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ABOUT THE NATURALITY OF BEATTIE'S DECOMPOSITION THEOREM WITH RESPECT TO A CHANGE OF HOPF ALGEBRAS

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RESUME

Dans cet article, en partant d'un morphisme entre deux algèbres de Hopf finies et commutatives G et H dans une catégorie fermée symétrique C avec objet base projective, les auteurs construisent un homomomorphisme de groupes abéliens entre $Gal_C(H)$ et $Gal_C(G)$ (les groupes des classes d'isomorphismes des H-objets et des G-objets de Galois, respectivement). La restriction de cet homomorphisme permet d'établir un homomorphisme entre les groupes des classes d'isomorphismes des H-objets et des G-objets de Galois avec base normale $N_C(H)$ et $N_C(G)$, en obtenant deux suites exactes qui relient ces groupes avec $G(H^*)$ et $G(G^*)$.

Finalement, ils construisent un diagramme commutatif qui rattache les morphismes précédents à d'autres suites, comme par exemple la dérivée du Théorème de décomposition de Beattie.

1 Preliminaries

In what follows, $\mathcal{C}$ denotes a symmetric closed category [6] with equalizers, coequalizers and basic object K. The natural symmetry isomorphisms in $\mathcal{C}$ are represented by τ. We denote by α_M and β_M the unit and the counit, respectively, of the $\mathcal{C}$-adjunction

$$M \otimes - \dashv HOM(M, -) : \mathcal{C} \to \mathcal{C}$$

If M is an object of $\mathcal{C}$ we denote by M^* the dual object $HOM(M, K)$ of M and by E_M the object $HOM(M, M)$.

Definition 1.1 An object P of $\mathcal{C}$ is called finite if the morphism

$$\nabla_{PKP} := HOM(P, \beta_P(K) \otimes P) \circ \alpha_P(P^* \otimes P) : P^* \otimes P \to E_P$$

is an isomorphism, equivalentely $HOM(P, -) \approx P^* \otimes -$. If P is finite we denote by a_P and b_P the unit and the counit, respectively, of the $\mathcal{C}$-adjunction $P \otimes - \dashv P^* \otimes - : \mathcal{C} \to \mathcal{C}$

Definition 1.2 Let P be a finite object in $\mathcal{C}$. If the factorization ∇_{KPK} of $\beta_P(K) : P \otimes P^* \to K$ through the coequalizer of the morphisms $\beta_P(P) \otimes P^*$ and $P \otimes (HOM(P, \beta_P(K) \circ [\beta_P(P) \otimes P^*]) \circ \alpha_P(E_P \otimes P^*)) : P \otimes E_P \otimes P^* \to P \otimes P^*$ is an isomorphism, we say that P is a progenerator in $\mathcal{C}$. Equivalentely, P is a progenerator in $\mathcal{C}$ if the diagram

$$P \otimes P^* \otimes P \otimes P^* \; \underset{P \otimes P^* \otimes\, b_P(K)}{\overset{b_P(K) \otimes P \otimes P^*}{\rightrightarrows}} \; P \otimes P^* \xrightarrow{\;b_P(K)\;} K$$

is a coequalizer diagram in $\mathcal{C}$.

Definition 1.3 An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$, $\mu_A : A \otimes A \to A$ are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. If $\mu_A \circ \tau_{A,A} = \mu_A$, then we will say that A is a commutative algebra.

Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$.

Examples 1.4

a) If $A = (A, \eta_A, \mu_A)$ is an algebra in $\mathcal{C}$ then $A^{op} = (A, \eta_A, \mu_A \circ \tau_{A,A})$ is an algebra in $\mathcal{C}$ that we will call opposite algebra of A.

b) If A, B are algebras in $\mathcal{C}$, we define the algebra product by

$$AB = (A \otimes B, \eta_A \otimes \eta_B, (\mu_A \otimes \mu_B) \circ (A \otimes \tau_{B,A} \otimes B)).$$

c) Each M of $\mathcal{C}$ determines an algebra in $\mathcal{C}$, $E_M = (E_M, \eta_{E_M}, \mu_{E_M})$ with $\eta_{E_M} := \alpha_M(K)$ and

$$\mu_{E_M} := HOM(M, \beta_M(M) \circ [\beta_M(M) \otimes E_M]) \circ \alpha_M(E_M \otimes E_M).$$

Definition 1.5 Let $A = (A, \eta_A, \mu_A)$ an algebra. (M, φ_M) is a left A-module if M is an object in $\mathcal{C}$ and $\varphi_M : A \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. With ${}_A\mathcal{C}$ we denote the category of left A-modules with morphisms those of $\mathcal{C}$ that preserve the structure. Similar definitions for right A-modules. Note that when $K = (K, \eta_K, \mu_K)$ is the trivial algebra in $\mathcal{C}$, then ${}_K\mathcal{C} = \mathcal{C}$

Examples 1.6

a) For all M of $\mathcal{C}$, $(M, \beta_M(M))$ is a right E_M-module.

b) M^* is a left E_M-module in $\mathcal{C}$ with structure

$$\varphi_{M^*} = HOM(M, \beta_M(K) \circ (\beta_M(M) \otimes M^*)) \circ \alpha_M(E_M \otimes M^*)$$

Definition 1.7 A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$, $\delta_D : D \to D \otimes D$ are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $\tau_{D,D} \circ \delta_D = \delta_D$, then we will say that D is a cocommutative coalgebra.

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$.

Definition 1.8 Let $H_1 = (H, \eta_H, \mu_H)$ be an algebra and $H_2 = (H, \varepsilon_H, \delta_H)$ a coalgebra and let $\lambda : H \to H$ be a morphism. Then $(H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda)$ is a Hopf algebra in $\mathcal{C}$ if ε_H and δ_H are algebra morphisms (equivalently η_H and μ_H are coalgebra morphisms) and λ is such that

$$\mu_H \circ (H \otimes \lambda) \circ \delta_H = \varepsilon_H \otimes \eta_H = \mu_H \circ (\lambda \otimes H) \circ \delta_H$$

If H_1 is commutative we say that H is commutative. Analogously, H is cocommutative if H_2 is cocommutative.

If H is finite then H is a progenerator [9]. As a consequence, H is faithfully flat.

1.9 If $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda)$ is a finite Hopf algebra (i.e. H is a finite object in $\mathcal{C}$) we will denote the dual Hopf algebra of H by $H^* = (H^*, \eta_{H^*}, \mu_{H^*}, \varepsilon_{H^*}, \delta_{H^*}, \lambda^*)$ where $\eta_{H^*} = (H^* \otimes \varepsilon_H) \circ a_H(K)$, $\mu_{H^*} = (H^* \otimes b_H(K)) \circ (H^* \otimes H \otimes b_H(K) \otimes H^*) \circ (H^* \otimes (\tau_{H,H} \circ \delta_H) \otimes H^* \otimes H^*) \circ (a_H(K) \otimes H^* \otimes H^*)$, $\varepsilon_{H^*} = b_H(K) \circ (\eta_H \otimes H^*)$, $\delta_{H^*} = ((H^* \otimes H^*) \otimes (b_H(K) \circ ((\mu_H \circ \tau_{H,H}) \otimes H^*))) \circ (H^* \otimes a_H(K) \otimes H \otimes H^*) \circ (a_H(K) \otimes H^*)$ and $\lambda^* = (H^* \otimes b_H(K)) \circ (H^* \otimes \lambda \otimes H^*) \circ (a_H(K) \otimes H^*)$.

Definition 1.10 Let H be a Hopf algebra. $(A; \varphi_A) = ((A, \eta_A, \mu_A); \varphi_A)$ is a left H-module algebra if $A = (A, \eta_A, \mu_A)$ is an algebra in $\mathcal{C}$, (A, φ_A) is a left H-module and η_A and μ_A are morphisms of left H-modules. Let $(A; \varphi_A)$, $(B; \varphi_B)$ be left H-module algebras. A morphism of left H-module monoids $f : A \to B$ is a morphism of algebras and left H-modules.

Examples 1.11

a) Let H be a cocommutative Hopf algebra. If $(A; \varphi_A)$ and $(B; \varphi_B)$ are left H-module algebras then $(AB; \varphi_{A \otimes B})$ and $(A^{op}; \varphi_A)$ are left H-module algebras where $\varphi_{A \otimes B} = (\varphi_A \otimes \varphi_B) \circ (H \otimes \tau_{H,A} \otimes B) \circ (\delta_H \otimes A \otimes B)$. Moreover, $\tau_{A,B} : A \otimes B \to B \otimes A$ is a morphism of left H-module algebras.

b) If (M, φ_M) and (N, φ_N) are left H-modules then

$$(HOM(M, N), \varphi_{HOM(M,N)})$$

is a left H-module where

$$\varphi_{HOM(M,N)} = HOM(M, \varphi_N \circ (H \otimes \beta_M(N)) \circ ([\tau_{M,H} \circ (\varphi_M \otimes H) \circ$$
$$(\tau_{M,H} \otimes H) \circ (M \otimes \tau_{H,H}) \circ (M \otimes H \otimes \lambda) \circ (M \otimes \delta_H)] \otimes HOM(M,N))) \circ$$
$$\alpha_M(H \otimes HOM(M,N))$$

With this structure, if H is a cocommutative Hopf algebra,

$$(E_M; \varphi_{E_M}) = ((E_M, \eta_{E_M}, \mu_{E_M}); \varphi_{E_M})$$

is a left H-module algebra.

Definition 1.12 Let H be a Hopf algebra. $(B; \rho_B) = ((B, \eta_B, \mu_B); \varphi_B)$ is a right H-comodule algebra if, $B = (B, \eta_B, \mu_B)$ is an algebra in $\mathcal{C}$, $(B; \rho_B)$ is a right H-comodule in $\mathcal{C}$ (i.e. $(\rho_B \otimes H) \circ \rho_B = (B \otimes \delta_H) \otimes \rho_B$ and $(B \otimes \varepsilon_H) \circ \rho_B = id_B$) and $\rho_B : B \to B \otimes H$ is an algebra morphism from B to the algebra product BH.

Example 1.13 If H is a commutative Hopf algebra and $(A; \rho_A)$, $(B; \rho_B)$ are right H-comodule algebras, then $((A^{op}; \rho_A)$ and $(AB; \rho_{A \otimes B})$ are right H-comodule algebras with $\rho_{A \otimes B} = (A \otimes B \otimes \mu_H) \circ (A \otimes \tau_{H,B} \otimes H) \circ (\rho_A \otimes \rho_B)$.

2 Galois H-objects with a normal basis and functoriality

In the next sections, H denotes a finite Hopf algebra in $\mathcal{C}$. We will suppose too that the basic object K is projective.

Definition 2.1 A right H-comodule algebra $(B; \rho_B)$ is said to be a Galois H-object if and only if:

i) The morphism $\gamma_B := (\mu_B \otimes H) \circ (B \otimes \rho_B) : B \otimes B \to B \otimes H$ is an isomorphism.

ii) B is a progenerator in $\mathcal{C}$.

Let $(B_1;\rho_{B_1})$, $(B_2;\rho_{B_2})$ be Galois H-objects. A morphism of Galois H-objects $f : B_1 \to B_2$ is a morphism of algebras and right H-comodules. Note that all morphisms of Galois H-objects are isomorphisms (see [7]).

Definition 2.2 If $(A;\rho_A)$, $(B;\rho_B)$ are right H-comodule algebras, then $A \circ_H B$, defined by the following equalizer diagram

$$A \circ_H B \xrightarrow{i^H_{AB}} A \otimes B \underset{\partial^{2H}_{AB}}{\overset{\partial^{1H}_{AB}}{\rightrightarrows}} A \otimes B \otimes H$$

$$\partial^{1H}_{AB} = (A \otimes \tau_{H,B}) \circ (\rho_A \otimes B) \qquad \partial^{2H}_{AB} = A \otimes \rho_B$$

is a right H-comodule algebra , to be denoted by $(A \circ_H B;\rho_{A\circ_H B})$, where $\mu_{A\circ_H B}$ (resp. $\eta_{A\circ_H B}$) is the factorization of the morphism $\mu_{A\otimes B} \circ (i^H_{AB} \otimes i^H_{AB})$ (resp. $\eta_A \otimes \eta_B$) through i^H_{AB} and $\rho_{A\circ_H B}$ is the factorization of $\partial^{1H}_{AB} \circ i^H_{AB}$ through the equalizer $i^H_{AB} \otimes H$.

When $(A;\rho_A)$, $(B;\rho_B)$ are Galois H-objects then $(A \circ_H B;\rho_{A\circ_H B})$ is also a Galois H-object (see (4.4.2) of [7]).

Definition 2.3 The set of isomorphism classes of Galois H-objects, with the operation defined in (2.2), is an abelian group to be denoted by $Gal_C(H)$. The unit element is the class of $(H;\delta_H)$ and the inverse of $[(B;\rho_B)]$ is $[(B^{op};(B \otimes \lambda) \circ \rho_B)]$.

Example 2.4 In the case of a finitely generated projective and cocommutative Hopf algebra H over a commutative ring R, $Gal_C(H)$ generalizes the group obtained by S. Chase and M. Sweedler in [5]. See [4] for more details.

Proposition 2.5 *Let $\varphi : G \to H$ be a morphism of Hopf algebras. If $(B;r_B)$ is a a right G-comodule algebra then $(B;\rho_B = (B \otimes \varphi) \circ r_B)$ is a right H-comodule algebra.*

Proof: Straightforward.

Remark 2.6 As a particular instance of 2.5 we obtain that $(G; \rho'_G = (G \otimes \varphi) \circ \delta_G)$ is a right H-comodule algebra.

Proposition 2.7 *Let $\varphi : G \to H$ be a morphism of finite Hopf algebras where G is cocommutative. Let $(A; \rho_A)$ be a right H-comodule algebra and $(B; r_B)$, $(C; r_C)$ be right G-comodule algebras. If A and C are flat, then*

$$A \circ_H (B \circ_G C) \approx (A \circ_H B) \circ_G C$$

as right G-comodule algebras.

Proof: Using the cocommutativity of G is not difficult to see that $(A \circ_H B, r_{A\circ_H B})$ is a G-comodule algebra, where $r_{A\circ_H B} : A \circ_H B \to A \circ_H B \otimes G$ is the factorization, through the equalizer $i^H_{AB} \otimes G$, of the morphism $(A \otimes r_B) \circ\, i^H_{AB}$. Analogously, let $r_{A\circ_H(B\circ_G C)}$ be the G-comodule structure for $A \circ_H (B \circ_G C)$. Note that, in this case $r_{A\circ_H(B\circ_G C)}$, satisfies the equality

$$(i^H_{A(B\circ_G C)} \otimes G) \circ r_{A\circ_H(B\circ_G C)} = (A \otimes r_{B\circ_G C}) \circ i^H_{A(B\circ_G C)}$$

being $r_{B\circ_G C}$ the G-comodule structure for $B \circ_G C$.

Now we prove that $A \circ_H (B \circ_G C)$ is the equalizer of the morphisms $\partial^{1G}_{(A\circ_H B)C}$ and $\partial^{2G}_{(A\circ_H B)C}$. Indeed, in the diagram

$$\begin{array}{ccccc}
A \circ_H (B \circ_G C) & \xrightarrow{i^H_{A(B\circ_G C)}} & A \otimes (B \circ_G C) & \overset{\partial^{1H}_{A(B\circ_G C)}}{\underset{\partial^{2H}_{A(B\circ_G C)}}{\rightrightarrows}} & A \otimes (B \circ_G C) \otimes H \\
 & & \downarrow {\scriptstyle A \otimes i^G_{BC}} & & \downarrow {\scriptstyle \Upsilon} \\
(A \circ_H B) \otimes C & \xrightarrow{i^H_{AB} \otimes C} & A \otimes B \otimes C & \overset{\partial^{1H}_{AB} \otimes C}{\underset{\partial^{2H}_{AB} \otimes C}{\rightrightarrows}} & A \otimes B \otimes H \otimes C \\
{\scriptstyle \partial^{1G}_{(A\circ_H B)C}} \downarrow\downarrow {\scriptstyle \partial^{2G}_{(A\circ_H B)C}} & & {\scriptstyle A \otimes \partial^{1G}_{BC}} \downarrow\downarrow {\scriptstyle A \otimes \partial^{2G}_{BC}} & & \\
(A \circ_H B) \otimes C \otimes G & \xrightarrow{i^H_{AB} \otimes C \otimes G} & A \otimes B \otimes C \otimes G & &
\end{array}$$

where $\Upsilon = (A \otimes B \otimes \tau_{C,H}) \circ (A \otimes i^G_{BC} \otimes H)$, we have that

$$(\partial^{1H}_{AB} \otimes C) \circ (A \otimes i^{G}_{BC}) = \Upsilon \circ \partial^{1H}_{A(B\circ_G C)}$$

$$(\partial^{2H}_{AB} \otimes C) \circ (A \otimes i^{G}_{BC}) = \Upsilon \circ \partial^{2H}_{A(B\circ_G C)}$$

$$(A \otimes \partial^{1G}_{BC}) \circ (i^{H}_{AB} \otimes C) = (i^{H}_{AB} \otimes C \otimes G) \circ \partial^{1G}_{(A\circ_H B)C}$$

$$(A \otimes \partial^{2G}_{BC}) \circ (i^{H}_{AB} \otimes C) = (i^{H}_{AB} \otimes C \otimes G) \circ \partial^{2G}_{(A\circ_H B)C}$$

and therefore,

$$(\partial^{1H}_{AB} \otimes C) \circ (A \otimes i^{G}_{BC}) \circ i^{H}_{A(B\circ_G C)} =$$

$$(A \otimes B \otimes \tau_{C,H}) \circ (A \otimes i^{G}_{BC} \otimes H) \circ \partial^{1H}_{A(B\circ_G C)} \circ i^{H}_{A(B\circ_G C)} =$$

$$(A \otimes B \otimes \tau_{C,H}) \circ (A \otimes i^{G}_{BC} \otimes H) \circ \partial^{2H}_{A(B\circ_G C)} \circ i^{H}_{A(B\circ_G C)} =$$

$$(\partial^{2H}_{AB} \otimes C) \circ (A \otimes i^{G}_{BC}) \circ i^{H}_{A(B\circ_G C)}$$

As a consequence, there exists a morphism $g : A \circ_H (B \circ_G C) \to (A \circ_H B) \otimes C$ satisfying $(i^{H}_{AB} \otimes C) \circ g = (A \otimes i^{G}_{BC}) \circ i^{H}_{A(B\circ_G C)}$. Moreover,

$$(i^{H}_{AB} \otimes C \otimes G) \circ \partial^{1G}_{(A\circ_H B)C} \circ g = (A \otimes \partial^{1G}_{BC}) \circ (i^{H}_{AB} \otimes C) \circ g =$$

$$(A \otimes \partial^{1G}_{BC}) \circ (A \otimes i^{G}_{BC}) \circ i^{H}_{A(B\circ_G C)}$$

$$(A \otimes \partial^{2G}_{BC}) \circ (A \otimes i^{G}_{BC}) \circ i^{H}_{A(B\circ_G C)} = (A \otimes \partial^{2G}_{BC}) \circ (i^{H}_{AB} \otimes C) \circ g =$$

$$(i^{H}_{AB} \otimes C \otimes G) \circ \partial^{2G}_{(A\circ_H B)C} \circ g$$

and then, since C and G are finite, $\partial^{1G}_{(A\circ_H B)C} \circ g = \partial^{2G}_{(A\circ_H B)C} \circ g$. Hence, there exists an unique

$$g' : A \circ_H (B \circ_G C) \to (A \circ_H B) \circ_G C$$

such that $i^{G}_{(A\circ_H B)C} \circ g' = g$. It is an standard calculus to prove that g' is a morphism of G-comodule algebras. Next we show that g' is an isomorphism.

Let $l : Q \to (A \circ_H B) \otimes C$ be a morphism such that $\partial^{1G}_{(A\circ_H B)C} \circ l = \partial^{2G}_{(A\circ_H B)C} \circ l$. Then $(A \otimes \partial^{1G}_{BC}) \circ (i^{H}_{AB} \otimes C) \circ l = (A \otimes \partial^{2G}_{BC}) \circ (i^{H}_{AB} \otimes C) \circ l$ and there exists a unique map $h : Q \longrightarrow A\otimes(B\circ_G C)$ satisfying $(i^H_{AB}\otimes C)\circ l = (A\otimes i^G_{BC})\circ h$. Moreover, $\partial^{1H}_{A(B\circ_G C)}\circ h = \partial^{2H}_{A(B\circ_G C)}\circ h$. Let $f : Q \longrightarrow A\circ_H(B\circ_G C)$ be the unique morphism such that $i^H_{A(B\circ_G C)}\circ f = h$. For this morphism it is easy to see that $g\circ f = l$.

If $s : Q \longrightarrow A\circ_H(B\circ_G C)$ verifies $g\circ s = l$, by the equalities

$$(A\otimes i^G_{BC})\circ i^H_{A(B\circ_G C)}\circ s = (i^H_{AB}\otimes C)\circ g\circ s =$$

$$(i^H_{AB}\otimes C)\circ l = (A\otimes i^G_{BC})\circ h = (A\otimes i^G_{BC})\circ i^H_{A(B\circ_G C)}\circ f$$

we obtain that $s = f$. Therefore g' is an isomorphism.

The next result is a generalization of the one obtained by Wenninger in [11].

Proposition 2.8 *Let $\varphi : G \longrightarrow H$ be a morphism of finite Hopf algebras where G is cocommutative. If $(A;\rho_A)$ is a Galois H-object then the pair $(A\circ_H G; r_{A\circ_H G})$, where $r_{A\circ_H G}$ is the morphism defined in the proof of Proposition 2.7, is a Galois G-object.*

Proof: The diagrams

$$A\otimes A\circ_H G \xrightarrow{A\otimes i^H_{AG}} A\otimes A\otimes G \underset{A\otimes\partial^{2H}_{AG}}{\overset{A\otimes\partial^{1H}_{AG}}{\rightrightarrows}} A\otimes A\otimes G\otimes H$$

and

$$A\otimes G \xrightarrow{A\otimes\rho_G} A\otimes G\otimes H \underset{A\otimes\partial^{2H}_{GH}}{\overset{A\otimes\partial^{1H}_{GH}}{\rightrightarrows}} A\otimes G\otimes H\otimes H$$

are equalizer diagrams. On the other hand, by the cocommutativity of G,

$$(A\otimes\partial^{1H}_{GH})\circ(A\otimes\tau_{H,G})\circ(\gamma_A\otimes G)\circ(A\otimes i^H_{AG}) =$$

$$(A\otimes\partial^{2H}_{GH})\circ(A\otimes\tau_{H,G})\circ(\gamma_A\otimes G)\circ(A\otimes i^H_{AG})$$

and then there exists a morphism $f : A\otimes A \circ_H G \to A \otimes G$ such that

$$(A \otimes \rho_G) \circ f = (A \otimes \tau_{H,G}) \circ (\gamma_A \otimes G) \circ (A \otimes i^H_{AG})$$

Trivially, $f = (\mu_A \otimes G) \circ (A \otimes i^H_{AG})$. Moreover, f is an isomorphism with inverse the factorization through the equalizer $A \otimes i^H_{AG}$ of the morphism $(\gamma_A^{-1} \otimes G) \circ (A \otimes \tau_{G,H}) \circ (A \otimes \rho_G)$.

For the morphism $\gamma_{A\circ_H G}$ we have that:

$$(A\otimes((\mu_G\otimes G)\circ(G\otimes\delta_G)\circ\tau^G_G))\circ(f\otimes G)\circ(A\otimes\tau_{G,A\circ_H G})\circ(f\otimes A\circ_H G) =$$

$$(\mu_A \otimes (\mu_G \circ \tau_{G,G}) \otimes G) \circ (A \otimes A \otimes G \otimes \tau_{G,G})\circ$$

$$(A\otimes((i^H_{AG}\otimes G)\circ r_{A\circ_H G})\otimes G)\circ(\mu_A\otimes\tau_{G,A\circ_H G})\circ(A\otimes i^H_{AG}\otimes A\circ_H G) =$$

$$(\mu_A \otimes G \otimes G) \circ (A \otimes \mu_{A\otimes G} \otimes G) \circ (A \otimes i^H_{AG} \otimes i^H_{AG} \otimes G)\circ$$

$$(A \otimes A \circ_H G \otimes r_{A\circ_H G}) = (f \otimes G) \circ (A \otimes \gamma_{A\circ_H G})$$

and then, since A is finite, $\gamma_{A\circ_H G}$ is an isomorphism. Finally $A \circ_H G$ is a progenerator because $f : A\otimes A\circ_H G \to A\otimes G$ is an isomorphism and A, G are progenerators.

Proposition 2.9 *Let $\varphi : G \to H$ be a morphism of finite cocommutative Hopf algebras. There exists a morphism of abelian groups $Gal(\varphi) : Gal_C(H) \to Gal_C(G)$ defined by*

$$Gal(\varphi)([(A;\rho_A)]) = [(A \circ_H G; r_{A\circ_H G})]$$

Proof: Let (A, ρ_A) and (B, ρ_B) be Galois H-objects. Then, by 2.7 we obtain that:

$$(A \circ_H G) \circ_G (B \circ_H G) \approx (A \circ_H G) \circ_G (G \circ_H B) \approx$$

$$A \circ_H (G \circ_G (G \circ_H B)) \approx A \circ_H ((G \circ_H B) \circ_G G) \approx$$

$$A \circ_H ((B \circ_H G) \circ_G G) \approx A \circ_H ((B \circ_H (G \circ_G G)) \approx$$

$$A \circ_H (B \circ_H G) \approx (A \circ_H B) \circ_H G$$

Therefore $Gal(\varphi)$ is a group morphism.

Definition 2.10 Let be H be a finite cocommutative Hopf algebra. We say that a Galois H-object $(A;\rho_A)$ has a normal basis if is isomorphic with H as an H-comodule.

The set of isomorphisms classes of Galois H-objects with a normal basis $N_{\mathcal{C}}(H)$ is a subgroup of $Gal_{\mathcal{C}}(H)$ (see 2.5 of [1]).

Proposition 2.11 *Let $\varphi : G \to H$ be a morphism of finite Hopf algebras where G is cocommutative. If $(A;\rho_A)$ is a Galois H-object with a normal basis then $(A \circ_H G; r_{A\circ_H G})$ is a Galois G-object with a normal basis.*

Proof: We define $h := (f \otimes G) \circ (\varphi \otimes G) \circ \delta_G$, where $f : H \to A$ is the H-comodule isomorphism wich exists because $(A,\, \rho_A)$ has a normal basis. Using the cocommutativity of G, h factorizes through the equalizer i^H_{AG}. Let $g : H \to A \circ_H G$ be this factorization. An straightforward verification yields that g is an isomorphism of G-comodule algebras with inverse $g^{-1} = ((\varepsilon_H \circ f^{-1}) \otimes G) \circ i^H_{AG}$.

Remark 2.12 Let $\varphi : G \to H$ be a morphism of cocommutative Hopf algebras. As a consequence of 2.9 and 2.11 we obtain that there exists a commutative diagram of abelian groups:

$$\begin{array}{ccc} N_{\mathcal{C}}(H) & \overset{i}{\hookrightarrow} & Gal_{\mathcal{C}}(H) \\ {\scriptstyle N(\varphi)}\downarrow & & \downarrow{\scriptstyle Gal(\varphi)} \\ N_{\mathcal{C}}(G) & \overset{i'}{\hookrightarrow} & Gal_{\mathcal{C}}(G) \end{array}$$

where $N(\varphi)$ is the restriction of $Gal(\varphi)$.

2.13 With $G(\mathcal{C}, H)$ we will denote the category whose objects are the Galois H-objects and whose morphisms are the morphisms of Galois H-objects. The product of Galois H-objects defines a product, in the sense of Bass [2], in $G(\mathcal{C}, H)$ and it is easy to prove that $Gal_{\mathcal{C}}(H) \approx K_0G(\mathcal{C}, H)$. Analogously we construct the category $N(\mathcal{C}, H)$ of Galois H-objects with a normal basis. This category has a product too and $N_{\mathcal{C}}(H) \approx K_0N(\mathcal{C}, H)$.

The category $\mathcal{C}(H) = \{(H;\delta_H)\}$ is a cofinal subcategory of $G(\mathcal{C}, H)$ and $N(\mathcal{C}, H)$, then we have that $K_1\mathcal{C}(H) \approx K_1G(\mathcal{C}, H) \approx K_1N(\mathcal{C}, H)$.

Moreover, $K_1\mathcal{C}(H)$ is isomorphic with $(G(H^*), *)$, the commutative group of grouplike morphisms of H^*; that is, the set of morphisms $h : K \to H^*$ such that $\delta_{H^*} \circ h = h \otimes h$, $\varepsilon_{H^*} \circ h = id_K$ with the operation of convolution $h * h' = \mu_{H^*} \circ (h \otimes h')$ (see [8]).

Let $\varphi : G \to H$ be a morphism of cocommutative Hopf algebras. There exists functors

$$\mathcal{G}(\varphi) : G(\mathcal{C}, H) \to G(\mathcal{C}, G) \qquad \mathcal{N}(\varphi) : N(\mathcal{C}, H) \to N(\mathcal{C}, G)$$

defined by $\mathcal{G}(\varphi)((A, \rho_A)) = (A \circ_H G; r_{A \circ_H G})$ and $\mathcal{N}(\varphi) = \mathcal{G}(\varphi)_{|N(\mathcal{C},H)}$. These functors preserve the product and are cofinal because if (B, r_B) is a Galois G-object then

$$(B \circ_G B^{op}, r_{B \circ_G B^{op}}) \approx (G, \delta_G) \approx \mathcal{G}(\varphi)((H; \delta_H))$$

Therefore, using K-theoretical arguments we obtain a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} G(H^*) & \xrightarrow{Gr(\varphi)} & G(G^*) & \longrightarrow & K_1N(\varphi) & \longrightarrow & N_{\mathcal{C}}(H) & \xrightarrow{N(\varphi)} & N_{\mathcal{C}}(G) \\ & & & \searrow & \downarrow j & & \downarrow i & & \downarrow i' \\ & & & & K_1G(\varphi) & \longrightarrow & Gal_{\mathcal{C}}(H) & \xrightarrow[Gal(\varphi)]{} & Gal_{\mathcal{C}}(G) \end{array}$$

where $Gr(\varphi) : G(H^*) \to G(G^*)$ is defined by

$$Gr(\varphi)(g) = (H \otimes b_H) \circ (\delta_H \otimes [f^{-1} \circ (((H^* \otimes (\varepsilon_H \circ g)) \circ a_H) \circ_H G) \circ f])$$

where f the isomorphism between G and $H \circ_H G$.

3 Naturality of Beattie's decomposition theorem

Definition 3.1 An algebra $A = (A, \eta_A, \mu_A)$ is said to be Azumaya if A is a progenerator in $\mathcal{C}$ and the morphism $\mathcal{X}_A : A \otimes A \to E_A$ between the algebras $A^{op}A$ and E_A defined by

$$\mathcal{X}_A := HOM(A, \mu_A \circ (A \otimes \mu_A) \circ (\tau_{A,A} \otimes A)) \circ \alpha_A(A \otimes A)$$

is an isomorphism.

Definition 3.2 On the set of isomorphism classes of left H-module Azumayan algebras we define the equivalence relation: $(A;\varphi_A) \sim (B;\varphi_B)$ if there exist (M,φ_M), (N,φ_N) left H-module progenerators in $\mathcal{C}$ and an isomorphism of left H-module algebras between $(AE_M^{op};\varphi_{A\otimes E_M})$ and $(BE_N^{op};\varphi_{A\otimes E_N})$.

The set of equivalence classes of left H-module Azumayan algebras is a group under the operation induced by the tensor product. The unit element is the class of the left H-module Azumayan algebra $(E_M;\varphi_{E_M})$, for some progenerator H-module (M,φ_M), and the inverse of $(A;\varphi_A)$ is $(A^{op};\varphi_A)$. This group is denoted by $BM(\mathcal{C},H)$ and the class of $(A;\varphi_A)$ by $[(A;\varphi_A)]$.

If $H=(1,1,\tau^K,1)$ is the trivial Hopf algebra in $\mathcal{C}$, then $BM(\mathcal{C},H)$ is the Brauer group, $B(\mathcal{C})$, of the symmetric closed category $\mathcal{C}$ (see [10], [7]).

Definition 3.3 For each Hopf algebra H and each left H-module algebra $(A;\varphi_A)$, the smash product $A\sharp H$ is defined by

$$A\sharp H = (A\otimes H, \eta_{A\sharp H}, \mu_{A\sharp H})$$

where

$$\eta_{A\sharp H} = \eta_A \otimes \eta_H$$

$$\mu_{A\sharp H} = (\mu_A\otimes\mu_H)\circ(A\otimes\varphi_A\otimes H\otimes H)\circ(A\otimes H\otimes\tau_{H,A}\otimes H)\circ (A\otimes\delta_H\otimes A\otimes H)$$

Definition 3.4 Let $(A;\varphi_A)$ a left H-module Azumayan algebra. We define the object $\Pi(A)$ by the equalizer diagram

$$\Pi(A) \xrightarrow{j_{A\sharp H}} A\otimes H \underset{n_{A\sharp H}}{\overset{m_{A\sharp H}}{\rightrightarrows}} HOM(A, A\otimes H)$$

$$m_{A\sharp H} = HOM(A,\mu_{A\sharp H}\circ(A\otimes\eta_H\otimes A\otimes H))\circ\alpha_A(A\otimes H) =$$

$$HOM(A,\mu_A\otimes H)\circ\alpha_A(A\otimes H)$$

$$n_{A\sharp H} = HOM(A,\mu_{A\sharp H}\circ(A\otimes\tau_{A,H}\otimes H)\circ(\tau_{A,A}\otimes H\otimes\eta_H))\circ$$

$\alpha_A(A \otimes H) =$

$HOM(A, (\mu_A \otimes H) \circ (A \otimes (\varphi_A \circ \tau_{A,H}) \otimes H) \circ (\tau_{A,A} \otimes \delta_H)) \circ \alpha_A(A \otimes H)$

$(\Pi(A) = (\Pi(A), \eta_{\Pi(A)}, \mu_{\Pi(A)}); \rho_{\Pi(A)})$ is a Galois H-object where $\mu_{\Pi(A)}$ (resp. $\eta_{\Pi(A)}$) is the factorization through the equalizer $j_{A\sharp H}$ of the morphism $\mu_{A\sharp H} \circ (j_{A\sharp H} \otimes j_{A\sharp H})$ (resp. $\eta_{A\sharp H}$) and $\rho_{\Pi(A)}$ is the factorization through the equalizer $j_{A\sharp H} \otimes H$ of the morphism $(A \otimes \delta_H) \circ j_{A\sharp H}$ (see [7]).

3.5 There is an epimorphism of abelian groups $\Pi : BM(\mathcal{C}, H) \rightarrow Gal_{\mathcal{C}}(H)$ given by $\Pi([(A; \varphi_A)]) = [(\Pi(A); \rho_{\Pi(A)})]$.

If $[(B; \rho_B)] \in Gal_{\mathcal{C}}(H)$, then $[(B\sharp H^*; \varphi_{B\sharp H^*} = (B \otimes H^* \otimes b_H) \circ (B \otimes \tau_{H,H^*} \otimes H^*) \circ (\tau_{H,B} \otimes \delta_{H^*}))]$ is in $BM(\mathcal{C}, H)$ and there is an isomorphism of Galois H-objects between B and $\Pi(B\sharp H^*)$.

The sequence (Beattie's decomposition theorem [3])

$$0 \rightarrow B(\mathcal{C}) \xrightarrow{i_H} BM(\mathcal{C}, H) \xrightarrow{\Pi} Gal_{\mathcal{C}}(H) \rightarrow 0$$

is split exact, where the morphism i_H is given by $i_H([A]) = [(A; \varepsilon_H \otimes A)]$ and the morphism $j : BM(\mathcal{C}, H) \rightarrow B(\mathcal{C})$ defined by $j([(A; \varphi_A)]) = [A]$ is a retraction (see [7]).

Definition 3.6 For an algebra $A = (A, \eta_A, \mu_A)$ and a coalgebra $D = (D, \varepsilon_D, \delta_D)$, we denote by $Reg(D, A)$ the group of invertible elements in the set of morphisms in $\mathcal{C}$ $f : D \rightarrow A$. The operation in this group is the convolution given by $f \wedge g = \mu_A \circ (f \otimes g) \circ \delta_D$. The unit element is $\varepsilon_D \otimes \eta_A$.

Definition 3.7 Let H be a Hopf algebra and $A = (A, \eta_A, \mu_A)$ be an algebra. We say that an action φ_A of H in A is inner if there exists a morphism $f \in Reg(H, A)$ such that

$$\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ \tau_{A,A})) \circ (f \otimes f^{-1} \otimes A) \circ (\delta_H \otimes A) : H \otimes A \rightarrow A$$

where f^{-1} is the convolution inverse of f.

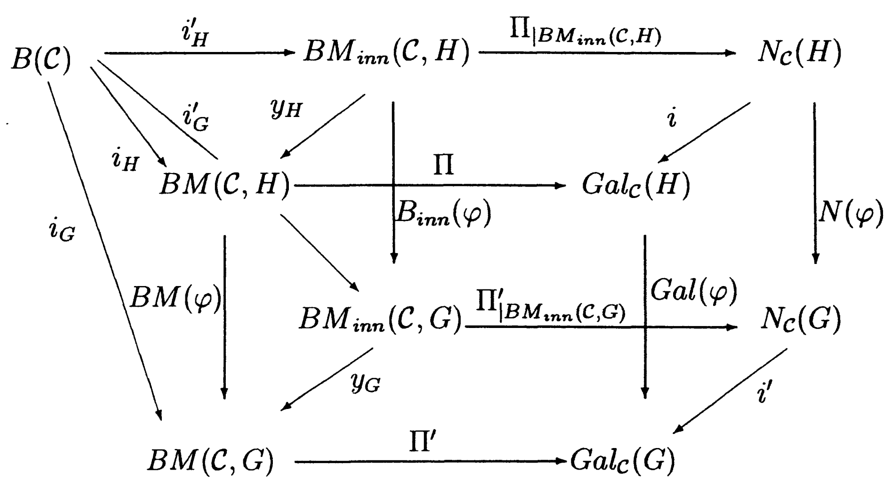
Definition 3.8 Let H be a finite cocommutative Hopf algebra. We denote by $BM_{inn}(C,H)$ the subset of $BM(C,H)$ built up with the equivalence classes that can be represented by an H-module Azumayan algebra with inner action.

3.9 The set $BM_{inn}(C,H)$ is a subgroup of $BM(C,H)$ (4.4 of [1]). We denote by y_H the inclusion morphism. Finally, the sequence (Beattie's decomposition theorem for inner actions)

$$0 \to B(C) \stackrel{i'_H}{\to} BM_{inn}(C,H) \stackrel{\Pi_{|BM_{inn}(C,H)}}{\longrightarrow} N_C(H) \to 0$$

is split exact (see 4.5 of [1]).

3.10 Finally, using the results of section two and the decomposition theorems of 3.5 and 3.9, for a morphism $\varphi : G \to H$ between finite cocommutative Hopf algebras there exists a commutative diagram of abelian groups:



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