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A NOTE ON THE HOLONOMY OF CONNECTIONS IN TWISTED BUNDLES

by *Marco MACKAAY*

RESUME. Les fibrés vectoriels tordus avec connexions ont été considérés à différents endroits (cf. Bouwknegt & Mathai [2], Kapustin [8] et les références qu'ils donnent). Dans cette note l'auteur considère les fibrés principaux tordus avec connexions et il étudie leur holonomie, qui se formule le plus naturellement possible en termes de foncteurs entre groupes catégoriques.

Introduction

Let M be a connected finite-dimensional smooth manifold. The holonomy map corresponding to a connection in a principal G -bundle over M yields a smooth group homomorphism

$$\mathcal{H}: \pi_1^{\text{thin}}(M) \rightarrow G,$$

where $\pi_1^{\text{thin}}(M)$ is the *thin fundamental group* of M and G a Lie group. In Section 1 the reader can find the precise definition of thin homotopy, but intuitively a homotopy between two loops is thin if it does not sweep out any area. This formulation of holonomy is due to Barrett [1] (see also Caetano and Picken's work [5]), who also proved that one can reconstruct both the bundle and the connection from the holonomy.

A natural question is whether there are generalizations of Barrett's results which involve "higher thin homotopy types" of M . Caetano and Picken [6] defined higher thin homotopy groups of M , denoted $\pi_n^{\text{thin}}(M)$. One way to describe the full homotopy 2-type of M , which contains

more information than just $\pi_1(M)$ and $\pi_2(M)$, is by means of the *fundamental categorical group* of M , denoted $C_2(M)$ (see Section 1). Recall that a categorical group is a group object in the category of groupoids. Picken and the present author [9] defined the *thin fundamental categorical group*, denoted $C_2^2(M)$, which encodes the information about the thin homotopy 2-type of M . In that same paper we showed that if M is simply connected, then a smooth group homomorphism $\pi_2^2(M) \rightarrow U(1)$ corresponds precisely to the holonomy map of a $U(1)$ -gerbe with gerbe-connection. If M is not simply-connected we showed that the gerbe-holonomy can be described as a smooth functor between $C_2^2(M)$ and a certain categorical group over $\pi_1^1(M)$, derived from the canonical line-bundle over the loop space of M corresponding to the gerbe.

This paper is about the next question: given an arbitrary categorical Lie group, \mathcal{G} , what geometrical structure on M yields a holonomy functor between $C_2^2(M)$ and \mathcal{G} ? Theorem 3.6 shows that, for transitive categorical Lie groups, i.e. for those which contain an arrow between any two objects, the answer is *twisted principal bundles with connection*, which we define in Section 2.

1 Categorical groups

Definition 1.1 *A categorical group is a group object in the category of groupoids. This means that it is a groupoid with a monoidal structure (a multiplication) which satisfies the group laws strictly. A categorical Lie group is a group object in the category of Lie groupoids, which means that the underlying groupoid is a Lie groupoid and that the tensor product defines a smooth operation with smooth inverses.*

For some general theory about categorical groups we refer to [3]. Throughout the paper, let M be a connected finite-dimensional smooth manifold and let $*$ be a base-point in M . Our first example in this paper is the *fundamental categorical group* of M , denoted $C_2(M)$. We want to work with smooth loops and homotopies in M , but the problem is that their composites need not be smooth in general. However, there is a subset of smooth loops and homotopies whose composites are smooth:

Definition 1.2 [5, 6, 9] A based loop $\ell: [0, 1] \rightarrow M$ is said to have a sitting point at $t_0 \in [0, 1]$, if there exists an $\epsilon > 0$ such that ℓ is constant on $[t_0 - \epsilon, t_0 + \epsilon]$. We denote the set of all smooth based loops in M with 0 and 1 as sitting points by $\Omega^\infty(M)$.

Similarly, a based homotopy $H: [0, 1] \times [0, 1] \rightarrow M$, which I call a cylinder, has a sitting point (s_0, t_0) , if there exists an $\epsilon > 0$ such that H is constant on the disc with centre (s_0, t_0) and radius ϵ in $[0, 1] \times [0, 1]$. The set of all smooth based homotopies with all points in the boundary of $[0, 1] \times [0, 1]$ being sitting points is denoted by $\Omega_2^\infty(M)$.

In order to define $C_2(M)$, we need to introduce the notion of thin homotopy:

Definition 1.3 [1, 5] Two loops, ℓ and ℓ' , are called thin homotopic if there exists a homotopy between them whose rank is at most equal to 1 everywhere, which is denoted by $\ell \stackrel{1}{\sim} \ell'$.

Definition 1.4 [1, 5] The thin fundamental group of M , denoted $\pi_1^1(M)$, consists of all thin homotopy classes of elements in $\Omega^\infty(M)$. The group operation is induced by the usual composition of loops.

We can define $C_2(M)$ as follows:

Definition 1.5 The objects of $C_2(M)$ are the elements of $\pi_1^1(M)$, which we temporarily denote by $[\gamma]$.

For any $\alpha, \beta, \gamma, \mu \in \Omega^\infty(M)$ and for any homotopies $G: \alpha \rightarrow \beta$ and $H: \gamma \rightarrow \mu$, we say that G and H are equivalent if there exist thin homotopies $A: \alpha \rightarrow \gamma$ and $B: \beta \rightarrow \mu$ such that

$$AHB^{-1} \sim G. \tag{1}$$

The morphisms between $[\gamma]$ and $[\mu]$ are the equivalence classes of

$$\bigcup_{\alpha, \beta} \{H: \alpha \rightarrow \beta \mid [\alpha] = [\gamma], [\beta] = [\mu]\}$$

modulo this equivalence relation.

The usual compositions of loops and homotopies define the structure of a categorical group on $C_2(M)$, as proved in [9].

Remark 1.6 *The usual definition of the fundamental categorical group of M yields a weak monoidal groupoid, because the objects are taken to be the loops themselves rather than their thin homotopy classes. In [9] Picken and the author defined this strict model.*

Similarly we can define the *thin fundamental categorical group* of M , denoted $C_2^2(M)$.

Definition 1.7 [6, 9] *Two cylinders, c and c' , are called thin homotopic if there exists a homotopy between them whose rank is at most equal to 2 everywhere, which is denoted by $c \stackrel{2}{\sim} c'$.*

Definition 1.8 [9] *The categorical group $C_2^2(M)$ is defined exactly as $C_2(M)$ except that the equivalence relation (1) is now*

$$AHB^{-1} \stackrel{2}{\sim} G.$$

Next we show how to construct a categorical group from any central extension of groups,

$$1 \rightarrow H \rightarrow E \xrightarrow{\pi} G \rightarrow 1. \quad (2)$$

We first construct the underlying groupoid, denoted $E \times E/H \rightrightarrows G$. This is a well-known construction due to Ehresmann (see [10] for references).

Definition 1.9 *The objects of*

$$E \times E/H \rightrightarrows G \quad (3)$$

are the elements of G , the morphisms are equivalence classes in $E \times E/H$, where the action of H is defined by $(e_1, e_2)h = (e_1h, e_2h)$. Let us denote such an equivalence class by $[e_1, e_2]$, and consider it to be a morphism from $\pi(e_1)$ to $\pi(e_2)$. Composition is defined by $[e_1, e_2h][e_2, e_3] = [e_1, e_3h]$, where $h \in H$. The identity morphism or unit of $g \in G$ is taken to be $1_g = [e, e]$, for any $e \in E$ such that $\pi(e) = g$. The inverse of $[e_1, e_2]$ is $[e_2, e_1]$.

Lemma 1.10 *The group operations on G and E induce a monoidal structure on (3). The tensor product on objects is simply the group operation on G . On morphisms the tensor product is defined by $[e_1, e_2] \otimes [e_3, e_4] = [e_1 e_3, e_2 e_4]$. Because G and E are groups, this makes (3) into a categorical group.*

Proof: Since the extension is central, the composition and tensor product satisfy the *interchange law*, i.e.

$$([e_1, e_2] \otimes [e_3, e_4])([e'_1, e'_2] \otimes [e'_3, e'_4]) = ([e_1, e_2][e'_1, e'_2]) \otimes ([e_3, e_4][e'_3, e'_4]),$$

whenever both sides of the equation make sense. The other requirements for a monoidal structure follow immediately from the group axioms in G and E . \square

Lemma 1.11 *If (2) is a central extension of Lie groups, then (3) yields a categorical Lie group.*

Proof: It is well-known that $E \xrightarrow{\pi} G$ defines a principal H -bundle. See [10] for a proof that (3) is a locally trivial Lie groupoid for any principal H -bundle. Clearly the tensor product and the inverses are smooth as well. \square

Clearly we can recover E from (3) by considering the subgroup of all morphisms of the form $[1, e]$ with the tensor product as group operation. The target map then defines the projection onto G with kernel H . There is a simple characterization of categorical groups coming from central extensions.

Definition 1.12 *A categorical group is called transitive, if there is a morphism between any two objects.*

Lemma 1.13 *There is a bijective correspondence between transitive categorical (Lie) groups and central extensions of (Lie) groups.*

Proof: An arbitrary transitive categorical group, \mathcal{G} , corresponds, in the way explained above, to the central extension

$$\mathcal{G}_1(1, \bullet) \xrightarrow{t} \mathcal{G}_0,$$

where $\mathcal{G}_1(1, \bullet)$ is the set of all 1-morphisms starting at the unit object, \mathcal{G}_0 is the set of all objects and t is the target map. The transitivity ensures that t is surjective. In any categorical group t is a group homomorphism and the interchange law, mentioned already in the proof of Lemma 1.10, ensures that $\mathcal{G}_1(1, 1)$ is central in $\mathcal{G}_1(1, \bullet)$. \square

Note that $C_2(M)$ and $C_2^2(M)$ are transitive if and only if M is simply connected.

Remark 1.14 *If \mathcal{G} is not transitive, then it does not correspond to a central extension, but to something more general called a crossed module. For an explanation we refer to [3].*

2 Twisted principal bundles and connections

A *twisted bundle* is a geometric structure whose failure to be a bundle is defined by an abelian Čech 2-cocycle. They appear in the literature in several places [2, 8]. In this section I have tried to give a systematic exposition of some basic facts about twisted bundles and connections, using Brylinski's construction [4] of the abelian gerbe which expresses the obstruction to lifting a principal G -bundle to a central extension E of G . Nothing in this section is new strictly speaking, but I hope that writing out everything explicitly is useful for the reader.

Let $\mathcal{U} = \{U_i : i \in \mathcal{I}\}$ be a *good* covering of M of open sets, i.e. all intersections

$$U_{i_1 \dots i_n} = U_{i_1} \cap \dots \cap U_{i_n}$$

of elements of \mathcal{U} are contractible or empty. From now on we fix a central extension of Lie groups, denoted as in (2).

Definition 2.1 *A twisted principal E -bundle, usually denoted \mathcal{P} , consists of a principal G -bundle, P , and a set of local principal E -bundles $Q_i \xrightarrow{q_i} U_i$, which allow for the natural projections $Q_i \xrightarrow{p_i} Q_i/H$, together with a set of bundle isomorphisms $\theta_i: Q_i/H \rightarrow P_i = P|_{U_i}$ and a set of bundle isomorphisms $\phi_{ij}: Q_i|_{U_{ij}} \rightarrow Q_j|_{U_{ij}}$ such that $\phi_{ji} = \phi_{ij}^{-1}$ holds and*

the following diagram commutes:

$$\begin{array}{ccc}
 Q_i & \xrightarrow{\phi_{ij}} & Q_j \\
 p_i \downarrow & & \downarrow p_j \\
 Q_i/H & \xrightarrow{\theta_j^{-1}\theta_i} & Q_j'/H
 \end{array} \tag{4}$$

Two twisted principal E -bundles, denoted $\mathcal{P} = (P, Q_i, \theta_i, \phi_{ij})$ and $\mathcal{P}' = (P', Q'_i, \theta'_i, \phi'_{ij})$, are equivalent if there are bundle isomorphisms $\psi: P \rightarrow P'$ and $\phi_i: Q_i \rightarrow Q'_i$ such that the following diagram commutes:

$$\begin{array}{ccc}
 Q_i & \xrightarrow{\phi_i} & Q'_i \\
 p_i \downarrow & & \downarrow p'_i \\
 Q_i/H & \xrightarrow{\theta_i'^{-1}\psi\theta_i} & Q'_i/H
 \end{array} \tag{5}$$

The following lemma is an easy consequence of our definitions and we leave its proof as an exercise.

Lemma 2.2 *The commutativity of (4) implies that there exists a smooth Čech 2-cocycle on M with values in H , given by local functions $h_{i,j,k}: U_{i,j,k} \rightarrow H$, such that*

$$\phi_{ki}\phi_{jk}\phi_{ij}(q) = qh_{i,j,k}(q_i(q)),$$

holds, for any $q \in Q_i|_{U_{i,j,k}}$.

The commutativity of (5) implies that there exists a Čech 1-cochain on M with values in H , given by local functions $h_{ij}: U_{ij} \rightarrow H$, such that

$$\phi'_{ij}(q) = \phi_j\phi_{ij}\phi_i^{-1}(q)h_{ij}(q_i(q))$$

holds, for any $q \in Q_i|_{U_{ij}}$. Furthermore, the equation

$$h'_{i,j,k} \equiv h_{i,j,k}h_{ij}h_{jk}h_{ki}$$

holds on $U_{i,j,k}$.

Remark 2.3 *Brylinski [4] shows that, given a principal G -bundle and the central extension, there is a canonical H -gerbe associated to them, whose equivalence class is represented by h_{ijk} in the previous lemma. The Q_i in Def. 2.1 are local trivializations of that gerbe and each ϕ_{ij} is an isomorphism between two different trivializations. As he shows, one can always choose (Q_i, ϕ_{ij}) which define a twisted E -bundle and any two choices lead to equivalent twisted E -bundles.*

Remark 2.4 *Choosing trivializations of all Q_i yields a definition of the twisted E -bundle in terms of smooth functions $e_{ij}: U_{ij} \rightarrow E$ such that $c_{ji} = c_{ij}^{-1}$ and*

$$e_{ij}e_{jk}e_{ki} \equiv h_{ijk}$$

holds on U_{ijk} . Similarly, one can express the equivalence of twisted E -bundles by smooth functions $e_i: U_i \rightarrow E$ satisfying

$$e'_{ij} \equiv e_i^{-1}e_{ij}e_jh_{ij}$$

on U_{ij} .

Definition 2.5 *A twisted principal E -bundle $\mathcal{P} = (P, Q_i, \theta_i, \phi_{ij})$ is called flat if the h_{ijk} in Lemma 2.2 are constant functions. Two flat twisted principal E -bundles are called flat equivalent if there exists an equivalence (ψ, ϕ_i) between them, such that the h_{ij} in Lemma 2.2 are constant functions.*

Remark 2.6 *From Brylinski's study [4] of the obstruction gerbe already mentioned we deduce at once that a flat twisted principal E -bundle is equivalent, in the sense of our Definition 2.1, to an ordinary principal E -bundle, but not necessarily equal to one.*

Remark 2.7 *If M is simply-connected, then any transitive Lie algebroid, with fibre $\mathcal{L}(E)$, can be integrated to a flat twisted principal E -bundle according to Mackenzie's results in[10] on the obstruction theory for integrating transitive Lie algebroids to Lie groupoids. ¹ His results*

¹I thank Mackenzie for making this remark after a talk I gave in Sheffield on twisted bundles.

also show that equivalent Lie algebroids yield flat equivalent flat twisted principal bundles, at least if the choice of central extension is the same (one can always mod out H and E by a discrete central subgroup, which makes no difference for the corresponding Lie algebras of course). There is a good notion of a connection in a transitive Lie algebroid [10] and it seems likely that such a connection can be integrated to a flat connection in the corresponding flat twisted principal bundle, as defined below. In that case the results in this paper would provide a notion of holonomy for connections in transitive Lie algebroids, even if they cannot be integrated to true principal bundles.

Next let us explain what a connection in a twisted principal E -bundle, $\mathcal{P} = (P, Q_i, \theta_i, \phi_{ij})$, is. Following Chatterjee's terminology for connections in gerbes [7], we distinguish between 0- and 1-connections.

Definition 2.8 *A 0-connection in \mathcal{P} consists of a G -connection, ω , in the principal G -bundle P and E -connections, η_i , in the local principal E -bundles Q_i , such that*

$$\theta_i^* p_i^*(\eta_i) = \omega_i = \omega|_{P_i} \tag{6}$$

holds, where $*$ denotes the push-forward for connections.

Two twisted principal E -bundles, \mathcal{P} and \mathcal{P}' , with 0-connections, (ω, η_i) and (ω', η'_i) respectively, are equivalent if there exists an equivalence

$$(\psi, \phi_i): \mathcal{P} \rightarrow \mathcal{P}'$$

such that

$$\psi^*(\omega) = \omega'. \tag{7}$$

Remark 2.9 *It might seem that too many 0-connections are equivalent according to the definition above, but that is because we have not yet defined 1-connections nor the equivalence between twisted principal bundles with both 0- and 1-connection.*

In the following lemma we derive two easy consequences of (6) and (7), the proof of which we omit. Note that the adjoint action of E

on $\mathcal{L}(H)$ is trivial and, therefore, for any $i \in \mathcal{I}$, the associated bundle $Q_i \times \mathcal{L}(H) / \sim$ is canonically isomorphic to the trivial bundle $U_i \times \mathcal{L}(H)$. Thus any form on Q_i , with values in the associated bundle above, that vanishes on vertical vectorfields, can be canonically identified with a form on U_i with values in $\mathcal{L}(H)$.

Lemma 2.10 *Equation (6) implies that there exists a 1-form on each U_{ij} with values in $\mathcal{L}(H)$, denoted A_{ij} , such that*

$$\eta_j - \phi_{ij}^*(\eta_i) \equiv A_{ij}$$

holds. Furthermore, we have $A_{ji} = -A_{ij}$ and

$$A_{ij} + A_{jk} + A_{ki} \equiv -h_{ijk}^{-1} dh_{ijk}$$

on U_{ijk} . Using the trivializations of Lemma 2.2 we get

$$A_j - e_{ij}^{-1} A_i e_{ij} - e_{ij}^{-1} de_{ij} \equiv A_{ij}$$

on U_{ij} .

Equation (7) implies that there exists a 1-form on each U_i with values in $\mathcal{L}(H)$, denoted B_i , such that

$$\eta_i - \phi_i^*(\eta_i) = B_i$$

holds. Furthermore, we have

$$A'_{ij} \equiv A_{ij} + B_j - B_i - h_{ij}^{-1} dh_{ij}$$

on U_{ij} . Using local trivializations we get

$$A'_i - e_i^{-1} A_i e_i - e_i^{-1} de_i = B_i.$$

Remark 2.11 *Given a G -connection in P , Brylinski [4] constructs a connective structure for the canonical gerbe mentioned in Remark 2.3. This consists of a local $\underline{A}_{U_i, \mathcal{L}(H)}^1$ -torsor on each U_i together with some data relating these different local torsors. A 0-connection in a twisted bundle is nothing but an object in each torsor. Brylinski shows that 0-connections always exist.*

Definition 2.12 Let \mathcal{P} be a twisted bundle with a 0-connection η_i . A 1-connection in (\mathcal{P}, η_i) consists of 2-forms F_i on U_i with values in $\mathcal{L}(H)$ satisfying

$$F_j - F_i \equiv dA_{ij}, \quad (8)$$

on U_{ij} .

A connection in \mathcal{P} consists of a 0-connection, (η_i) , and a 1-connection in (\mathcal{P}, η_i) .

Two twisted E -bundles, \mathcal{P} and \mathcal{P}' , with connections, (η_i, F_i) and (η'_i, F'_i) respectively, are equivalent if there exists an equivalence

$$(\psi, \phi_i): (\mathcal{P}, \eta_i) \rightarrow (\mathcal{P}', \eta'_i)$$

such that

$$F'_i = F_i + dB_i, \quad (9)$$

where B_i was defined in Lemma 2.10.

Remark 2.13 A 1-connection is what Brylinski [4] calls a curving and he shows that there always exists one for a given abelian gerbe with connective structure.

Definition 2.14 Let $(\mathcal{P}, \eta_i, F_i)$ be a twisted bundle with connection, then the global 3-form on M with values in $\mathcal{L}(H)$ defined by

$$G|_{U_i} = dF_i$$

is called the curvature. The connection is called flat if

$$G \equiv 0$$

on M .

The following theorem follows directly from Brylinski's [4] analogous result for abelian gerbes.

Lemma 2.15 A twisted bundle is flat if and only if it admits a flat connection.

3 Holonomy

Let \mathcal{G} be the categorical Lie group associated to a central extension of Lie groups (2). In this section I first show how the holonomy of a connection in a twisted principal E -bundle assembles nicely into a functor

$$\mathcal{H}: C_2^2(M) \rightarrow \mathcal{G},$$

and then I show that this functor contains all the information about the twisted bundle and its connection.

Throughout this section a principal bundle with connection is always considered in terms of local forms $\mathcal{P} = (e_{ij}, g_{ij}, h_{ijk}, A_i, D_i, A_{ij}, F_i)$, where (g_{ij}, D_i) defines a principal G -bundle with connection, (e_{ij}, A_i, F_i) define the local bundles Q_i and the 0- and 1-connection in \mathcal{P} and h_{ijk} and A_{ij} were defined in the Lemmas 2.2 and 2.10.

Given a smooth cylinder

$$c: [0, 1]^2 \rightarrow M,$$

such that $c(s, 0) = c(s, 1) = *$ for all $s \in [0, 1]$, choose an open covering of the image of c in M . Let $V_i = c^{-1}(U_i)$, where U_i is an open set in the covering of $c([0, 1]^2)$. Next choose a rectangular subdivision of $[0, 1]^2$ such that each little rectangle R_i is contained in at least one open set, which for convenience I take to be V_i . Denote the edge $R_i \cap R_j$ by E_{ij} , and the vertex $R_i \cap R_j \cap R_k \cap R_l$ by V_{ijkl} . Let $\epsilon(c) \in U(1)$ be the following complex number:

$$\begin{aligned} \epsilon(c) = & \prod_{\alpha} \exp \int_{R_{\alpha}} c^* F_{\alpha} \cdot \prod_{\alpha, \beta} \exp \int_{E_{\alpha\beta}} c^* A_{\alpha\beta} \\ & \times \prod_{\alpha, \beta, \gamma, \delta} h_{\alpha\beta\gamma}(c(V_{\alpha\beta\gamma\delta})) h_{\alpha\delta\gamma}(c(V_{\alpha\beta\gamma\delta}))^{-1}. \end{aligned} \quad (10)$$

The last two products are to be taken over the labels of contiguous faces in the rectangular subdivision only and in such a way that each face, edge and vertex appears only once. The convention for the order of the labels is indicated in Fig. 1, and the orientation of the surfaces in that picture is to be taken counterclockwise. If c is closed, then ϵ is exactly equal to the gerbe-holonomy as Picken and I showed in [9],

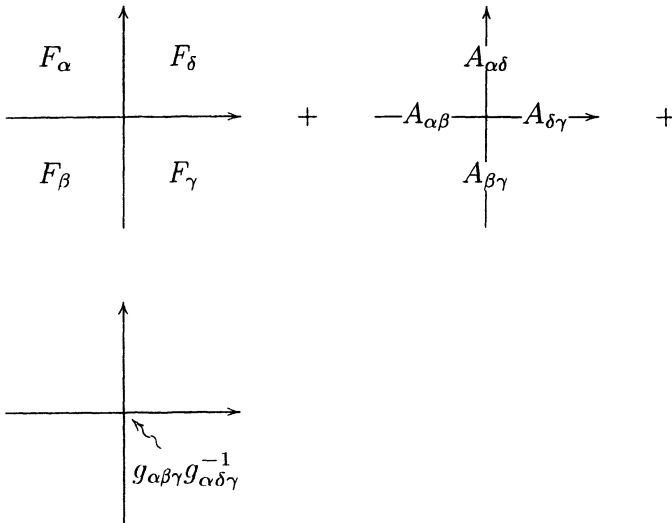


Figure 1: concrete formula for gerbe-holonomy

so, in that case, its value does not depend on any of the choices that were made for its definition. In general ϵ depends on the choices that we made in (10), of course. As a matter of fact, its value only depends on the choice of covering of the boundary of $c([0, 1]^2)$, because changes in the covering of the “middle” of $c([0, 1]^2)$ do not affect ϵ , which can be shown by repeated use of Stokes’ theorem. Let us give one more ill-defined definition. Let

$$\ell: [0, 1] \rightarrow M$$

be a loop, based at $*$, in M . Since g_{ij}, D_i is an honest principal G -bundle with connection, one can define in the usual way their holonomy along ℓ , denoted by $\mathcal{H}_0(\ell) \in H$. When one tries to do the same for c_{ij}, A_i , the usual formula for the holonomy is not well-defined. However, this should not stop us. Let the image of ℓ be covered by certain U_i again, and choose a subdivision of $[0, 1]$ such that each subinterval I_i is contained in the inverse image of at least one open set, taken to be V_i .

Let $V_{i,i+1}$ be the vertex $I_i \cap I_{i+1}$. Define $\mathcal{H}_1(\ell) \in E$ as

$$\mathcal{H}_1(\ell) = \prod_i \mathcal{P}\exp \int_{I_i} \ell^* A_i \cdot e_{i,i+1}(\ell(V_{ij})). \quad (11)$$

In (11) $\mathcal{P}\exp \int$ means the path-ordered integral, which one has to use because E is non-abelian in general. For the same reason the order in the product is important. We are now ready for the definition of the holonomy functor \mathcal{H} , which of course has to be independent of all choices.

Definition 3.1 *Let ℓ represent a class in $\pi_1^1(M)$ and define*

$$\mathcal{H}(\ell) = \mathcal{H}_0(\ell) \in H.$$

As remarked already this is well-defined.

Let $c: [0, 1]^2 \rightarrow M$ represent a class in $C_2^2(M)([\ell], [\ell'])$. Define

$$\mathcal{H}(s) = [\mathcal{H}_1(\ell), \epsilon(s)\mathcal{H}_1(\ell')] \in E \times E/H. \quad (12)$$

The next lemma shows that this is well-defined indeed.

Lemma 3.2 *The holonomy functor \mathcal{H} , as defined in Def. 3.1, is a well-defined functor between categorical Lie groups*

$$\mathcal{H}: C_2^2(M) \rightarrow \mathcal{G},$$

which is independent of all the choices that we made for its definition.

Proof: Showing that \mathcal{H} preserves the categorical Lie group structures is very easy, once it has been established that it is well-defined. Therefore I only show the latter. To prove well-definedness one has to show two things: that (12) does not depend on the choice of covering of c , and that (12) is equal for all representatives of the equivalence class of c . Let us first prove the first of these two statements. As far as $H_1(\ell)$ and $H_1(\ell')$ are concerned, it is clear that only the choice of covering of the boundary of c affects their value. We already argued that the same is true for the value of c , due to Stokes' theorem. It now suffices to see what happens when we introduce an extra vertical line in our rectangular subdivision

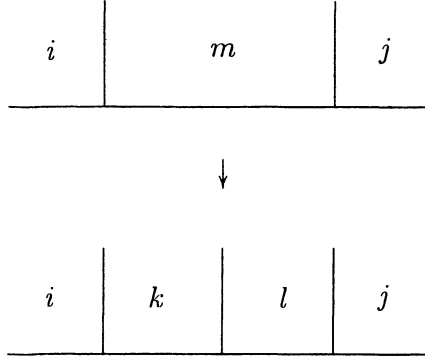


Figure 2: change in covering

and a new covering of the new (smaller) rectangles at the boundary. In Fig. 2 one can see such a change.

The notation is as indicated in those two pictures. Let us first compare the values of $H_1(\ell)$ in the two pictures. In the calculations below the pull-back ℓ^* has been suppressed to simplify the notation. In the first picture the part of $\mathcal{H}_1(\ell)$ that matters is equal to

$$e_{im}(V_{im}) \cdot \mathcal{P} \exp \int_{I_m} A_m \cdot e_{mj}(V_{mj}), \quad (13)$$

and in the second picture that part becomes

$$e_{ik}(V_{ik}) \cdot \mathcal{P} \exp \int_{I_k} A_k \cdot e_{kl}(V_{kl}) \cdot \mathcal{P} \exp \int_{I_l} A_l \cdot e_{lj}(V_{lj}). \quad (14)$$

Now, we can rewrite (14) to obtain (13) times an abelian factor. In order to do this we have to use the transformation rule for the path-ordered integral:

$$\mathcal{P} \exp \int_{[a,b]} (e^{-1} A e + e^{-1} d e) = e(a)^{-1} \cdot \mathcal{P} \exp \int_{[a,b]} A \cdot e(b). \quad (15)$$

Using the equations satisfied by the A_i as a connection in a twisted

principal E -bundle, we see that (14) can be rewritten as

$$\begin{aligned}
 & e_{ik}(V_{ik}) \cdot \mathcal{P} \exp \int_{I_k} (e_{mk}^{-1} A_m e_{mk} + e_{mk}^{-1} d e_{mk}) \cdot e_{kl}(V_{kl}) \\
 & \times \mathcal{P} \exp \int_{I_l} (e_{ml}^{-1} A_m e_{ml} + e_{ml}^{-1} d e_{ml}) \cdot e_{mj}(V_{mj}) \cdot \exp \int_{I_k} A_{mk} \cdot \exp \int_{I_l} A_{ml}.
 \end{aligned}$$

Using (15) we see that this is equal to

$$\begin{aligned}
 & e_{ik}(V_{ik}) e_{km}(V_{ik}) \cdot \mathcal{P} \exp \int_{I_k} A_m \cdot e_{mk}(V_{kl}) e_{kl}(V_{kl}) e_{lm}(V_{kl}) \\
 & \times \mathcal{P} \exp \int_{I_l} A_m \cdot e_{ml}(V_{mj}) e_{lj}(V_{mj}) \cdot \exp \int_{I_k} A_{mk} \cdot \exp \int_{I_l} A_{ml}.
 \end{aligned}$$

Finally, using that the coboundary of e is equal to h , we get

$$\begin{aligned}
 & e_{im}(V_{im}) \cdot \mathcal{P} \exp \int_{I_k \cup I_l = I_m} A_m \cdot e_{mj}(V_{mj}) \\
 & \times h_{ikm}(v_{im}) h_{mkl}(V_{kl}) h_{mlj}(V_{mj}) \cdot \exp \int_{I_k} A_{mk} \cdot \exp \int_{I_l} A_{ml} \quad (16) \\
 & = (8) \times \text{abelian factor.}
 \end{aligned}$$

(Of course a similar calculation can be made for $\mathcal{H}_1(\ell')$. A straightforward calculation using Stokes' theorem, which I omit, now shows that the inverse of the extra abelian factor in (16) times the extra abelian factor in $\mathcal{H}_1(\ell')$ cancel against the extra (abelian) factor in $\epsilon(s)$ which appears when it is computed for the same change in covering.

Next let us prove that (12) is constant on thin homotopy classes. Let

$$c_1, c_2: [0, 1]^2 \rightarrow M$$

be two cylinders which represent the same class in $C_2^2(M)([\ell], [\ell'])$. We have to show that

$$[\mathcal{H}_1(\ell_1), \epsilon(c_1)\mathcal{H}_1(\ell'_1)] = [\mathcal{H}_1(\ell_2), \epsilon(c_2)\mathcal{H}_1(\ell'_2)], \quad (17)$$

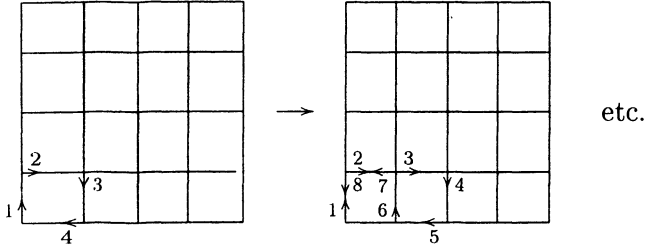


Figure 3: snake

where $\ell_i \overset{1}{\sim} \ell$ and $\ell'_i \overset{1}{\sim} \ell'$ for $i = 1, 2$. Without loss of generality we may assume that $\ell_2 = \ell$ and $\ell'_2 = \ell'$. Let A be a thin homotopy between ℓ_1 and ℓ and let B be a thin homotopy between ℓ'_1 and ℓ' , such that

$$Ac_2B^{-1} \overset{2}{\sim} c_1.$$

If the same covering of the boundaries is used, then $\epsilon(Ac_2B^{-1}) = \epsilon(c_1)$, which follows from the general theory of gerbe holonomy developed in [9]. Therefore we have

$$\begin{aligned} & [\mathcal{H}_1(\ell_1), \epsilon(c_1)\mathcal{H}_1(\ell'_1)] \\ &= [\mathcal{H}_1(\ell_1), \epsilon(Ac_2B^{-1})\mathcal{H}_1(\ell'_1)] \\ &= [\mathcal{H}_1(\ell_1)\epsilon(A)^{-1}, \epsilon(c_2)\mathcal{H}_1(\ell'_1)\epsilon(B)^{-1}] \end{aligned}$$

It only remains to prove that, for a fixed covering of the boundaries, we have

$$\mathcal{H}_1(\ell_1)\epsilon(A)^{-1} = \mathcal{H}_1(\ell) \quad \text{and} \quad \mathcal{H}_1(\ell'_1)\epsilon(B)^{-1} = \mathcal{H}_1(\ell').$$

Given a rectangular subdivision of $[0, 1]^2$ as above, one can write the loop around the boundary of $[0, 1]^2$ as the composite of loops which just go around the boundary of one little rectangle R_i at a time and are connected with the basepoint via a tail lying on some of the edges. See Fig. 3 for an example, which is like a snake (follow the numbers, such that each arrow is numbered on the left-hand side).

Note that the contribution of the 2-forms F_i for $\epsilon(A)$ and $\epsilon(B)$ is trivial, because both A and B are thin. Using the transformation rule for path-ordered integrals (15) in the same way as above it is not hard to see that $\mathcal{H}_1(\text{snake})$ equals

$$\mathcal{H}_1(\ell_1)\epsilon(A)^{-1}\mathcal{H}_1(\ell)^{-1} \tag{18}$$

in the first case and

$$\mathcal{H}_1(\ell'_1)\epsilon(B)^{-1}\mathcal{H}_1(\ell')^{-1} \tag{19}$$

in the second case. Now recall that on each open set U_i we have an honest principal E -bundle with an honest connection A_i , because only globally these data do not match up. Therefore in both cases the value of \mathcal{H}_1 around the boundary of each R_i equals 1, because A and B are thin. The conclusion is that the expressions in (18) and (19) are both equal to 1 as well. \square

Clearly the connection in \mathcal{P} is flat if and only if \mathcal{H} is constant on ordinary homotopy classes of cylinders, which happens if and only if \mathcal{H} defines a functor $\mathcal{H}: C_2(M) \rightarrow \mathcal{G}$.

Note that the lemma above implies that the element

$$\mathcal{H}_1(\ell)(\mathcal{H}_1(\ell')\epsilon(c))^{-1} \in E$$

is well-defined for any $c: [0, 1]^2 \rightarrow M$. Kapustin [8] studied the special case in which $E = \text{GL}(n, \mathbb{C})$ and $c(0, t) = *$ equals the trivial loop at the basepoint. His main mathematical result about the holonomy of connections in twisted vector bundles seems to be that $\text{tr}(\mathcal{H}_1(\ell')\epsilon(c))$ is a well-defined complex number in that particular case.

In order to understand the sequel, one should note that G acts by conjugation both on itself and on E .

Definition 3.3 *Given a holonomy functor, \mathcal{H} , one can define the conjugate holonomy functor $\mathcal{H}^g = g^{-1}\mathcal{H}g$, for any $g \in G$, by*

$$\mathcal{H}_0^g(\ell) = g^{-1}\mathcal{H}_0(\ell)g$$

and

$$\mathcal{H}^g(c) = [g^{-1}\mathcal{H}_1(\ell_1)g, \epsilon(c)g^{-1}\mathcal{H}_1(\ell_2)g].$$

As a matter of fact there is a natural isomorphism between \mathcal{H} and \mathcal{H}^g defined by

$$[\mathcal{H}_1(\ell), g^{-1}\mathcal{H}_1(\ell)g]: \mathcal{H}_0(\ell) \rightarrow \mathcal{H}_0^g(\ell),$$

for any loop ℓ . Note that this natural isomorphism is well-defined indeed.

Lemma 3.4 *Equivalent twisted principal E -bundles with connection give rise to conjugate holonomy functors.*

Proof: Recall that two twisted principal E -bundles with connections, denoted $g_{ij}, e_{ij}, h_{ijk}, A_i, D_i, A_{ij}, F_i$ and $g'_{ij}, e'_{ij}, h'_{ijk}, A'_i, D'_i, A'_{ij}, F'_i$ respectively, are equivalent if

$$\begin{aligned} g'_{ij} &= g_i^{-1}g_{ij}g_j \\ e'_{ij} &= e_i^{-1}e_{ij}e_jh_{ij} \\ D'_i &= g_i^{-1}D_i g_i + g_i^{-1}dg_i \\ A'_i &= e_i^{-1}A_i e_i + B_i + e_i^{-1}de_i \\ F'_i &= F_i + dB_i, \end{aligned}$$

where $p_i \circ e_i = g_i$. Thus we see that

$$\mathcal{H}'_0(\ell) = (\pi g_0(*))^{-1}\mathcal{H}_0(\ell)\pi g_0(*),$$

holds, for any loop ℓ , where by convention U_0 is the open set that covers the basepoint.

We now have to show that the identity

$$\mathcal{H}'(c) = g_0(*)^{-1}\mathcal{H}(c)g_0(*)$$

holds, for any cylinder $c: \ell_1 \rightarrow \ell_2$. Using the transformation rule (15) again we get

$$\begin{aligned} \mathcal{H}'_1(\ell_1) &= \prod_i \mathcal{P}\exp \int_{I_i} \ell_1^* A'_i \cdot e'_{i,i+1}(\ell_1(V_{ij})) \\ &= e_0(*)^{-1} \cdot \prod_i \mathcal{P}\exp \int_{I_i} \ell_1^* A_i \cdot e_{i,i+1}(\ell_1(V_{ij})) \cdot e_0(*) \\ &\quad \times \prod_i \exp \int_{I_i} \ell_1^* B_i \cdot h_{i,i+1}(\ell_1(V_{ij})) \\ &= e_0(*)^{-1}\mathcal{H}_1(\ell_1)e_0(*) \times \text{abelian factor.} \end{aligned}$$

Analogously we get

$$\mathcal{H}'_1(\ell_2) = e_0(*)^{-1}\mathcal{H}_1(\ell_2)e_0(*) \times \text{abelian factor.}$$

Using Stokes' theorem it is now easy to see that $\epsilon'(c)$ cancels these two abelian factors so that

$$\begin{aligned} \mathcal{H}'(c) &= [\mathcal{H}'_1(\ell_1), \epsilon'(c)\mathcal{H}'_1(\ell_2)] \\ &= [e_0(*)^{-1}\mathcal{H}_1(\ell_1)e_0(*), \epsilon(c)e_0(*)^{-1}\mathcal{H}_1(\ell_2)e_0(*)] \\ &= e_0(*)^{-1}\mathcal{H}(s)e_0(*). \end{aligned}$$

Finally, the result follows from the observation that conjugation by $g_0(*)$ is equal to conjugation by $e_0(*)$, because E is a *central* extension. \square

Putting together Barrett's [1] results about the reconstruction of bundles with connections from their holonomies and Picken and my [9] analogous results for gerbes, one now almost immediately gets the following lemma.

Lemma 3.5 *Given a smooth functor of categorical Lie groups*

$$\mathcal{H}: C_2^2(M) \rightarrow \mathcal{G},$$

there exists a twisted principal E -bundle with connection whose holonomy functor is equal to \mathcal{H} .

If \mathcal{H} is the holonomy functor of a given twisted principal E -bundle with connection, then the twisted principal E -bundle with connection constructed from \mathcal{H} is equivalent to the given one.

Proof: Barrett's results [1] allow us to construct g_{ij} and D_i for a given holonomy functor. The rest of the proof relies on the same techniques as employed in [9]. We only sketch the construction here. Let us see what e_{ij} and A_i are in terms of the holonomy functor \mathcal{H} . In [9] we chose a fixed point in each open set, called $x_i \in U_i$, and a fixed point in each double overlap, $x_{ij} \in U_{ij}$. We picked a path from the basepoint $* \in M$ to each x_i and showed how to fix paths in U_i from x_i to any other point in U_i and from x_{ij} to any other point in U_{ij} . We also showed how to

fix homotopies inside U_i between any two homotopic paths in U_i . In particular we got a fixed loop

$$* \rightarrow x_i \rightarrow y \rightarrow x_j \rightarrow *,$$

for any point $y \in U_{ij}$. Call this loop $\ell_{ij}(y)$. We also got a fixed homotopy, $c_{ij}(y)$, between $\ell_{ij}(x_{ij})$ and $\ell_{ij}(y)$. Now consider

$$\mathcal{H}(c_{ij}(y)) \in E \times E/H. \quad (20)$$

Take a representative of $\mathcal{H}(c_{ij}(x_{ij}))$ in $E \times E$, which of course is of the form (e, e) . For any $y \in U_{ij}$, take the unique representative of (20) of the form (e, e') and define

$$e_{ij}(y) = e' \in E.$$

The choice which this reconstruction of e_{ij} involves, corresponds to gauge fixing. By convention we fix the representative of $\mathcal{H}(c_{ji}(x_{ji}))$ to be (e^{-1}, e^{-1}) . Then the identity

$$e_{ji}(y) = e_{ij}(y)^{-1}$$

follows immediately. Because we have

$$\ell_{ij}(y) \star \ell_{jk}(y) \star \ell_{ki}(y) \stackrel{1}{\sim} c_{\star},$$

we can define the 2-cocycle

$$h_{ijk}(y) = e_{ij}(y)e_{jk}(y)e_{ki}(y) \in \ker \pi \cong H.$$

Note that this definition of the 2-cocycle is equal to the one given in [9]. Different choices of representatives of $\mathcal{H}(s_{ij}(x_{ij}))$ in $E \times E$ yield an equivalent twisted principal E -bundle.

Analogously we can reconstruct the A_i . Given a vector $v \in T_y(U_i)$, we can represent it by a small path $q(t)$ in U_i , whose derivative at $t = 0$ is equal to v . Then there is a loop

$$* \rightarrow x_i \rightarrow y \xrightarrow{q} q(t) \rightarrow x_i \rightarrow *. \quad (21)$$

Call it $\ell_i(q(t))$. For each value of t , we can use the fixed homotopy in U_i to get a homotopy, $c_i(q(t))$, from the trivial loop at $*$ to $\ell_i(q(t))$. Consider

$$\mathcal{H}(c_i(q(t))) = [e(t), e'(t)] \in E \times E/H. \quad (22)$$

Define

$$A_i(v) = \frac{d}{dt} e(t)^{-1} e'(t)|_{t=0}.$$

It is not hard to see that

$$A_j(v) - e_{ij}(y)^{-1} A_i(v) e_{ij}(y) - e_{ij}^{-1} d e_{ij}(v) = A_{ij}(v)$$

is exactly the abelian 1-form that we reconstructed in [9]. The fact that the definition of $A_i(v)$ does not depend on the particular choice of $q(t)$ follows precisely from the same arguments that we used in that paper.

The reconstruction of the 1-connection F_i is exactly the same as in [9], because it only depends on the value of the holonomy functor around closed cylinders.

The rest of the proof is similar to the proofs of the analogous results for bundles and gerbes in [1] and [9] and we omit the details. \square

Choosing a different basepoint in M and a path from that basepoint to $*$ yields an equivalence between the two respective thin fundamental categorical groups. This equivalence induces an equivalence relation on those holonomy functors \mathcal{H} which correspond to the same equivalence class of twisted principal E -bundle with connection. Just as for ordinary connections, two holonomy functors are equivalent if and only if they are conjugate by an element in G . Together with Lem. 3.2, Lem. 3.4 and Lem. 3.5 these remarks prove the following theorem:

Theorem 3.6 *There is a bijective correspondence:*

$$\{\text{twisted principal } E\text{-bundles on } M \text{ with connection}\} / \sim$$

$$\longleftrightarrow$$

$$\text{Hom}(C_2^2(M), \mathcal{G}) / \mathcal{G}_0.$$

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