form a Galois connection between $Q$ and $M$ since, for all $a \in Q$ and all $M \in M$, $a \leq (a \twoheadrightarrow_r n) \twoheadrightarrow_l n$ and $m \leq (m \twoheadrightarrow_l n) \twoheadrightarrow_r n$. So we have that $m \twoheadrightarrow_l n = ((m \twoheadrightarrow_l n) \twoheadrightarrow_r n) \twoheadrightarrow_l n$ and $a \twoheadrightarrow_r n = ((a \twoheadrightarrow_r n) \twoheadrightarrow_l n) \twoheadrightarrow_r n$.

A simple calculation gives us the following two lemmas.

**Lemma 1.2** *Let $Q$ be a quantale. If $M$ is a (strong) $Q$-module then we have a (strong) homomorphism $f_M : Q \to \mathcal{Q}(M)$, $f_M(a)(x) = a \bullet x$. Conversely, let $f : Q \to \mathcal{Q}(M)$ be a (strong) homomorphism of quantales. Then $M$ is a (strong) $Q$-module with the module action $a \bullet_f x = f(a)(x)$.*

**Lemma 1.3** *Let $Q$ be a quantale, $a \not\leq b$. If $M$ is a $Q$-module separating $a$ from $b$ then we have a separating homomorphism $f_M : Q \to \mathcal{Q}(M)$. Conversely, let $f : Q \to \mathcal{Q}(M)$ be a homomorphism separating $a$ from $b$. Then $M$ is a separating $Q$-module.*

The following theorem (see [6]) is our main motivation for studying quantale modules.

**Theorem 1.4** *Let $Q$ be a quantale. Then the following conditions are equivalent:*

1. *$Q$ is spatial i.e. $Q$ is a strong subquantale of a product of simple quantales.*
2. *Each element of $Q$ is an intersection of primes.*
3. *$Q$ has enough strong separating left $Q$-modules.*

**Proof.** (1) $\Leftrightarrow$(2) by [6]. (1) $\Leftrightarrow$(3) by lemma 1.2 and by lemma 1.3. $\square$

# 2 Factorization on $Q$-modules

In the following, we shall assume that $Q$ is an arbitrary, fixed quantale.

**Definition 2.1** *Let $M$ be a left $Q$-module, $j : M \to M$ an operator on $M$ satisfying:*

| | |
|---|---|
| $(i)$ | $m \leq j(m)$ |
| $(ii)$ | $m \leq n$ *implies* $j(m) \leq j(n)$ |
| $(iii)$ | $a \bullet j(m) \leq j(a \bullet m)$ |

*for all $a \in Q$, $m, n \in M$. We say that $j$ is a* module prenucleus *on $M$. We put $M_j = \{m \in M : m = j(m)\}$. Evidently, $M_j$ is a closure system* *in $M$, that is, closed under arbitrary meets in $M$. Note that $\bigvee_j = j \circ \bigvee$. We let $\nu(j)$ be the associated closure operator, so that*

$$\nu(j)(n) = \bigwedge\{m \in M_j : m \geq n\}.$$

*We say that $\nu(j)$ is* generated *by $j$. If $j$ satisfies*

(iv) $\qquad j(m) = j(j(m))$

*for all $m \in M$ we shall say that $j$ is a* module nucleus *on $M$. We shall denote by $N(M)$ the lattice of all module nuclei on $M$.*

Note that any module prenucleus (nucleus) on a $Q$-module $M$ is a module prenucleus (nucleus) on a $Q[e]$-module $M$ and conversely.

**Lemma 2.2** *Let $j$ be a module nucleus on a left $Q$-module $M$. Then $M_j$ is a left $Q$-module with the join $\bigvee_j$ and the action $\bullet_j : Q \times M_j \to M_j$ defined as follows*

$$a \bullet_j m = j(a \bullet m)$$

*for all $a \in Q$ and all $m \in M_j$. Furthermore, $M_j$ is unital (strong) if $M$ is.*

**Proof.** Let $a, b \in Q$, $m \in M_j$. Then $(a \cdot b) \bullet_j m = j((a \cdot b) \bullet m) = j(a \bullet (b \bullet m)) = j(a \bullet j(b \bullet m)) = a \bullet_j (b \bullet_j m)$. Similarly, let $S \subseteq Q$, $m \in M_j$. Then $(\bigvee S) \bullet_j m = j((\bigvee S) \bullet m) = j(\bigvee\{s \bullet m : s \in S\}) = j(\bigvee\{j(s \bullet m) : s \in S\}) = \bigvee_j\{s \bullet_j m : s \in S\}$. Now, let $a \in Q$, $N \subseteq M$. Then $a \bullet_j (\bigvee_j N) = j(a \bullet j(\bigvee N)) = j(a \bullet \bigvee N) = j(\bigvee\{a \bullet n : n \in N\}) = j(\bigvee\{j(a \bullet n) : n \in N\}) = \bigvee_j\{a \bullet_j n : n \in N\})$. $\square$

Similarly as for frames (see [2]) we have

**Proposition 2.3** *Let $j$ be a module prenucleus on a left $Q$-module $M$. Then $\nu(j)$ is the smallest module nucleus greater than $j$. Moreover, the module homomorphism $\nu(j) : M \to M_j$ is universal among all module homomorphisms $f : M \to N$ for which $f(m) = f(j(m))$ for all $m \in M$.*

**Proof.** Evidently, $\nu(j)$ is inflationary, idempotent and order-preserving. We have to show that, for all $a \in Q$ and $m \in M$, $a \bullet \nu(j)(m) \leq \nu(j)(a \bullet m)$. We put $E_{a,m} = \{m' \in M : m \leq m' \leq \nu(j)(m), a \bullet m' \leq \nu(j)(a \bullet m)\}$. Then $m \in E_{a,m}$, $m' \in E_{a,m}$ implies $j(m') \in E_{a,m}$ - $m \leq m' \leq \nu(j)(m)$ implies $m \leq j(m) \leq j(m') \leq j(\nu(j)(m)) = \nu(j)(m)$ and $a \bullet m' \leq \nu(j)(a \bullet m)$ implies $a \bullet j(m') \leq j(a \bullet m') \leq j(\nu(j)(a \bullet m)) = \nu(j)(a \bullet m)$. Moreover, for any non-void $M' \subseteq E_{a,m}$, $\bigvee M' \in E_{a,m}$ by the left distribution law of modules. Hence $m_0 = \bigvee E_{a,m} \in E_{a,m}$. This gives us $j(m_0) \leq m_0 \leq j(m_0)$ i.e. $m \leq m_0 \leq \nu(j)(m) \leq m_0$. So we have $a \bullet \nu(j)(m) \leq a \bullet m_0 \leq \nu(j)(a \bullet m)$.

Let us prove the second part of the lemma. Note that $\nu(j)(j(m)) = \nu(j)(m)$ for all $m \in M$. Let $f : M \to N$ be a module homomorphism such that $f(m) = f(j(m))$ for all $m \in M$. We have to show that $f(\nu(j)(m)) = f(m)$ for all $m \in M$. We put $F_m = \{m' \in M : m \leq m' \leq \nu(j)(m), f(m') = f(m)\}$. Then $m \in F_m$, $m' \in F_m$ implies $j(m') \in F_m$ by the condition on $f$ and, for all non-void $F' \subseteq F_m$, $\bigvee F' \in F_m$ since $f$ is a join-preserving. We put $m_1 = \bigvee F_m$. Then $j(m_1) \leq m_1 \leq j(m_1)$ i.e. $m \leq m_1 \leq \nu(j)(m) \leq \nu(j)(m_1) = m_1$ i.e. $f(m) \leq f(\nu(j)(m)) \leq f(m_1) = f(m)$. We then have a uniquely determined map $\hat{f} : M_j \to N$ defined by $\hat{f}(m) = f(m)$ for all $m \in M_j$ i.e. $\hat{f} \circ j = f$. Evidently, for all $a \in Q$, $m \in M_j$ and $S \subseteq M_j$, we have $\hat{f}(\bigvee_{M_j} S) = \hat{f}(j(\bigvee S)) = f(\bigvee S) = \bigvee_{s \in S} f(s) = \bigvee_{s \in S} \hat{f}(j(s)) = \bigvee_{s \in S} \hat{f}(s)$ and $\hat{f}(a \bullet_{M_j} m) = \hat{f}(j(a \bullet m)) = f(a \bullet m) = a \bullet f(m) = a \bullet \hat{f}(j(m)) = a \bullet \hat{f}(m)$ i.e. $\hat{f}$ is a module homomorphism. $\square$

Another way of obtaining $\nu(j)$ is to iterate the prenucleus $j$ until it converges by the following procedure:

$$\begin{array}{lcl} j^0(m) & = & m, \\ j^{\sigma+1}(m) & = & j(j^\sigma(m)) \\ j^\lambda(m) & = & \bigvee\{j^\sigma(m) : \sigma < \lambda\} \end{array}$$

for all $m \in M$, all ordinals $\sigma$ and all limit ordinals $\lambda$. Then each $j^\sigma$ is a module prenucleus with the same fixpoints as $j$ and, for sufficiently large ordinal $\sigma$, $j^\sigma$ has to be idempotent.

Following Macnab (see [8]) for Heyting algebras we have

**Theorem 2.4** *Let $Q$ be a unital quantale, $M$ a unital left $Q$-module, $j : M \to M$ an operator on $M$. The following conditions are equivalent:*

1. *$j$ is a module nucleus on $M$.*
2. *For all $m, n$ in $M$, $m \twoheadrightarrow_l j(n) = j(m) \twoheadrightarrow_l j(n)$.*

**Proof.** $1 \Longrightarrow 2$. We have, for all $a \in Q$, $a \leq m \twoheadrightarrow_l j(n)$ iff $a \bullet m \leq j(n)$ iff $j(a \bullet m) \leq j(n)$ iff $a \bullet j(m) \leq j(n)$ iff $a \leq j(m) \twoheadrightarrow_l j(n)$ i.e. the condition 2 holds.

$2 \Longrightarrow 1$. We have, for all $m \in M$, $j(m) \leq j(m)$ i.e. $e \bullet j(m) \leq j(m)$ i.e. $e \leq j(m) \twoheadrightarrow_l j(m)$ i.e. $e \leq m \twoheadrightarrow_l j(m)$ i.e. $e \bullet m \leq j(m)$ i.e. $m \leq j(m)$ i.e. $j$ is inflationary.

Now, let $m, n \in M$, $m \leq n$. Then $m \leq n \leq j(n)$ i.e. $e \bullet m \leq j(n)$ i.e. $e \leq m \twoheadrightarrow_l j(n) = j(m) \twoheadrightarrow_l j(n)$ i.e. $e \bullet j(m) \leq j(n)$ i.e. $j(m) \leq j(n)$.

Let $a \in Q$, $m \in M$. We have $a \bullet m \leq j(a \bullet m)$ i.e. $a \leq m \twoheadrightarrow_l j(a \bullet m)$ i.e. $a \leq j(m) \twoheadrightarrow_l j(a \bullet m)$ i.e. $a \bullet j(m) \leq j(a \bullet m)$.

Let us show that $j$ is idempotent. Assume $m \in M$. We have $j(m) \leq j(m)$ i.e. $e \bullet j(m) \leq j(m)$ i.e. $e \leq j(m) \twoheadrightarrow_l j(m)$ i.e. $e \leq j(j(m)) \twoheadrightarrow_l j(m)$ i.e. $e \bullet j(j(m)) \leq j(m)$ i.e. $j(j(m)) \leq j(m) \leq j(j(m))$ i.e. $j(j(m)) = j(m)$. So it follows that $j$ is a module nucleus on $M$. $\square$

**Proposition 2.5** *Let $M$ be a left $Q$-module, $n \in M$. Then the map $w_n : M \to M$ defined by $w_n(m) = (m \twoheadrightarrow_l n) \twoheadrightarrow_r n$ is a module nucleus. Similarly, the map $W_n : Q \to Q$ defined by $W_n(a) = (a \twoheadrightarrow_r n) \twoheadrightarrow_l n$ is a right module nucleus on $Q$, seen as a right module over itself. Moreover, for all $a \in Q$ and $m \in M$, we have $a \bullet m \leq n$ iff $W_n(a) \bullet w_n(m) \leq n$.*

**Proof.** Clearly, the Galois connection makes $m \leq w_n(m) = w_n(w_n(m))$ and $m_1 \leq m_2$ implies $w_n(m_1) \leq w_n(m_2)$ for all $m, m_1, m_2 \in M$. Let $a \in Q$, $m \in M$. Then $y \leq a \bullet m \twoheadrightarrow_l n$ iff $(y \cdot a) \bullet m = y \bullet (a \bullet m) \leq n$ iff $(y \cdot a) \leq m \twoheadrightarrow_l n$ iff $(y \cdot a) \twoheadrightarrow_r n \geq w_n(m)$ iff $(y \cdot a) \bullet w_n(m) \leq n$ iff $y \bullet (a \bullet w_n(m)) \leq n$ iff $y \leq (a \bullet w_n(m)) \twoheadrightarrow_l n$. Hence $a \bullet m \twoheadrightarrow_l n = (a \bullet w_n(m)) \twoheadrightarrow_l n$, and thus $w_n(a \bullet m) = w_n(a \bullet w_n(m))$, i.e. $w_n(a \bullet m) \geq$

$a \bullet w_n(m)$. To prove the second part of the lemma note that $Q$ is a right module over itself. By similar arguments as above, we have that $W_n$ is a right module nucleus.

Finally, we show that $a \bullet m \leq n \iff W_n(a) \bullet w_n(m) \leq n$: $a \bullet m \leq n$ iff $a \leq m \twoheadrightarrow_l n$ iff, by the Galois connection, $a \leq w_n(m) \twoheadrightarrow_l n$ iff $a \bullet w_n(m) \leq n$ iff, by similar steps using $\twoheadrightarrow_r$, $W_n(a) \bullet w_n(m) \leq n$. $\square$

**Lemma 2.6** *Let $Q$ be a quantale, $j_\alpha, \alpha \in \Lambda$ module prenuclei (nuclei) on a left $Q$-module $M$. Then the operator $\bigwedge_{\alpha \in \Lambda} j_\alpha$ is a module prenucleus (nucleus).*

**Proof.** Evidently, for all $m, n \in M$, $a \in Q$, we have $m \leq \bigwedge_{\alpha \in \Lambda} j_\alpha(m) = \left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(m)$, $m \leq n$ implies $j_\alpha(m) \leq j_\alpha(n)$ i.e. $\left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(m) \leq \left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(n)$ and $a \bullet \left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(m) \leq a \bullet j_\beta(m) \leq j_\beta(a \bullet m)$ for all $\beta \in \Lambda$ i.e. $a \bullet \left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(m) \leq \left(\bigwedge_{\alpha \in \Lambda} j_\alpha\right)(a \bullet m)$. Let $j_\alpha, \alpha \in \Lambda$ be module nuclei. Then $\bigwedge_{\alpha \in \Lambda} j_\alpha(\bigwedge_{\beta \in \Lambda} j_\beta(m)) \leq j_\gamma(j_\gamma(m)) = j_\gamma(m)$ for all $\gamma \in \Lambda$ and all $m \in M$ i.e. $\bigwedge_{\alpha \in \Lambda} j_\alpha$ is idempotent. $\square$

We have then that, for any set $\Lambda$, $\bigwedge\{j_\lambda \in N(M) : \lambda \in \Lambda\}$ (with $\bigwedge$ defined pointwise), exists and it is a module nucleus i.e. $N(M)$ is a complete lattice. Because the meet in $N(M)$ is pointwise so also is the order on it. We shall denote by $\bigsqcup$ the join in $N(M)$ and by $\bigvee$ the pointwise join, which in general is only a prenucleus. Let $j_1, j_2 \in N(M)$. Then $j_1 \circ j_2, j_2 \circ j_1 \leq j_1 \sqcup j_2$. Note that $j_1 \circ j_2$ is a module prenucleus. So it follows that if $j_1 \circ j_2$ is idempotent then $j_1 \circ j_2 = j_1 \sqcup j_2$. If $j_1 \circ j_2 \leq j_2 \circ j_1$ then $j_2 \circ j_1$ is idempotent i.e. a module nucleus.

Recall the following notions from [3]. Let $Q$ be a semigroup, $a \in Q$. We then put $Q\langle a\rangle = \{a^k : k \in \mathbf{N}\}$. We say that $Q\langle a\rangle$ is a *cyclic subsemigroup* of $Q$ generated by the element $a$. For any $a \in Q$, we have either that $Q\langle a\rangle$ is infinite or that $Q\langle a\rangle$ is finite i.e. there exist $k_1, k_2 \in \mathbf{N}$ minimal such that $a^{k_1+k_2} = a^{k_2}$; here $k_1 = \mathbf{p}(a)$ is the *period* of $a$ and $k_2 = \mathbf{i}(a)$ the *index* of $a$. We say that an element $a \in Q$ is *periodic* if $Q\langle a\rangle$ is finite. We shall denote by $Q_{\mathbf{P}} = \{a \in Q : a \text{ is periodic}\}$ and $Q_{\mathbf{E}} = \{a \in Q : a \text{ is idempotent}\}$. If $Q = Q_{\mathbf{P}}$ ($Q = Q_{\mathbf{E}}$) we say that $Q$ is *periodic* (*idempotent*).

Now we generalise the nuclei $w_n$ of Proposition 2.5.

**Lemma 2.7** *Let $Q$ be a unital quantale, $j$ a module prenucleus on a unital left $Q$-module $M$, $k \in \mathbf{N}$, $n \in M$, $z \leq j(n) \twoheadrightarrow_l n$. Then the operator $\nu^k_{z,n} : M \to M$ defined by the prescription*

$$\nu^k_{z,n}(m) = \left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n$$

*is a module prenucleus on $M$ greater than $j$. Especially, the operator $\nu^k_n(j) = \nu^k_{j(n) \twoheadrightarrow_l n, n}$ is a module prenucleus on $M$ greater than $j$. Moreover, if $z \in Q_{\mathbf{P}}$ and $k \geq \mathbf{i}(z)$ we have that $\nu^k_{z,n}$ is a module nucleus greater than $j$.*

**Proof.** We first show that, for all $m \in M$, $j(m) \leq \nu^k_{z,n}(m)$ – we have then $m \leq \nu^k_{z,n}(m)$. It suffices to show that

$$\left(z^k(m \twoheadrightarrow_l n)\right) \bullet j(m) \leq n.$$

We have

$$\begin{aligned}
\left(z^k(m \twoheadrightarrow_l n)\right) \bullet j(m) &= z^k \bullet ((m \twoheadrightarrow_l n) \bullet j(m)) \\
&\leq z^k \bullet j\left((m \twoheadrightarrow_l n) \bullet m\right) \\
&\leq z^k \bullet j(n) \\
&\leq n,
\end{aligned}$$

where the last inequality follows by induction on $k$:

- $z \bullet j(n) \leq (j(n) \twoheadrightarrow_l n) \bullet j(n) \leq n$,
- $z^{k+1} \bullet j(n) = z \bullet (z^k \bullet j(n)) \leq z \bullet n \leq z \bullet j(n) \leq n$.

Note that the function $\left(z^k(- \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n$ is order-preserving. Now, let us show that $a \bullet \nu^k_{z,n}(m) \leq \nu^k_{z,n}(a \bullet m)$ for all $a \in Q$ and $m \in M$, that is, we check that $a \bullet [\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n] \leq \left(z^k((a \bullet m) \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n$. Evidently, $(a \bullet m \twoheadrightarrow_l n) \cdot a \leq m \twoheadrightarrow_l n$. We have

$$\begin{aligned}
\left[\left(z^k((a \bullet m) \twoheadrightarrow_l n)\right) \cdot a\right] \bullet \left[\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n\right] &\leq \\
\left(z^k(m \twoheadrightarrow_l n)\right) \bullet \left[\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n\right] &\leq n.
\end{aligned}$$

So we have shown that $\nu^k_{z,n}$ is a module prenucleus.

Now assume that $z \in Q_{\mathbf{P}}$ and $k \geq \mathbf{i}(z)$. We show that $\nu^k_{z,n}(\nu^k_{z,n}(m)) \leq \nu^k_{z,n}(m)$. This is equivalent to the following inequality $(\star)$

$$\left(z^k(m \twoheadrightarrow_l n)\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \leq n.$$

Since $- \twoheadrightarrow_l n$ and $- \twoheadrightarrow_r n$ form a Galois connection and $z$ is periodic the inequality $(\star)$ follows from the inequality $(\star\star_i)$, $i \in \mathbf{N}$

$$\begin{aligned}&\left[z^{i\mathbf{p}(z)} \cdot \left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right)\right] \bullet \\ &\quad \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \leq n.\end{aligned}$$

Let us prove $(\star\star_i)$ implies $(\star)$. We have

$$\begin{aligned}\left(z^k(m \twoheadrightarrow_l n)\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n &= \\ \left[z^{i\mathbf{p}(z)} \cdot \left(z^k(m \twoheadrightarrow_l n)\right)\right] \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n &= \\ z^{i\mathbf{p}(z)} \bullet \left(\left(z^k(m \twoheadrightarrow_l n)\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n\right) &\leq \\ z^{i\mathbf{p}(z)} \bullet \left(\twoheadrightarrow_l n\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n\right) &\leq \\ z^{i\mathbf{p}(z)} \bullet n &\leq n.\end{aligned}$$

(in fact, we also have $(\star)$ implies $(\star\star_i)$ since we obtain by $(\star)$ and the Galois connection that

$$\left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \leq n.$$

Then

$$\begin{aligned}&\left[z^{i\mathbf{p}(z)} \cdot \left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right)\right] \bullet \\ &\quad \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n = \\ &\quad = z^{i\mathbf{p}(z)} \bullet \left(\left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right) \bullet \right. \\ &\quad \left. \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \ \right) \leq z^{i\mathbf{p}(z)} \bullet n \leq n.)\end{aligned}$$

Now, for a large enough $i$, such that $i\mathbf{p}(z) = i_0 + k$ for some $i_0 \in \mathbf{N}$, the condition $(\star\star_i)$ becomes equivalent to

$$z^{i_0} \bullet [z^k \left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right) \bullet \\ \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n] \leq n.$$

The last inequality evidently holds since $z^{i_0} \bullet n \leq n$ and

$$z^k \left(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n\right) \bullet \left(z^k(\left(z^k(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n \leq n.$$

We have then that $\nu^k_{z,n}$ is a module nucleus. $\square$

Johnstone gives in ([4]) an explicit formula for $\nu(j)$ for frames. The following proposition is a slight modification of it for $Q$-modules.

**Proposition 2.8** *Let $Q$ be a unital periodic quantale, $j$ a module prenucleus on a unital left $Q$-module $M$. Then the operator $\nu_{\mathbf{p}}(j) : M \to M$ defined by the prescription*

$$\nu_{\mathbf{p}}(j)(m) = \bigwedge\{\left((j(n) \twoheadrightarrow_l n)^{\mathbf{i}(j(n) \twoheadrightarrow_l n)}(m \twoheadrightarrow_l n)\right) \twoheadrightarrow_r n : n \in M\}$$

*is the associated module nucleus $\nu(j)$.*

**Proof.** Note that $\nu_{\mathbf{p}}(j) = \bigwedge_{n\in M}\{\nu_n^{\mathbf{i}(j(n) \twoheadrightarrow_l n)}(j)\}$ i.e. $\nu_{\mathbf{p}}(j)$ is a module nucleus greater than $j$. Let $j(m) = m$. We put

$$m' = \left((m \twoheadrightarrow_l m)^{\mathbf{i}(m \twoheadrightarrow_l m)}(m \twoheadrightarrow_l m)\right) \twoheadrightarrow_r m, \ k = \mathbf{i}(m \twoheadrightarrow_l m).$$

We then have $\nu_{\mathbf{p}}(j)(m) \leq m'$ and

$$\begin{aligned} m' = (e \cdot e) \bullet m' &\leq (e \cdot (m \twoheadrightarrow_l m)) \bullet m' \\ &\leq \left((m \twoheadrightarrow_l m)^k (m \twoheadrightarrow_l m)\right) \bullet m' \\ &\leq m \end{aligned}$$

i.e. $\nu_{\mathbf{p}}(j)(m) = j(m)$. Hence, $\nu_{\mathbf{p}}(j)$ has the same fixed points as $j$ i.e. $\nu_{\mathbf{p}}(j)$ is the least module nucleus greater than $j$. $\square$

**Corollary 2.9** *Let $Q$ be a unital idempotent quantale, $j$ a module prenucleus on a unital left $Q$-module $M$. Then the operator $\nu_{\mathbf{e}}(j) : M \to M$ defined by the prescription*

$$\nu_{\mathbf{e}}(j)(m) = \bigwedge\{((j(n) \twoheadrightarrow_l n)(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n : n \in M\}$$

*is the associated module nucleus $\nu(j)$.*

**Theorem 2.10** *Let $Q$ be a unital periodic quantale, $M$ a unital left $Q$-module. Then, for any set $\Lambda$, $\bigsqcup\{j_\lambda \in N(M) : \lambda \in \Lambda\} = \bigsqcup^{\uparrow}\{\nu_{\mathbf{p}}(j_{\lambda_1} \circ \ldots \circ j_{\lambda_s}) : \{\lambda_1, \ldots, \lambda_s\} \subseteq \Lambda\} = \nu_{\mathbf{p}}(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\})$.*

**Proof.** The first equality follows from the proposition 2.8 and from the fact that a directed join of module nuclei is computed pointwise. The second one follows from an easy observation that the pointwise join of module prenuclei is a module prenucleus. □

**Corollary 2.11** *Let $Q$ be a unital idempotent quantale, $M$ a unital left $Q$-module. Then, for any set $\Lambda$, $\bigsqcup\{j_\lambda \in N(M) : \lambda \in \Lambda\} = \bigsqcup^{\uparrow}\{\nu_{\mathbf{e}}(j_{\lambda_1} \circ \ldots \circ j_{\lambda_s}) : \{\lambda_1, \ldots, \lambda_s\} \subseteq \Lambda\} = \nu_{\mathbf{e}}(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\})$.*

**Proposition 2.12** *Let $Q$ be a unital quantale, $j$ a module prenucleus on a unital left $Q$-module $M$. Then the operators $\nu_{\mathbf{P}}(j), \nu_{\mathbf{E}}(j) : M \to M$ defined by the prescriptions*

$$\begin{aligned}
\nu_{\mathbf{P}}(j) &= \bigwedge\{\nu^{z,n}_{\mathbf{i}(z)}(j) : n \in M, z \in Q_{\mathbf{P}}, z \leq j(n) \twoheadrightarrow_l n\} \\
\nu_{\mathbf{E}}(j) &= \bigwedge\{\nu^{z,n}_{1}(j) : n \in M, z \in Q_{\mathbf{E}}, z \leq j(n) \twoheadrightarrow_l n\}
\end{aligned}$$

*coincide with the associated nucleus $\nu(j)$.*

**Proof.** By 2.6 and 2.7 we know that both operators are module nuclei on $M$ greater than $j$.

Now, let $j(m) = m$. Then $e \in Q_{\mathbf{E}} \subseteq Q_{\mathbf{P}}$ and $e \leq m \twoheadrightarrow_l m$. We then have

$$\begin{aligned} j(m) &\leq \nu_{\mathbf{P}}(j)(m) \leq \nu_{\mathbf{E}}(j)(m) \leq (e \cdot (m \twoheadrightarrow_l m)) \twoheadrightarrow_r m \\ &= e \bullet [(m \twoheadrightarrow_l m) \twoheadrightarrow_r m] \\ &\leq (m \twoheadrightarrow_l m) \bullet [(m \twoheadrightarrow_l m) \twoheadrightarrow_r m] \\ &\leq m = j(m) \end{aligned}$$

i.e. $\nu_{\mathbf{P}}(j)(m) = \nu_{\mathbf{E}}(j)(m) = j(m)$.

Since both $\nu_{\mathbf{E}}(j)$ and $\nu_{\mathbf{E}}(j)$ have the same fixed points as $j$ we have

$$\nu_{\mathbf{E}}(j) = \nu(j) = \nu_{\mathbf{P}}(j). \ \square$$

**Theorem 2.13** *Let $Q$ be a unital quantale, $M$ a unital left $Q$-module. Then, for any set $\Lambda$, $\bigsqcup\{j_\lambda \in N(M) : \lambda \in \Lambda\} = \bigsqcup^{\uparrow}\{\nu_{\mathbf{P}}(j_{\lambda_1} \circ \ldots \circ j_{\lambda_s}) : \{\lambda_1, \ldots, \lambda_s\} \subseteq \Lambda\} = \bigsqcup^{\uparrow}\{\nu_{\mathbf{E}}(j_{\lambda_1} \circ \ldots \circ j_{\lambda_s}) : \{\lambda_1, \ldots, \lambda_s\} \subseteq \Lambda\} = \nu_{\mathbf{P}}\left(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\}\right) = \nu_{\mathbf{E}}\left(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\}\right)$.*

**Proof.** Similar to the proof of theorem 2.10, with $\nu_{\mathrm{p}}$ replaced by $\nu_{\mathbf{P}}$ and $\nu_{\mathbf{E}}$ and using proposition 2.12 instead of proposition 2.8. $\square$

Another way how to construct new module nuclei is the method of $R$-saturated elements introduced for frames by Kříž (see [7]).

**Definition 2.14** *Let $M$ be a left $Q$-module, $R \subseteq M \times M$ a relation on $M$. We shall say that an element $m \in M$ is* $R$-saturated *if*

$$a \bullet n_1 \leq m \textit{ iff } a \bullet n_2 \leq m \quad \textit{and} \quad n_1 \leq m \textit{ iff } n_2 \leq m$$

*holds for all $a \in Q$, $n_1, n_2 \in M$ such that $(n_1, n_2) \in R$. We shall denote by $M_R$ the set of all $R$-saturated elements.*

Note that $M_R$ coincides with the sup-lattice quotient of $M$ by the relation $R' = R \cup \{(a \bullet m_1, a \bullet m_2) : a \in Q, (m_1, m_2) \in R\}$ [5]. Equivalently the map $j_R : M \to M$ defined by $j_R(m) = \bigwedge\{n \in M_R : m \leq n\}$ is a closure operator on $M$, $M_{j_R} = M_R$, and the sup-lattice homomorphism $j_R : M \to M_R$ is universal amongst all sup-lattice homomorphisms $f : M \to N$ for which $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R'$.

**Lemma 2.15** *Let $a \in Q$, $m \in M$ and $m$ is $R$-saturated. Then the element $a \twoheadrightarrow_r m = \bigvee\{n \in M : a \bullet n \leq m\}$ is $R$-saturated.*

**Proof.** Let $n_1, n_2 \in M$, $(n_1, n_2) \in R$, $c \in Q$. Then $c \bullet n_1 \leq a \twoheadrightarrow_r m$ iff $a \bullet (c \bullet n_1) \leq m$ iff $(a \cdot c) \bullet n_1 \leq m$ iff $(a \cdot c) \bullet n_2 \leq m$ iff $a \bullet (c \bullet n_2) \leq m$ iff $c \bullet n_2 \leq a \twoheadrightarrow_r m$. Similarly, $n_1 \leq a \twoheadrightarrow_r m$ iff $n_2 \leq a \twoheadrightarrow_r m$. $\square$

**Theorem 2.16** *Let $M$ be a left $Q$-module, $R \subseteq M \times M$. Then the closure operator $j_R$ is a module nucleus, and the module homomorphism $j_R : M \to M_R$ is universal amongst all module homomorphisms $f : M \to N$ for which $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R$.*

**Proof.** Assume $a \in Q$, $m \in M$. Then $a \bullet m \leq j_R(a \bullet m)$ i.e. $m \leq a \twoheadrightarrow_r j_R(a \bullet m)$ i.e. $j_R(m) \leq a \twoheadrightarrow_r j_R(a \bullet m)$ i.e. $a \bullet j_R(m) \leq j_R(a \bullet m)$. Hence, $j_R$ is a module nucleus.

Now, let $f : M \to N$ be a module homomorphism such that $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R, m \in M$. Then $f$ factors uniquely through $j_R$ and a sup-lattice homomorphism $f' : M_R \to N$, because $(m_1, m_2) \in R'$ implies $f(m_1) = f(m_2)$. Let us prove that $f'$ preserves the action. Denoting by $\bullet_{j_R}$ the action on $M_R$ and by $\bullet_N$ the action on $N$ we have $f'(a \bullet_{j_R} j_R(m)) = f'(j_R(a \bullet j_R(m))) = f(a \bullet j_R(m)) = a \bullet_N f(j_R(m)) = a \bullet_N f'(m)$. $\square$

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