

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

A. BARKHUDARYAN

R. EL BASHIR

V. TRNKOVÁ

Endofunctors of set and cardinalities

Cahiers de topologie et géométrie différentielle catégoriques, tome 44, n° 3 (2003), p. 217-239

http://www.numdam.org/item?id=CTGDC_2003__44_3_217_0

© Andrée C. Ehresmann et les auteurs, 2003, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ENDOFUNCTORS OF Set AND CARDINALITIES

by A. BARKHUDARYAN, R. EL BASHIR and V. TRNKOVÁ

RESUME. Les foncteurs $F: \mathbf{K} \rightarrow \mathbf{H}$ qui sont naturellement équivalents à tout foncteur $G: \mathbf{K} \rightarrow \mathbf{H}$ tel que FX soit isomorphe à GX pour tout X sont appelés foncteurs DVO. Les auteurs étudient les foncteurs DVO dans la catégorie Set des ensembles et applications. Ils introduisent une hypothèse (EUCE) sur la théorie des ensembles (relativement consistante avec (ZFC+GCH)) et, sous l'hypothèse (GCH+EUCE), caractérisent les classes W de cardinaux de la forme

$$W = \{ |X|; |FX| = |X| \} \text{ pour un } F: \text{Set} \rightarrow \text{Set}.$$

Les résultats obtenus résolvent divers problèmes posés par Rhi-neghost [15] et Zmrzlina [19].

1. Introduction

In late sixties and early seventies, the categories of bialgebras $A(F, G)$ were one of the main topics of the Prague "Seminar on general mathematical structures". Let us recall that, if $F, G: \mathfrak{K} \rightarrow \mathfrak{H}$ are functors, the objects of the category $A(F, G)$ are the pairs (X, o) , where X is an object of \mathfrak{K} and o is an \mathfrak{H} -morphism $FX \rightarrow GX$; the \mathfrak{K} -morphisms $f: X \rightarrow X'$ with $G(f) \circ o = o' \circ F(f)$ are precisely the morphisms $(X, o) \rightarrow (X', o')$ in $A(F, G)$. At the Prague seminar, only the case $F, G: \text{Set} \rightarrow \text{Set}$ (where Set denotes the category of all sets and mappings) was studied in that period and the categories $A(F, G)$ were called *generalized algebraic categories*. The obtained results about them were published in [1], [2], [3], [10], [11], [18].

Financial support of the Grant agency of the Czech Republic under the grant No 201/99/0310 and of the Grant Agency of the Charles University under the grant No 5/1998/B MAT/MFF is gratefully acknowledged by the first and the third authors. The second author acknowledges the financial support of the Grant Agency of the Charles University under the grant No 169/1999/B MAT/MFF. Also supported by MSM 113200007.

It occurred (as it could be expected) that the properties of the category $A(F, G)$ heavily depend on the behavior of the functors F and G . This attracted the attention of the participants of the seminar to examining the properties of the endofunctors of the category Set themselves. The results are contained in the papers [7], [8], [9], [13], [16], [17].

After thirty years, Y. T. Rhineghost and Alois Zmrzlina refreshed this field of problems and gave several results and interesting questions about functors $\text{Set} \rightarrow \text{Set}$, see [14], [15], [19]. The present paper was inspired by these papers. We solve, *inter alia*, several problems of these papers, some of them only under a set-theoretical hypothesis. The absolute results (in the sense they are proved in the Zermelo-Fraenkel set theory with the axiom of choice (ZFC)) in this paper concern the functors determined uniquely, up to natural equivalence, by their values on objects; see Definition 3.1, where such functors are called DVO functors. We will give here some examples of DVO functors and, in particular, solve the Problem 16 of [19](see Proposition 3.2 and the remarks following it). On the other hand, we are able to prove the claim that

every DVO functor is finitary

only under one of the set-theoretical assumptions ($GCH+EUCE$) and ($EUUR$). The hypothesis ($EUCE$) (= Every infinite Unattainable Cardinal is an Eliminator) is introduced in the present paper, see Section 4. It “sits” between the following set-theoretical statements:

$(EUUR) \Rightarrow (EUCE) \Rightarrow$ no measurable cardinals exist.

$EUUR$ is the abbreviation for

Every Uniform Ultrafilter is Regular.

(Let us recall that a filter \mathfrak{F} on the cardinal κ is called *uniform*, if $|A| = \kappa$ for every $A \in \mathfrak{F}$. An ultrafilter \mathcal{U} on κ is called *regular*, if there exists a subset $Z \subseteq \mathcal{U}$ of cardinality κ such that $\bigcap Z' = \emptyset$ whenever $Z' \subseteq Z$ is infinite.)

The relative consistency of ($EUUR$) with ($ZFC+GCH$) was proved by H.-D. Donder in [5]. The implication $(EUUR) \Rightarrow (EUCE)$ is precisely Proposition 4.3 of the present paper, and the second implication is Proposition 4.4.

In Theorem 5.7 we characterize under ($GCH+EUCE$) the classes W of cardinal numbers for which there exists a functor $F : \text{Set} \rightarrow \text{Set}$ such

that

$$W = \{|X|; |FX| = |X|\}.$$

The Theorem 5.8 states that if we require the uniqueness (up to natural equivalence) of F in the above formula, then W must be the class of all cardinals and F is the identity functor. This is again true under the hypotheses ($GCH+EUCE$) and not known generally. The above mentioned theorems solve the Problems 8 and 9 of [15].

The present paper is a full and extended version of the preliminary paper [4], where some of the results of the present paper were announced (mostly without any proof).

2. Preliminaries

Let us start with notation, definitions and auxiliary statements.

Notation. For a cardinal number α , denote by $\text{Set}_{<\alpha}$ the category of sets of cardinality less than α and all their mappings. Similarly, $\text{Set}_{\leq\alpha} = \text{Set}_{<\alpha+}$, $\text{Set}_{>\alpha} = \text{Set} \setminus \text{Set}_{\leq\alpha}$, $\text{Set}_{\geq\alpha} = \text{Set} \setminus \text{Set}_{<\alpha}$.

By C_M we denote the constant functor with $F(X) = M$ for any set X and $F(f) = \text{id}_M$ for any mapping f . The functor $C_{0,M}$ ($C_{N,1}$) only differs from C_M (C_1) in that $C_{0,M}(0) = 0$ ($C_{N,1}(0) = N$). All these functors are called constant functors.

The symbol Id will, as usual, denote the identical functor.

Suppose the functors $F, G : \text{Set} \rightarrow \text{Set}$ have a common subfunctor H . By $(F + G)/H$ we denote the result of "gluing" F and G along H , i. e. the quotient of the disjoint union $F + G$ that unifies H in F and G . We will often use this in cases when H is not a subfunctor but only has an isomorphic copy in F and G . In such cases $(F + G)/H$ denotes any of the functors obtained from the disjoint union $F + G$ by gluing two copies of H in F and G (i. e. any push-out of F, G with monotransformations $\mu_F : H \rightarrow F, \mu_G : H \rightarrow G$). Analogously we use $(\sum_{a \in A} F_a)/H$.

The following *fundamental* lemma was used implicitly quite often in the older papers and is explicitly formulated in [19]:

Lemma 2.1. *Suppose $F : \text{Set} \rightarrow \text{Set}$ is a functor different from C_0 . Then F contains either an isomorphic copy of Id or an isomorphic copy of $C_{0,1}$.*

A functor $F : \text{Set} \rightarrow \text{Set}$ is said to be *connected*, if $F(1)$ is a one-point set. Each functor F different from C_0 is the coproduct of its *connected components*:

$$F = \sum_{a \in F(1)} F_a,$$

where F_a for $a \in F(1)$ is the subfunctor of F defined by $F_a(X) = [F(f_X)]^{-1}(a)$, f_X being the unique map $f_X : X \rightarrow 1$. Due to Lemma 2.1, each of these components contains either an isomorphic copy of $C_{0,1}$ or an isomorphic copy of Id . It is easily seen that each component can contain only *one* copy of only *one* of these functors.

Let $F : \text{Set} \rightarrow \text{Set}$ be a functor and $\alpha \geq 2$ a cardinal number. For any set X put

$$F^\alpha(X) = \bigcup \{[Ff](Y); |Y| < \alpha, f : Y \rightarrow X\}.$$

Obviously F^α is a subfunctor of F .

The following definition can be found in [7].

Definition 2.2. *We say that α is an unattainable cardinal for the functor F if*

$$F^\alpha(\alpha) \neq F(\alpha).$$

The following proposition is proved in [7]:

Theorem 2.3. *If $|X|$ is an infinite unattainable cardinal for F , then $|F(X)| > |X|$.*

For $F : \text{Set} \rightarrow \text{Set}$ let $f : \text{Cn} \setminus 2 \rightarrow \text{Cn}$ be its *increase function*, i. e.

$$f(\alpha) = |F(\alpha) \setminus F^\alpha(\alpha)|$$

for $\alpha \geq 2$. We then define $f(0)$ to be the number of isomorphic copies of $C_{0,1}$ that are contained in F ; similarly, $f(1)$ is the number of isomorphic copies of Id contained in F . This function is sufficient for determining the cardinality of $F(X)$ for finite sets X . Actually, Koubek proved that

$$(2.1) \quad |F(\kappa)| = \sum_{\lambda \leq \kappa} \binom{\kappa}{\lambda} f(\lambda)$$

for $0 < \kappa < \omega$ (see [7]). Note that once F contains an isomorphic copy of $C_{0,1}$, $F(0)$ can be of any size; hence we restrict to $0 < \kappa$.

Under suitable set-theoretical hypotheses (namely $(EUUR)$ and/or $(EUCE+GCH)$), the formula (2.1) holds also for every infinite κ (Theorem 4.1). The meaning of $\binom{\kappa}{\lambda}$ in the infinite case is the same as in the finite case: it is the cardinality of the set of λ -point subsets of κ , i. e. $\binom{\kappa}{\lambda} = \kappa^\lambda$.

We will also need another proposition of Koubek (see [7]):

Proposition 2.4. *Let $F : \text{Set} \rightarrow \text{Set}$ be a functor and assume that $|F(X)| < |X|$. If X is finite, then F is constant on $\text{Set}_{<|X|}$. If X is infinite, then F is constant on $\text{Set}_{\leq|X|}$.*

Suppose all the unattainable cardinals of $F : \text{Set} \rightarrow \text{Set}$ are less than $\kappa \in \text{Cn}$. The construction of the left Kan extension as a point-wise colimit (see e. g. [12]) implies that F is the left Kan extension of its restriction $F|_{\text{Set}_{<\kappa}}$. This implies the following

Proposition 2.5. *If all the unattainable cardinals of the functors $F, G : \text{Set} \rightarrow \text{Set}$ are less than $\kappa \in \text{Cn}$, then F and G are naturally equivalent iff their restrictions $F|_{\text{Set}_{<\kappa}}$ and $G|_{\text{Set}_{<\kappa}}$ are naturally equivalent.*

Take any sets X and Y , and take $a \in F(X) \setminus F^{|X|}(X)$, $b \in F(Y) \setminus F^{|Y|}(Y)$. We say that a dominates b (symbolically, $a \rightarrow b$), if there exists a mapping $f : X \rightarrow Y$ such that $[Ff](a) = b$. The relation \rightarrow is clearly transitive. On the set $F(X) \setminus F^{|X|}(X)$ it gives rise to two equivalence relations: the one generated by it, to be denoted by \sim_X , and its symmetric part, i. e. $\approx_X = \rightarrow \cap \rightarrow^{-1}$. These relations will be useful when showing that two functors are not naturally equivalent.

3. Finitary DVO functors

In [15], Y. T. Rhineghost questioned whether the equations

$$(3.1) \quad |F(X)| = \begin{cases} 1 + \binom{|X|}{2}, & \text{if } |X| < \omega, \\ |X|, & \text{if } |X| \geq \omega \end{cases}$$

define a unique functor (up to natural equivalence, of course) or not. This was answered by A. Zmrzlina who asked more general questions. Before we formulate and answer the questions of A. Zmrzlina, let us give a general definition.

Definition 3.1. A functor $F : \mathfrak{K} \rightarrow \mathfrak{H}$ is called a DVO functor (Defined by Values on Objects), if F is naturally equivalent to any functor $G : \mathfrak{K} \rightarrow \mathfrak{H}$ for which $G(x)$ is isomorphic to $F(x)$ for every $x \in \text{obj } \mathfrak{K}$.

This definition was inspired by [19], where it is shown that many “small” functors are DVO functors.

Here we will bring even more examples of DVO functors.

Let \mathfrak{P} be the (covariant) power-set functor. Recall that for any cardinal κ , \mathfrak{P}^κ denotes the subfunctor of \mathfrak{P} defined by

$$\mathfrak{P}^\kappa(X) = \{Y \subseteq X; |Y| < \kappa\}.$$

The functor B_κ is then obtained from \mathfrak{P}^{κ^+} by collapsing its subfunctor \mathfrak{P}^κ onto C_1 . Thus,

$$B_\kappa(X) = \{Y \subseteq X; |Y| = \kappa\} \cup \{0\}$$

for any set X , and for $f : X_1 \rightarrow X_2$ we have

$$[B_\kappa(f)](Y) = \begin{cases} f(Y), & \text{if } |f(Y)| = \kappa, \\ 0, & \text{if } |f(Y)| < \kappa; \end{cases}$$

$$[B_\kappa(f)](0) = 0.$$

Note that B_2 satisfies the equation (3.1). Thus Problem 6 in [15] asks whether B_2 is a DVO functor. A. Zmrzlina answered the question positively (see [19]) and asked if the functors B_κ for $\kappa > 2$ are DVO or not.

Proposition 3.2. The functor B_κ with $\kappa > 2$ is DVO if and only if $\kappa < \omega$.

Proof. Let $\kappa > \omega$. Define $G = (B_\kappa + B_\kappa)/C_{0,1}$. Clearly $|G(X)| = |B_\kappa(X)|$ for any set X . The functors B_κ and G are not naturally equivalent for the following reason: all the elements of $B_\kappa(\kappa) \setminus B_\kappa^\kappa(\kappa)$ are \sim -equivalent, whereas in $G(\kappa) \setminus G^\kappa(\kappa)$ we have two equivalence classes of \sim .

Now suppose $\kappa = n < \omega$. The main ideas of the proof that B_n is DVO are borrowed from A. Zmrzlina’s paper and are also used with some modifications for the proof of some other functors being DVO.

Suppose we have a functor $G : \text{Set} \rightarrow \text{Set}$ with $|G(X)| = |B_n(X)|$ for all X . Let f be the increase function of B_n ; then $f(0) = f(n) = 1$, $f(k) = 0$ otherwise. Let g be the increase function of G .

1. According to the formula (2.1) for finite κ , we have

$$\sum_{k \in n} \binom{n}{k} f(k) = \sum_{k \in n} \binom{n}{k} g(k)$$

for each $n \in \omega$. On the other hand, according to Theorem 2.3, $g(k) = f(k) = 0$ for $k \geq \omega$. Now, one sees that actually $f = g$.

2. As $g(1) = 0$, G does not contain a copy of Id and thus it should contain a copy of $C_{0,1}$. Moreover, n is the only unattainable cardinal of B_n , $G(1) = G(2) = \dots = G(n-1) = \{0\}$ and $G(n) = \{0, a\}$ for some $a \neq 0$ (we will identify the copy of $C_{0,1}$ in G with $C_{0,1}$). Now, let $h : n \rightarrow n$ be a function. If $|\text{Im } h| = n$, then h is a bijection and hence so is $G(h)$. Since $[G(h)](0) = 0$, we have $[G(h)](a) = a$. If $|\text{Im } h| < n$ then h factors through $n-1$ and thus $[G(h)](a) = 0$. So, if we identify a with n , the restriction of G to the category of sets of cardinality at most n is identical to that of the functor B_n . According to Proposition 2.5, B_n and G are naturally equivalent. □

Using the above pattern, one can show that the functors $n \times \text{Id}$ are DVO for $n \in \omega$. The cases $n = 1$ and $n = 2$ were proved in [19].

In [19], A. Zmrzlina asked if the covariant hom-functors

$$Q_M = \text{Set}(M, -)$$

are DVO. The answer is:

NO for $|M| > 1$.

For M infinite, $(Q_M + Q_M)/Q_M^{|M|}$ is a functor that has isomorphic values on objects, but is not naturally equivalent to Q_M . For M finite we introduce another class of functors, first.

For any cardinal n one can define the functor R_n as a subfunctor of the power-set functor \mathfrak{P} with

$$R_n(X) = \{Y \subseteq X; 0 < |X| \leq n\}.$$

It is easy to see that $|Q_2(X)| = |(R_2 + R_2)/\text{Id}(X)|$ for every X , yet these functors are not naturally equivalent. Interestingly enough, Q_2 and $(R_2 + R_2)/\text{Id}$ are, up to a natural equivalence, the only functors $G : \text{Set} \rightarrow \text{Set}$ such that $|G(X)| = |Q_2(X)|$ for every X (i. e. Q_2 is a 2-DVO functor). Now, for $n > 2$, the functor $Q_{n-2} \times (R_2 + R_2)/\text{Id}$ shows that Q_n is not DVO. □

As to the functors R_n , they are DVO in most cases:

Proposition 3.3. *The functor $R_n : \text{Set} \rightarrow \text{Set}$ is DVO if and only if $n < \omega$ and $n \neq 3$.*

Proof. The proof of the fact that R_n is DVO for $n \neq 3$ follows the same pattern as those of Proposition 3.2 and Theorem 3.4 and is hence omitted. We only show that for $n = 3$ there are three distinct functors (in the sense that they are not naturally equivalent) which have values on objects isomorphic to those of R_3 .

For $i = 1, 2$ take

$$F_i(X) = X + \{Y \subseteq X; |Y| = 3\}.$$

For $f : X \rightarrow X'$ define $[F_i f](x) = f(x)$ for $x \in X$, $[F_i f](\{a, b, c\}) = f(\{a, b, c\})$ for $\{a, b, c\} \subseteq X$ if $|f(\{a, b, c\})| = 3$, otherwise if $f(a) = f(b)$, then $F_1(\{a, b, c\}) = f(a)$ and $F_2(\{a, b, c\}) = f(c)$. Thus we get functors F_1 and F_2 . Both are connected and contain an isomorphic copy of Id. It is clear that $|(R_2 + F_i)/\text{Id}(X)| = |R_3(X)|$ for any set X . Moreover, R_3 , $(R_2 + F_1)/\text{Id}$ and $(R_2 + F_2)/\text{Id}$ are up to a natural equivalence the only functors that obtain the same cardinalities as R_3 . In other words, R_3 is 3-DVO. \square

Let $B_{0,n}$ denote the functor B_n redefined at 0 by $B_{0,n}(0) = 0$.

Theorem 3.4. *Suppose $F : \text{Set} \rightarrow \text{Set}$ is a non-constant finitary functor which does not contain a subfunctor naturally equivalent to Id. Then F is DVO if and only if either*

- a) $F = C_{0,N} + B_{0,n}$ for some $2 \leq n < \omega$ and some finite set N ; or
- b) $F = (\sum_{n \in M} B_n)/C_1$, possibly redefined at 0, where $M \subseteq \omega \setminus 2$ is such that $n \in M$ implies $n + 1 \notin M$ and $n + 2 \notin M$.

Remark. *As already noted, it can be shown that, under the set-theoretical hypotheses (EUUR) or (GCH+EUCE), every DVO functor is finitary. Thus, under these assumptions, Theorem 3.4 characterizes all the DVO functors that do not contain an isomorphic copy of the identity functor. The case of functors containing Id is not known to the authors.*

Proof of Theorem 3.4. Suppose $F : \text{Set} \rightarrow \text{Set}$ is a non-constant finitary DVO functor, which does not contain an isomorphic copy of

Id. Let f be the increase function of F ; then $f(1) = 0$ and $f(0) \neq 0$. Consider the functor

$$G = (C_{F(0),1} + \sum_{n \in \omega \setminus 2} f(n) \times B_{0,n})/C_{0,1} + C_{0,|F(1)|-1}$$

(where $\kappa - 1$ is κ for $\omega \leq \kappa \in \text{Cn}$, $\kappa - 1$ for $0 < \kappa \in \omega$ and 0 for $\kappa = 0$). Let g be the increase function of G ; it is easily seen that $f = g$ and thus, according to the formula (2.1) for finite κ , $|F(X)| = |G(X)|$ for any set X . Thus, F and G are naturally equivalent and we may assume $F = G$.

Suppose $F(1)$ contains more than one point. Choose a $j \in \omega \setminus 2$ with $f(j) \neq 0$ and consider

$$G' = (C_{F(0),1} + \sum_{n \in \omega \setminus \{0,1,j\}} f(n) \times B_{0,n} + (f(j) - 1) \times B_{0,j})/C_{0,1} + B_{0,j} + C_{0,|F(1)|-2}.$$

Clearly $|F(X)| = |G'(X)|$ for every X and thus F and G' should be naturally equivalent; this is possible only if j is the only finite non-zero ordinal with $f(j) \neq 0$, $f(j) = 1$, and $F(0) = 0$. Thus $F = B_{0,j} + C_{0,|F(1)|-1}$. It remains to notice that $F(1)$ is finite: otherwise the functor $\text{Id} + C_{0,|F(1)|-1}$ would show that F is not DVO.

Now suppose $|F(1)| = 1$. Then, possibly after redefining at 0,

$$F = (\sum_{n \in \omega \setminus 2} f(n) \times B_{0,n})/C_{0,1}.$$

Suppose $f(n) > 1$ for some $n \in \omega \setminus 2$.

Recall that $Q_n = \text{Set}(n, -)$. Q_n has a subfunctor R defined by $R(X) = \{g : n \rightarrow X; |\text{Im } g| < n\}$. Collapsing this subfunctor to $C_{0,1}$ we obtain a functor, say Q . For two functions $g, h : n \rightarrow X$ put $g \equiv h$ if $g = h \circ \pi$ for some even permutation $\pi \in A_n$. This relation is a congruence for the functor Q and we may take the factorfunctor $S = Q/\equiv$. Note that S has only one component, contains $C_{0,1}$ as a subfunctor and has exactly one unattainable cardinal n with increase 2. Also note that the relation \rightarrow (as defined at the end of Section 2) is not identical on $S(n) \setminus S^n(n)$ in contrast with $(B_{0,n} + B_{0,n})/C_{0,1}$, hence these functors are not naturally equivalent.

Now, replace $(B_{0,n} + B_{0,n})/C_{0,1}$ in F by the functor S . We will get a functor G with $|G(X)| = |F(X)|$ for all X , yet G is not naturally equivalent to F .

This proves that $f(n) \leq 1$ for all n . Thus, there exists a set $M \subseteq \omega \setminus 2$ such that

$$F = \left(\sum_{n \in M} B_n \right) / C_1.$$

Let $n \in M$. We need to prove that then $n + 1 \notin M$ and $n + 2 \notin M$. This will be done by constructing two functors which can replace $(B_{0,n} + B_{0,n+1})/C_{0,1}$ and $(B_{0,n} + B_{0,n+2})/C_{0,1}$, respectively. The first one is obtained by collapsing in the functor R_{n+1} the subfunctor R_{n-1} to $C_{0,1}$. The resulting functor T then has one component, contains $C_{0,1}$ and the only point in $T(n+1) \setminus T^{n+1}(n+1)$ dominates the only point in $T(n) \setminus T^n(n)$, which is not the case for $(B_{0,n} + B_{0,n+1})/C_{0,1}$.

The construction of the second functor is a bit more complicated. For $g, h : n + 2 \rightarrow X$ put $g \equiv h$ if $|g^{-1}(x)| - |h^{-1}(x)|$ is even for any $x \in X$. The relation \equiv is a congruence of the functor Q_{n+2} and we can take $U = Q_{n+2} / \equiv$. Put $V(X) = \{r \in U(X); (\exists g \in r) | \text{Im } g| < n\}$; V is a subfunctor of U . By collapsing V to $C_{0,1}$, we obtain from U the desired functor W . It is connected, contains $C_{0,1}$ as its subfunctor, has exactly two unattainable cardinals — n and $n + 2$, and increase 1 at each of these cardinals. Again, the only point in $W(n+2) \setminus W^{n+2}(n+2)$ dominates the only point in $W(n) \setminus W^n(n)$, which is not the case for $(B_{0,n} + B_{0,n+2})/C_{0,1}$.

This completes the “only if” part of the proof.

Now we prove that the functors described in a) and b) are DVO. Let $F = C_{0,N} + B_{0,n}$. Suppose $G : \text{Set} \rightarrow \text{Set}$ is such that $|F(X)| = |G(X)|$ for all X . Then G has $|N| + 1$ components and one unattainable cardinal n with increase 1. Thus, $|N|$ of the components are constants, i. e. naturally equivalent to $C_{0,1}$, and one of them contains $C_{0,1}$, has exactly one unattainable cardinal n , and increase of 1 at this cardinal. As in the proof of Proposition 3.2, we conclude from Proposition 2.5 that this last component is naturally equivalent to $B_{0,n}$.

Let $F = \left(\sum_{n \in M} B_n \right) / C_1$, where M is as in b). Again, suppose $G : \text{Set} \rightarrow \text{Set}$ is such that $|F(X)| = |G(X)|$ for all X . As $|F(X)| = |X|$ for X infinite, G can have no infinite unattainable cardinals, due to Theorem

2.3. Let f and g be the increase functions of F and G , respectively. According to the formula (2.1) for finite κ we deduce $f = g$. We need to prove that F and G are naturally equivalent; according to Proposition 2.5 it suffices to show that their restrictions $F|\omega$ and $G|\omega$ to the full subcategory with finite ordinals as objects are naturally equivalent.

For every $n \in \omega$, let us denote by \bar{n} the full subcategory of Set with $n + 1 = \{0, 1, \dots, n\}$ as objects. Let $M = \{m_1, m_2, \dots, m_n, \dots\}$ be ordered increasingly. Suppose we have a natural equivalence

$$\tau^k : F|\bar{m}_k \rightarrow G|\bar{m}_k$$

(we surely have such an equivalence for $k = 1$, by Proposition 3.2). We will show that τ^k can be extended to a natural equivalence $\tau^{k+1} : F|\bar{m}_{k+1} \rightarrow G|\bar{m}_{k+1}$. Since both F and G have no unattainable cardinal between m_k and m_{k+1} , we can extend τ^k to a natural equivalence $\phi : F|\overline{(m_{k+1} - 1)} \rightarrow G|\overline{(m_{k+1} - 1)}$, by Proposition 2.5. Without any loss of generality, we may assume that $\phi = \text{id}$ and $F|\overline{(m_{k+1} - 1)} = G|\overline{(m_{k+1} - 1)}$, i. e. G restricted to $\overline{(m_{k+1} - 1)}$ is given by the above formula for F . Hence any $x \in G(s)$ with $s < m_{k+1}$ is either 0 (which means that x is in the copy of C_1 in G) or it is a subset of s inherited from some B_z with $z \leq s$ and $z \in M$, that is $|x| \in M$. Since the increase function g of G is equal to the increase function of F , we have $g(m_{k+1}) = 1$, i. e.

$$G(m_{k+1}) \setminus G^{m_{k+1}}(m_{k+1}) = \{a\}$$

for some a . We show that $G|\bar{m}_{k+1}$ behaves with respect to a precisely as $F|\bar{m}_{k+1}$ behaves with respect to $m_{k+1} \in F(m_{k+1})$.

Let $h : m_{k+1} \rightarrow m_{k+1} = \{0, \dots, m_{k+1} - 1\}$ be a bijection; hence $[F(h)](m_{k+1}) = m_{k+1}$. Clearly, $G(h)$ is also a bijection and

$$[G(h)](G^{m_{k+1}}(m_{k+1})) = G^{m_{k+1}}(m_{k+1}),$$

hence necessarily $[G(h)](a) = a$. Thus a is a fix-point of $G(h)$ and m_{k+1} is a fix-point of $F(h)$, for any bijection h .

Now, let l be a map $l : m_{k+1} \rightarrow m_{k+1} - 1 = \{0, \dots, m_{k+1} - 2\}$ identical on all $n \in (m_{k+1} - 1)$ and $l(m_{k+1} - 1) = m_{k+1} - 2$. Then $F(l)$ sends $m_{k+1} \in F(m_{k+1})$ into the copy of C_1 in F , i. e. $[F(l)](m_{k+1}) = 0$. We show that $[G(l)](a) = 0$ as well. Let us suppose the contrary. Since we suppose $F|\overline{(m_{k+1} - 1)} = G|\overline{(m_{k+1} - 1)}$, $[G(l)](a)$ is an element of the value of $(\sum_{m \in M} B_m) / C_1$ at $(m_{k+1} - 1)$. Since $[G(l)](a)$ is not 0, it is equal to some $A \subseteq (m_{k+1} - 1)$ with $|A| \in M$. However, $m_{k+1} - 1$ and $m_{k+1} - 2$

are not in M , by the condition b) in the Theorem, so that necessarily $|A| < m_{k+1} - 2$. Consequently there exist $i, j \in (m_{k+1} - 1)$ such that $i \in A$ and $j \notin A$. Denote by α (or by β) the transposition (i, j) in m_{k+1} (on in $(m_{k+1} - 1)$). Then we have $l = \beta l \alpha$ but

$$A = [G(l)](a) = [G(\beta)](G(l)(G(\alpha)(a))) = [G(\beta)](A) \neq A,$$

which is a contradiction. Thus $[G(l)](a) = 0$.

As any function $m_{k+1} \rightarrow n$ for $n \leq m_{k+1}$ is either a bijection or factors through a bijection followed by the above function l , after identifying $a \in G(m_{k+1})$ with $m_{k+1} \in F(m_{k+1})$, we get $F|\overline{m_{k+1}} = G|\overline{m_{k+1}}$. In other words, we are able to extend the natural equivalence ϕ to a natural equivalence $\tau^{k+1} : F|\overline{m_{k+1}} \rightarrow G|\overline{m_{k+1}}$.

After countably many steps we thus get a natural equivalence $\tau = \bigcup_{k \in \mathbb{N}} \tau^k : F|\omega \rightarrow G|\omega$. □

4. Infinitary functors and relevant set-theoretic statements

In this section we investigate the relations between the set-theoretical principles mentioned in the introduction, and the formula (2.1). First of all, recall that $(EUUR)$ stands for **E**very **U**niform **U**ltrafilter is **R**egular. As already noted, it was proved by Donder that $(EUUR)$ together with the generalized continuum hypothesis (GCH) is relatively consistent with the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) .

Let \mathfrak{F} be a regular ultrafilter on a set I of cardinality λ and let X be an arbitrary set. Then the cardinality of the ultraproduct $\prod_{I/\mathfrak{F}} X$ is known to be $|X|^\lambda$. On the other hand, every uniform filter can be extended to a uniform ultrafilter. Hence, $(EUUR)$ implies that if \mathfrak{F} is a uniform filter on I , then

$$|\prod_{I/\mathfrak{F}} X| = |X|^\lambda.$$

This fact comes out to be very handy in the following

Theorem 4.1. *(EUUR) For any functor $F : \text{Set} \rightarrow \text{Set}$ the formula (2.1) holds.*

Proof. We already know that the formula (2.1) holds for κ finite.

Let κ be an infinite cardinal. Then we have

$$\begin{aligned} |F(\kappa)| &= |F^\kappa(\kappa) \cup (F(\kappa) \setminus F^\kappa(\kappa))| = \\ &= |\bigcup\{[F(g)](F(\lambda) \setminus F^\lambda(\lambda); \lambda < \kappa, g : \lambda \rightarrow \kappa)\} + f(\kappa) \leq \\ &\leq \sum_{\lambda \leq \kappa} \binom{\kappa}{\lambda} f(\lambda). \end{aligned}$$

(Note that this inequality is true for any functor in the Zermelo-Fraenkel set theory with the axiom of choice.)

Now we only need to prove the converse inequality. Suppose $|F(\kappa)| < \kappa$. Then, according to Proposition 2.4, F is constant on $\text{Set}_{\leq |X|} \setminus \{0\}$, and the formula obviously holds for κ .

Suppose $|F(\kappa)| \geq \kappa$. Then we only need to show that $|F(\kappa)| \geq \binom{\kappa}{\lambda}$ for any infinite unattainable cardinal $\lambda \leq \kappa$, as $|F(\kappa)| \geq f(\lambda)$ is trivial. Fix some $a \in F(\lambda) \setminus F^\lambda(\lambda)$. We put

$$\mathfrak{F}_a = \{Y \subseteq \lambda; a \in \text{Im } F(\nu_Y^\lambda)\},$$

where $\nu_Y^\lambda : Y \rightarrow \lambda$ is the inclusion map.

Since endofunctors of Set preserve finite non-empty intersections, by [16], if $Y, Z \in \mathfrak{F}_a$ and $Y \cap Z \neq \emptyset$ then $Y \cap Z \in \mathfrak{F}_a$. On the other hand, if $Y \cap Z = \emptyset$ we will put $Y_1 = Y \cup D, Z_1 = Z \cup D$ for some $D \subseteq X$ with cardinality less than $|X|$. Then

$$a \in \text{Im } F(\nu_{Y_1}^\lambda) \cap \text{Im } F(\nu_{Z_1}^\lambda) = \text{Im } F(\nu_D^\lambda),$$

a contradiction with $a \in F(\lambda) \setminus F^\lambda(\lambda)$. We proved that \mathfrak{F}_a is a uniform filter on X (this was observed by Koubek in [7]).

Further, for $g : \lambda \rightarrow \kappa$ a monomorphism and $b = Fg(a) \in \mathfrak{F}_b = \{Y \subseteq \kappa; b \in \text{Im } F(\nu_Y^\kappa)\}$, if restricted to $\text{Im } g$, a uniform filter by the same argument. Hence \mathfrak{F}_b is a filter on κ , $\text{Im } g \in \mathfrak{F}_b$ and $\{Y \cap \text{Im } g; Y \in \mathfrak{F}_b\} = \{g(U); U \in \mathfrak{F}_a\}$. In particular, $\{g(U); U \in \mathfrak{F}_a\}$ is a basis of \mathfrak{F}_b . Of course, if for $b, c \in \kappa$, which are images of a in monomorphisms $\lambda \rightarrow \kappa$, the filters \mathfrak{F}_b and \mathfrak{F}_c are distinct, then b and c are distinct. Therefore the cardinality of $F(\kappa)$ is at least the number of filters on κ which have a basis that is the image of \mathfrak{F}_a under a monomorphism $\lambda \rightarrow \kappa$. This number is at least the cardinality of the filtered product $\prod_{\lambda/\mathfrak{F}_a} \kappa$. Indeed, we can fix a partition of κ into λ pieces of cardinality κ and consider the choice functions $\lambda \rightarrow \kappa$. Let $g : \lambda \rightarrow \kappa$ be such a

map and let us put $\mathfrak{B}_g = \{g(Y) \mid Y \in \mathfrak{F}_\alpha\}$. The filters generated by \mathfrak{B}_g and \mathfrak{B}_h are distinct whenever h does not coincide with g in $\prod_{\lambda/\mathfrak{F}_\alpha} \kappa$.

According to the remark above the theorem, the cardinality of this filtered product under (EUUR) is exactly κ^λ . \square

For any functor $F : \text{Set} \rightarrow \text{Set}$ denote by W_F the class of its “fix-points”, i. e.

$$W_F = \{\kappa \in \text{Cn}; |F(\kappa)| = \kappa\}.$$

Definition 4.2. Let $W \subseteq \text{Cn}$. An infinite cardinal α is called an eliminator of W if $\gamma \notin W$ whenever cf $\gamma \leq \alpha \leq \gamma$.

As a corollary of Theorem 4.1 we get the following

Proposition 4.3. (EUUR) Let $F : \text{Set} \rightarrow \text{Set}$ be a functor, let α be an infinite unattainable cardinal for F . Then α is an eliminator of W_F .

Proof. According to the formula (2.1) (see Theorem 4.1),

$$|F(\gamma)| \geq \gamma^\alpha \geq \gamma^{\text{cf} \gamma} > \gamma,$$

for cf $\gamma \leq \alpha \leq \gamma$. \square

The conclusion of the above proposition, namely, that Every infinite Unattainable Cardinal for a functor F is an Eliminator of W_F , will be denoted by (EUCE). Note that (EUCE) is not a set-theoretical statement, but rather a scheme of formulas, one for each functor F . It is a consistent set-theoretical assumption: (EUCE) is a consequence of (EUUR) by the above proposition. This assumption in its turn implies another one:

Proposition 4.4. (EUCE) implies that no measurable cardinals exist.

Proof. Let us assume that κ is an (uncountable) measurable cardinal. This assumption is equivalent (see e.g. [6]) to the existence of an elementary embedding $j : U \rightarrow M$ of the universal class into a transitive model of set theory, such that κ is the smallest ordinal with $j(\kappa) \neq \kappa$.

Since the notions of mapping, identity mapping and composition of mappings are described by formulas of set theory, j can be viewed as a functor from the category Set to the category \underline{M} , which has M as its object class and morphisms and composition are defined by the same formulas of set theory as they are in Set . As M is transitive, these definitions define Set -morphisms and Set -composition, so \underline{M} is a subcategory of Set and j can be viewed as an endofunctor of Set .

Since $j|_{\text{Set}_{<\kappa}} = \text{Id}|_{\text{Set}_{<\kappa}}$ and $j(\kappa) \neq \kappa$, κ is an unattainable cardinal of j .

Choose cardinals $\kappa_0 = \kappa, \kappa_1, \kappa_2, \dots$ such that

$$j(\kappa_0) < \kappa_1 \leq j(\kappa_1) < \kappa_2 < \dots < \kappa_n \leq j(\kappa_n) < \kappa_{n+1} \leq \dots .$$

Take $\beta = \sup_{n \in \omega} \kappa_n$; hence $\omega = \text{cf } \beta < \kappa < \beta$. If κ were an eliminator of W_j , necessarily $\beta \neq |j(\beta)|$. But, since j preserves supremas of sets of ordinals and $j(A) = \{j(a); a \in A\}$ whenever $|A| < \kappa$, it follows that

$$j(\beta) = \sup_{n \in \omega} j(\kappa_n) = \beta,$$

which is a contradiction. □

It is interesting to see that under the generalized continuum hypothesis, *(EUCE)* and the formula (2.1) are equivalent:

Theorem 4.5. *(GCH) The formula (2.1) holds for any functor $F : \text{Set} \rightarrow \text{Set}$ if and only if (EUCE) is valid.*

Proof. The proof is actually included in those of Theorem 4.1. and Proposition 4.3.

Suppose *(EUCE)* is valid, let $F : \text{Set} \rightarrow \text{Set}$ be a functor and let κ be an infinite cardinal. As noted in the proof of Theorem 4.1, only the inequality $|F(\kappa)| \geq \kappa^\lambda$ for infinite unattainable cardinals $\lambda \leq \kappa$ should be proved, in the case when $|F(\kappa)| \geq \kappa$. Due to *(EUCE)* every such λ is an eliminator of W_F . Thus, if $\text{cf } \kappa \leq \lambda$, we have $|F(\kappa)| \neq \kappa$, hence $F(\kappa) > \kappa$ and $|F(\kappa)| \geq \kappa^+ = \kappa^\lambda$. The case $\lambda < \text{cf } \kappa$ is taken care of by the generalized continuum hypothesis: $\kappa^\lambda = \kappa \leq |F(\kappa)|$.

As to the converse implication, note that in the proof of Proposition 4.3 only the validity of the formula (2.1) was needed. □

We finish this section by proving that under the mentioned set-theoretical assumptions no infinitary functor is DVO. Note that infinitary functors are those having an infinite unattainable cardinal.

Lemma 4.6. *Suppose the formula (2.1) holds for the DVO functor F . Then F is finitary.*

Proof. Let $F : \text{Set} \rightarrow \text{Set}$ have infinite unattainable cardinals. Denote $G = F^\omega$.

Suppose G contains a subfunctor naturally equivalent to $C_{0,1}$. For $i = 1, 2$ put

$$G_i = (G + \sum_{\omega \leq \kappa \in \mathbb{C}_n} f(\kappa) \times G_{i,\kappa})/C_{0,1},$$

where $G_{1,\kappa} = B_{0,\kappa}$ and $G_{2,\kappa} = B_{0,\kappa} \times B_{0,\kappa}$ (the gluing is done along one of the isomorphic copies of $C_{0,1}$ in G). It is easily seen that

$$|G_i(\kappa)| = \sum_{\lambda \leq \kappa} \binom{\kappa}{\lambda} f(\lambda) = |F(\kappa)|.$$

Note that G_1 and G_2 are not naturally equivalent. Really, take an unattainable cardinal κ for F . Consider the relations \rightarrow and \approx on $G_i(\kappa) \setminus G_i^\kappa(\kappa)$ (where \rightarrow and \approx are as in Section 2). For G_1 these relations coincide; not so for G_2 , as $(A, B) \rightarrow (A, A)$, but $(A, A) \not\rightarrow (A, B)$ for disjoint subsets $A, B \subseteq \kappa$, $|A| = |B| = \kappa$. This proves that F is not DVO.

In case G does not contain a copy of $C_{0,1}$ it contains a copy of Id and one should replace in the previous paragraph $C_{0,1}$ with Id and take e. g. $G_{1,\kappa} = \text{Id} \times B_\kappa$, $G_{2,\kappa} = \text{Id} \times B_\kappa \times B_\kappa$. Again, we need to prove that G_1 and G_2 are not naturally equivalent. Take an unattainable cardinal κ for F . Take a \sim -equivalence class A_i on $G_i(\kappa) \setminus G_i^\kappa(\kappa)$ and consider the equivalence \approx on A_i . (Actually, A_i is $G_{i,\kappa}(\kappa) \setminus G_{i,\kappa}^\kappa(\kappa)$ for some of the copies of $G_{i,\kappa}$ in G_i .) For $i = 1$ the equivalence \approx has precisely two equivalence classes: $\{(a, A); a \in A \subseteq \kappa, |A| = \kappa\}$ and $\{(a, A); a \in \kappa \setminus A, A \subseteq \kappa, |A| = \kappa\}$. For $i = 2$ we have more of them: (a, A, A) , (a, A, B) , (a, B, B) for $a \in A$, $a \notin B$ are not \approx -equivalent. Thus, again G_1 and G_2 are not naturally equivalent and thus F cannot be DVO. \square

The above lemma together with the Theorems 4.1 and 4.5 have an immediate consequence:

Proposition 4.7. *Suppose that either (EUUR) or (EUCE+GCH) holds. Then every DVO functor is finitary.*

5. Functorially definable classes of cardinal numbers

This section is devoted to the study of functorially definable classes of cardinals.

Definition 5.1. We say that a class $W \subseteq \text{Cn}$ is functorially definable (abbreviated *f*-definable) if there exists a functor $F : \text{Set} \rightarrow \text{Set}$ with $W_F = W$. The class W is uniquely functorially definable, if such a functor is unique up to natural equivalence.

In [15], Y. T. Rhineghost asks to characterize all the (uniquely) functorially definable classes of cardinals.

We characterize the *f*-definable classes under $(GCH+EUCE)$ (see Theorem 5.7 below). The proof is divided into lemmas 5.2–5.6 and Theorem 5.7.

Lemma 5.2. Let W be an *f*-definable class, denote $A = W \cap \omega$. Then either $|A \setminus \{0\}| \leq 2$ or $A \in \omega + 1$.

Proof. Let $F : \text{Set} \rightarrow \text{Set}$ and $W = W_F$. We will discuss two cases:

a) Suppose $\text{Id} \subseteq F$. If $|F(1)| > 1$, then clearly $|F(n)| > n$ for $0 < n < \omega$, thus $A \subseteq \{0\}$. If $|F(1)| = 1$, then F does not contain a copy of C_1 , thus $F(0) = 0$. Suppose $A \neq \omega$; let m be the least finite number such that $|F(m)| > m$. Then m is an unattainable cardinal for F and thus, for $n > m$,

$$|F(n)| \geq n + \binom{n}{m} > n,$$

hence $A = m$.

b) Suppose F does not contain a copy of Id . Let $n_1 < n_2$ be the first two members of $A \setminus \{0\}$ (if they exist—the other case does not interest us). As each component of F contains an isomorphic copy of $C_{0,1}$, there certainly exists an unattainable cardinal p for F with $2 \leq p \leq n_2$ (otherwise we would have $|F(n_1)| = |F(n_2)|$). Then, for $n > n_2$,

$$|F(n)| \geq 1 + \binom{n}{p} > n,$$

and thus $A \setminus \{0\} = \{n_1, n_2\}$. □

Let $B_{[\alpha,\beta]}$ denote the coproduct of β copies of B_α , glued along C_1 .

Lemma 5.3. Let $A \subseteq \omega$ be a set such that $|A \setminus \{0\}| \leq 2$, and let an infinite cardinal m be given. Then there exists a functor $F : \text{Set} \rightarrow \text{Set}$ such that

- a) $W_F \cap \omega = A$,
- b) $|F(n)| = m$ for all infinite $n \leq m$, and

c) $|F(n)| = n$ for all $n \geq m$.

Proof. Let us denote $A' = A \setminus \{0\}$. We will construct a functor F , which satisfies the conditions b), c) of the lemma and for which $W_F \cap \omega = A'$. Particularly, $F(0) \neq 0$. If $0 \notin A$, we are done, otherwise we will redefine F by $F(0) = 0$.

If $A' = \emptyset$, choose F to be the coproduct of Id and C_m .

If $A' = \{p\}$ or for some positive $p \in \omega$, we start with C_p and glue to it m copies of B_{p+1} along a copy of C_1 .

If $A' = \{n_1, n_2\}$, where $0 < n_1 < n_2$, then we choose F to be the coproduct of C_{n_1} , $B_{[n_2, n_2 - n_1]}$, and $B_{[n_2 + 1, m]}$, glued along C_1 . \square

Lemma 5.4. *Let a finite cardinal k and an infinite cardinal m be given. Then there exists a functor $F : \text{Set} \rightarrow \text{Set}$ such that*

- a) $W_F \cap \omega = k$,
- b) $|F(n)| = m$ for all infinite $n \leq m$, and
- c) $|F(n)| = n$ for all $n \geq m$.

Proof. For $k \leq 1$ one can use Lemma 5.3. If $k \geq 2$ we take $F = \text{Id} \times B_{[k, m]}$. \square

The following definition is the core of our characterization of f-definable classes.

Definition 5.5. *A class $W \subseteq \text{Cn}$ is called e-consistent if it satisfies the following conditions:*

- a) if $\alpha \in W$ is infinite and $\alpha^+ \notin W$, then α^+ is an eliminator of W ;
- b) if $\alpha = \sup_I \alpha_i$ for a set $\{\alpha_i; i \in I\}$ of smaller (possibly finite) cardinals, $\alpha_i \in W$ for all $i \in I$, but $\alpha \notin W$, then there exists an eliminator γ for W such that $\text{cf } \alpha \leq \gamma \leq \alpha$. If α itself is not an eliminator and $\alpha^+ \notin W$, then α^+ is an eliminator of W .

Lemma 5.6. *(GCH+EUCE) For every functor $F : \text{Set} \rightarrow \text{Set}$, W_F is e-consistent.*

Proof. a) Let $\alpha \in \text{Cn}$ be infinite, and suppose $\alpha \in W_F$, $\alpha^+ \notin W_F$. If α is the greatest member of W_F , then certainly α^+ is an eliminator of W_F . Hence suppose there exists $\beta \in W_F$, $\beta > \alpha$; we may assume β is the smallest possible.

If $|F(\alpha^+)| < \alpha^+$, then, according to Proposition 2.4, F is constant on $\text{Set}_{\leq \alpha^+}$. Thus F has an unattainable cardinal p such that $\alpha^+ < p \leq \beta$.

Then, by the assumption (*EUCE*), p is an eliminator of W_F , and clearly α^+ is also an eliminator of W_F .

Now suppose $|F(\alpha^+)| > \alpha^+$. But

$$|F^\alpha(\alpha^+)| = (\alpha^+)^\alpha \cdot |F(\alpha)| = \alpha^+,$$

thus α^+ is an unattainable cardinal for F and thus it is an eliminator of W_F .

b) Let $\alpha = \sup_I \alpha_i \notin W_F$, where $\alpha_i \in W_F$ and $\alpha_i < \alpha$ for each $i \in I$. Clearly $|F(\alpha)| > \alpha$. Let us suppose that no γ with $cf \alpha \leq \gamma \leq \alpha$ is an unattainable cardinal for F . Then

$$F(\alpha) = \bigcup \{[Ff](\beta); \beta < \alpha, f : \beta \rightarrow X\},$$

and hence $|F(\alpha)| \leq \sum_{\beta < cf \alpha} \alpha^\beta \cdot \alpha = \alpha$, which is a contradiction. Thus, there is an unattainable cardinal γ with $cf \alpha \leq \gamma \leq \alpha$, which is an eliminator for W_F according to our assumption.

Suppose α is not an unattainable cardinal for F . Then

$$|F^{\alpha^+}(\alpha^+)| \leq \sum_{\gamma < \alpha} (\alpha^+)^\gamma \cdot \alpha = \alpha^+.$$

Thus if $|F(\alpha^+)| > \alpha^+$, α^+ must be an unattainable cardinal for F and hence an eliminator of W_F . \square

Let $W \subseteq \text{Cn}$. For every eliminator α of W , let us denote by $\tilde{\alpha}$ the smallest $\beta > \alpha$ such that $\beta \in W$, if such a β exists, otherwise $\tilde{\alpha} = 1$.

Theorem 5.7. (*GCH+EUCE*) *A class $W \subseteq \text{Cn}$ is functorially definable if and only if it is e-consistent and either $W \cap \omega \in \omega + 1$ or $|W \cap \omega| \leq 2$.*

Proof. The “only if” part is already proved in Lemmas 5.2 and 5.6.

Suppose we are given a class $W \subseteq \text{Cn}$ satisfying the conditions of the Theorem. We will construct a functor $F : \text{Set} \rightarrow \text{Set}$ with $W_F = W$ in two steps.

Let m be the least infinite cardinal in W , if it exists, otherwise take $m = \omega$.

Step 1. First, we construct a functor $G : \text{Set} \rightarrow \text{Set}$ satisfying the following conditions:

- a) $W \cap m = W_G \cap m$,
- b) $|G(\alpha)| \geq \alpha$ for $\alpha \geq \omega$,

c) $|G(\alpha)| = \alpha$ for $\alpha \in W$.

If $|W \cap \omega| \leq 2$, let G be the functor given by Lemma 5.3. The conditions are trivially satisfied.

If $2 < W \cap \omega \in \omega$, let G be the functor given by Lemma 5.4. Again, all of the conditions are satisfied.

Suppose $W \cap \omega = \omega$. If $m = \omega$, take $G = \text{Id}$. The conditions a), b) and c) again obviously hold. Let $m > \omega$; take $G = \text{Id} \times B_{[\omega, m]}$. For any infinite cardinal $n < m$,

$$|G(n)| = n^\omega \cdot m \geq m > n,$$

thus condition a) holds. Condition b) holds since $B(X) \neq \emptyset$ for any non-empty X . Since W is e-consistent, $n \in W$ for $n \in \omega$ and $\omega \notin W$, condition b) in the definition of e-consistency implies that ω is an eliminator of W . Thus, for any infinite $\alpha \in W$ cf $\alpha > \omega$. Due to the Generalized Continuum Hypothesis and the fact that $\alpha \geq m$ we get

$$|G(\alpha)| = \alpha^\omega \cdot m = \alpha,$$

which is condition c).

Step 2. Suppose the functor G contains a copy of $C_{0,1}$. We then obtain the functor F by gluing a copy of $B_{[\alpha, \tilde{\alpha}]}$ to G along a copy of $C_{0,1}$ for every eliminator α of W . We need to prove that for $n \geq m$, $|F(n)| = n$ iff $n \in W$. Suppose $n \in W$, $n \geq m$. Then for any eliminator $\alpha < n$ we have cf $n > \alpha$ and thus, due to (GCH), $n^\alpha = n$. On the other hand, surely $\tilde{\alpha} \leq n$, hence

$$|F(n)| = |G(n)| + \sum \{n^\alpha \cdot \tilde{\alpha}; \alpha < n, \alpha \text{ is an eliminator of } W\} = n.$$

Let $n \notin W$, $n > m$. If there is no $\beta \in W$ with $\beta > n$ then n is an eliminator of W , $\tilde{n} = 1$ and thus

$$|F(n)| \geq |B_n(n)| = 2^n > n.$$

So let us suppose there exists $\beta > n$ with $\beta \in W$; suppose β is the least such cardinal. Denote $\alpha = \sup W \cap n$. Now, if $\alpha \in W$, then $\alpha^+ \notin W$ and, by a) in the definition of e-consistency, α^+ is an eliminator of W with $\alpha^+ = \beta > n$. Thus

$$|F(n)| \geq |B_{[\alpha^+, \beta]}(n)| = n^{\alpha^+} \cdot \beta \geq \beta > n.$$

Suppose $\alpha \notin W$ and $\alpha = n$. Then, according to b) in the definition of e-consistency, there exists an eliminator γ with $cf\ n \leq \gamma \leq n$. Then

$$|F(n)| \geq |B_\gamma(n)| = n^\gamma \geq n^{cf\ n} > n.$$

If $\alpha \notin W$ and $\alpha < n$, then either $\gamma = \alpha$ or $\gamma = \alpha^+$ is an eliminator of W , and again

$$|F(n)| \geq |B_{[\gamma, \beta]}(n)| \geq \beta > n.$$

Similarly, if G contains a copy of Id and does not contain a copy of $C_{0,1}$, then F is obtained by gluing a copy of $\text{Id} \times B_{[\alpha, \hat{\alpha}]}$ to G for every eliminator α of W . \square

The proof of the previous theorem actually is also a hint to its “uniquely functorially definable” counterpart:

Theorem 5.8. (*GCH+EUCE*) *A class $W \subseteq \text{Cn}$ of cardinal numbers is uniquely functorially definable if and only if $W = \text{Cn}$.*

Proof. The fact that Cn is uniquely functorially definable is equivalent to the fact that Id is a DVO functor, which was proved in [19].

Suppose W is a functorially definable class of cardinals, different from Cn . We will follow the proof of Theorem 5.7 and see that we can then construct two not naturally equivalent functors F_1 and F_2 with $W = W_{F_1} = W_{F_2}$. The functor F_1 is the one constructed in the proof of Theorem 5.7.

Suppose $W \cap \omega \neq \omega$. Then, as we know, there exists an $n \in \omega$ such that $W \cap \omega \subseteq n - 1$. In Lemmas 5.3 and 5.4 we construct a functor G with $W_G \cap \omega = W \cap \omega$. Replace this by the functor $G_2 = (G + B_{0,n})/C_{0,1}$ (or $G_2 = (G + \text{Id} \times B_{0,n})/\text{Id}$ in case of Lemma 5.4) and use it as the starting functor G in step 1 of the proof of Theorem 5.7. Thus we get a functor F_2 for which n is an unattainable cardinal; not so for F_1 .

Now suppose $W \cap \omega = \omega$. Then the functor F in Theorem 5.7 is constructed by gluing together some $\text{Id} \times B_\lambda$'s along Id (where all of the λ 's are infinite). The functor F_2 is obtained by exactly the same procedure, only using $\text{Id} \times B_\lambda \times B_\lambda$ instead of $\text{Id} \times B_\lambda$. The proof that F_1 and F_2 are not naturally equivalent is contained in the second part of the proof of Lemma 4.6. \square

References

- [1] J. Adámek, *Limits and colimits in generalized algebraic categories*, Czech. Math. J. **26** (1976), 55–64.
- [2] J. Adámek, V. Koubek, *Coequalizers in generalized algebraic categories*, Comment. Math. Univ. Carolinae **13** (1972), 311–324.
- [3] J. Adámek, V. Koubek, V. Pohlová, *Colimits in the generalized algebraic categories*, Acta Univ. Carolinae **13** (1972), No. 2, 29–40.
- [4] A. Barkhudaryan, R. El Bashir, V. Trnková, *Endofunctors of Set*, in: Proceedings of the Conference Categorical Methods in Algebra and Topology, Bremen 2000, eds: H. Herrlich and H.-E. Porst, Mathematik-Arbeitspapiere **54** (2000), 47–55.
- [5] H.-D. Donder, *Regularity of ultrafilters and the core model*, Israel Journal of Math. **63** (1988), No. 3, 289–322.
- [6] A. Kanamori, *The Higher Infinite*, Springer-Verlag, 1994.
- [7] V. Koubek, *Set functors*, Comment. Math. Univ. Carolinae **12** (1971), 175–195.
- [8] V. Koubek, *Set functors II — contravariant case*, Comment. Math. Univ. Carolinae **14** (1973), 47–59.
- [9] V. Koubek, J. Reiterman, *Set functors III — monomorphisms, epimorphisms, isomorphisms*, Comment. Math. Univ. Carolinae **14** (1973), 441–455.
- [10] V. Kůrková-Pohlová, *On sums in generalized algebraic categories*, Czech. Math. J. **23** (1973), 235–251.
- [11] V. Kůrková-Pohlová, V. Koubek, *When a generalized algebraic category is monadic*, Comment. Math. Univ. Carolinae **15** (1975), 577–587.
- [12] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.
- [13] J. Reiterman, *An example concerning set functors*, Comment. Math. Univ. Carolinae **12** (1971), 227–233.
- [14] Y. T. Rhineghost, *The Functor that Wouldn't be — A Contribution to the Theory of Things that Fail to Exist*, in: Categorical Perspectives, Birkhauser Verlag, Trends in Mathematics (2001), 29–36.
- [15] Y. T. Rhineghost *The Emergence of Functors — A Continuation of “The Functor that Wouldn't be”*, in: Categorical Perspectives, Birkhauser Verlag, Trends in Mathematics (2001), 37–46.
- [16] V. Trnková, *Some properties of set functors*, Comment. Math. Univ. Carolinae **10** (1969), 323–352.
- [17] V. Trnková, *On descriptive classification of set functors I and II*, Comment. Math. Univ. Carolinae **12** (1971), 143–175 and 345–357.
- [18] V. Trnková, P. Goralčík, *On products in generalized algebraic categories*, Comment. Math. Univ. Carolinae **10** (1969), 49–89.
- [19] A. Zmrzlina, *Too Many Functors, — A Continuation of “The Emergence of Functors”*, in: Categorical Perspectives, Birkhauser Verlag, Trends in Mathematics (2001), 47–62.

A. Barkhudaryan

Mathematical Institute of the Charles University
Sokolovská 83
186 75 Prague 8
Czech Republic
e-mail: artur@karlin.mff.cuni.cz

R. El Bashir

Department of Algebra
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Prague 8
Czech Republic
e-mail: bashir@karlin.mff.cuni.cz

V. Trnková

Mathematical Institute of the Charles University
Sokolovská 83
186 75 Prague 8
Czech Republic
e-mail: trnkova@karlin.mff.cuni.cz