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LOW-DIMENSIONAL COHOMOLOGY FOR CATEGORICAL GROUPS

by *B. A.R. GARZÓN and A. del RIO*

RESUME. Dans ce travail, les auteurs définissent les groupes catégoriques de cohomologie $H^i(\mathbf{G}, \mathbf{A})$, pour $i = 0, 1$, d'un groupe catégorique \mathbf{G} , à coefficients dans un groupe catégorique tressé (symétrique pour $i = 1$) \mathbf{A} muni d'une action cohérente à gauche de \mathbf{G} . Ces coefficients sont appelés \mathbf{G} -modules (symétriques). Ils montrent qu'à toute suite exacte courte de \mathbf{G} -modules symétriques on peut associer une suite exacte à 6 termes qui connecte H^0 et H^1 . Des groupes de cohomologie bien connus dans plusieurs contextes différents, ainsi que les suites exactes qui les relient, s'avèrent être des projections de cette théorie générale dans la catégorie des groupes abéliens, si l'on considère les groupes d'homotopie π_0 et π_1 de H^1 .

1 Introduction

It is well known [17, 22] that, if G is a group and A is a G -module, then A^G , the subgroup of invariant elements in A , is exactly $H^0(G, A)$, the zero-cohomology group of G with coefficients in A , and the quotient group $Der(G, A)/IDer(G, A)$, of the abelian group of derivations from G into A by the subgroup of inner derivations, is the first cohomology group $H^1(G, A)$. Actually, these cohomology groups $H^0(G, A)$ and $H^1(G, A)$ are, respectively, the kernel and the cokernel of the group homomorphism $A \rightarrow Der(G, A)$ given by inner derivations. Moreover, if $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is a short exact sequence of G -modules, then there is a group exact sequence

$$0 \rightarrow A'^G \rightarrow A^G \rightarrow A''^G \rightarrow \frac{Der(G, A')}{IDer(G, A')} \rightarrow \frac{Der(G, A)}{IDer(G, A)} \rightarrow \frac{Der(G, A'')}{IDer(G, A'')}$$

that actually extends to a long exact sequence connecting all $H^i(G, A)$, $i \geq 0$.

Categorical groups are monoidal groupoids in which each object is invertible, up to isomorphism, with respect to the tensor product [2, 18, 25, 27]. In this setting, group cohomology for Picard categories [28] and Schreier theories for the classification of extensions of categorical groups [2, 5, 6, 7, 12] have been developed. Moreover, in a recent

paper [14], we defined and studied the categorical group $\text{Der}(\mathbb{G}, \mathbb{A})$ of derivations from a categorical group \mathbb{G} into a \mathbb{G} -module \mathbb{A} . This categorical group could be regarded as a sort of cohomology categorical group at dimension zero but, following the classic group case recalled above, our aim in this paper is to define cohomology categorical groups which, at the lowest dimension, consist of suitably defined invariant objects under a given categorical action. More precisely, if \mathbb{G} is any categorical group acting coherently on a braided categorical group \mathbb{A} (i.e., \mathbb{A} is a \mathbb{G} -module), we define in this paper the cohomology categorical group $\mathcal{H}^0(\mathbb{G}, \mathbb{A})$ as the kernel of the homomorphism of categorical groups $\mathbb{A} \rightarrow \text{Der}(\mathbb{G}, \mathbb{A})$ given by inner derivations and then we recognize it as the categorical group $\mathbb{A}^{\mathbb{G}}$ of invariant objects (see Section 4). If \mathbb{A} is symmetric (i.e., \mathbb{A} is a symmetric \mathbb{G} -module), we define $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ as the cokernel of the homomorphism of symmetric categorical groups $\mathbb{A} \rightarrow \text{Der}(\mathbb{G}, \mathbb{A})$ given by inner derivations (see Section 5). When \mathbb{G} and \mathbb{A} are suitably specialized, the cohomological invariants π_0 and π_1 of these $\mathcal{H}^i(\mathbb{G}, \mathbb{A})$ are well-known cohomology groups (see examples in Sections 4 and 5). Furthermore, by using the notion of exactness in the context of categorical groups introduced in [19, 29], we show in Section 6 that, associated to any short exact sequence of symmetric \mathbb{G} -modules,

$$\mathbb{A}' \rightarrow \mathbb{A} \rightarrow \mathbb{A}'' ,$$

there exists a six-term exact sequence of categorical groups

$$\mathcal{H}^0(\mathbb{G}, \mathbb{A}') \rightarrow \mathcal{H}^0(\mathbb{G}, \mathbb{A}) \rightarrow \mathcal{H}^0(\mathbb{G}, \mathbb{A}'') \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{A}') \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{A}'')$$

By taking π_0 and π_1 in this exact sequence of categorical groups, we obtain group exact sequences which, in particular cases, specialize then to well-known group cohomology exact sequences.

First of all we dedicate a preliminary section to fix notations and recall the main notions and results we use throughout the paper.

2 Preliminaries

Monoidal categories and, in particular, categorical groups have been studied extensively in the literature and we refer to [1, 2, 18, 20, 21, 26, 27] for the background.

We recall that a *categorical group* is a monoidal groupoid $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ with

$$a = a_{x,y,z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z),$$

$$l = l_x : I \otimes X \rightarrow X, \quad r = r_x : X \otimes I \rightarrow X,$$

where every object X is invertible, that is, the functors $Y \mapsto X \otimes Y$ and $Y \mapsto Y \otimes X$ are equivalences for any $Y \in \mathbb{G}$. In this case it is possible to choose, for each $X \in \mathbb{G}$, an object $X^* \in \mathbb{G}$ (termed an *inverse* for X) and isomorphisms, $\gamma_x : X \otimes X^* \rightarrow I$ and $\vartheta_x : X^* \otimes X \rightarrow I$, such that $l_x \cdot (\gamma_x \otimes 1) = r_x \cdot (1 \otimes \vartheta_x) \cdot a_{x,x^*,x}$. The choice of a system of inverses $(X^*, \gamma_x, \vartheta_x)$, $X \in \mathbb{G}$, induces a (contravariant) endofunctor $(-)^* : \mathbb{G} \rightarrow \mathbb{G}, X \mapsto X^*$, such that the isomorphisms γ_x and ϑ_x are natural. There are also natural isomorphisms $t_x : X \rightarrow (X^*)^*$ and $v_{x,y} : (X \otimes Y)^* \rightarrow Y^* \otimes X^*$. Note that the natural isomorphisms l_x and r_x , of left and right unit, ensure that the unit object I is an invertible object and, since $r_I = l_I : I \otimes I \rightarrow I$, we choose (I, r_I, r_I) as an inverse for I .

A categorical group \mathbb{G} is said to be a *braided categorical group* if it is also equipped with a family of natural isomorphisms $c = c_{x,y} : X \otimes Y \rightarrow Y \otimes X$ (the braiding) that interacts with a, r and l , satisfying suitable coherence conditions [18]. A braided categorical group \mathbb{G} is called a *symmetric categorical group* if the condition $c^2 = 1$ is satisfied.

We will denote by \mathcal{CG} (respectively \mathcal{BCG} or \mathcal{SCG}) the 2-category of categorical groups (respectively of braided or symmetric categorical groups) whose arrows (here called *homomorphisms*), $\mathbf{T} = (T, \mu) : \mathbb{G} \rightarrow \mathbb{H}$, are functors $T : \mathbb{G} \rightarrow \mathbb{H}$ together with families of natural isomorphisms $\mu = \mu_{x,y} : T(X \otimes Y) \rightarrow T(X) \otimes T(Y)$, $X, Y \in \mathbb{G}$, such that the usual coherence condition holds. If \mathbb{G} and \mathbb{H} are braided (symmetric) categorical groups, compatibility with the braiding (symmetry) is also required. A 2-cell (here called *morphism*) from (T, μ) to (T', μ') consists of a natural transformation $\epsilon : T \rightarrow T'$ such that, for any objects $X, Y \in \mathbb{G}$, $(\epsilon_x \otimes \epsilon_y) \cdot \mu_{x,y} = \mu'_{x,y} \cdot \epsilon_{x \otimes y}$.

If $\mathbf{T} : \mathbb{G} \rightarrow \mathbb{H}$ is a homomorphism, there exists an isomorphism, $\mu_0 : T(I) \rightarrow I$, such that, for any $X \in \mathbb{G}$, the equalities $T(r_x) =$

$r_{T(X)} \cdot (1 \otimes \mu_0) \cdot \mu_{X,I}$ and $T(l_X) = l_{T(X)} \cdot (\mu_0 \otimes 1) \cdot \mu_{I,X}$ hold. In addition, once a system of inverses $(X^*, \gamma_X, \vartheta_X)$ for $X \in \mathbb{G}$ and $(Y^*, \gamma_Y, \vartheta_Y)$ for $Y \in \mathbb{H}$ has been chosen, there exist unique isomorphisms $\lambda_X : T(X^*) \rightarrow T(X)^*$, such that the equalities $\mu_0 \cdot T(\gamma_X) = \gamma_{T(X)} \cdot (1 \otimes \lambda_X) \cdot \mu_{X,X^*}$ and $\mu_0 \cdot T(\vartheta_X) = \vartheta_{T(X)} \cdot (\lambda_X \otimes 1) \cdot \mu_{X^*,X}$ hold.

Recall that, if $\mathbb{G} \in \mathcal{CG}$, then the set of connected components of \mathbb{G} , $\pi_0(\mathbb{G})$, has a group structure (which is abelian if $\mathbb{G} \in \mathcal{BCG}$) where the operation is given by $[X] \cdot [Y] = [X \otimes Y]$. Also, the abelian group $\pi_1(\mathbb{G}) = \text{Aut}_{\mathbb{G}}(I)$ is associated to \mathbb{G} .

The *kernel* $(K(\mathbf{T}), \mathbf{j}, \epsilon)$ of a homomorphism $\mathbf{T} : \mathbb{G} \rightarrow \mathbb{H}$ was defined in [19, 29] and we now recall an explicit description of this universal object. $K(\mathbf{T})$ is the categorical group whose objects are pairs (X, u_X) where $X \in \mathbb{G}$ and $u_X : T(X) \rightarrow I$ is an arrow in \mathbb{H} ; an arrow $f : (X, u_X) \rightarrow (Y, u_Y)$ is an arrow $f : X \rightarrow Y$ in \mathbb{G} such that $u_X = u_Y \cdot T(f)$; the tensor product is given by $(X, u_X) \otimes (Y, u_Y) = (X \otimes Y, u_X \cdot u_Y)$, where $u_X \cdot u_Y : T(X \otimes Y) \rightarrow I$ is the composite $T(X \otimes Y) \xrightarrow{\mu_{X,Y}} T(X) \otimes T(Y) \xrightarrow{u_X \otimes u_Y} I \otimes I \xrightarrow{r_I = l_I} I$; the unit object is (I, μ_0) and the associativity and left-unit and right-unit constraints are given by $a_{X,Y,Z}$, l_X and r_X respectively; an inverse for any object (X, u_X) is given by $(X^*, (u_X^*)^{-1} \lambda_X)$, where X^* is an inverse for X . As for $\mathbf{j} : K(\mathbf{T}) \rightarrow \mathbb{G}$, it is the strict homomorphism sending $f : (X, u_X) \rightarrow (Y, u_Y)$ to $f : X \rightarrow Y$. Finally, $\epsilon : \mathbf{Tj} \rightarrow \mathbf{0}$ is the morphism whose component at (X, u_X) is given by u_X . If \mathbb{G} and \mathbb{H} are braided (symmetric) categorical groups, then $K(\mathbf{T})$ is also a braided (symmetric) categorical group, where the braiding (symmetry) $c = c_{(X, u_X), (Y, u_Y)} : (X, u_X) \otimes (Y, u_Y) \rightarrow (Y, u_Y) \otimes (X, u_X)$ is exactly $c_{X,Y}$, and \mathbf{j} is a homomorphism of braided (symmetric) categorical groups. The categorical group $K(\mathbf{T})$ just described is a standard homotopy kernel and so it is determined, up to isomorphism, by the following strict universal property: given a homomorphism $\mathbf{F} : \mathbb{K} \rightarrow \mathbb{G}$ and a morphism $\tau : \mathbf{TF} \rightarrow \mathbf{0}$, there exists a unique homomorphism $\mathbf{F}' : \mathbb{K} \rightarrow K(\mathbf{T})$ such that $\mathbf{jF}' = \mathbf{F}$ and $\epsilon\mathbf{F}' = \tau$.

In [19], the following notion of exactness for homomorphisms of categorical groups was introduced. Let $\mathbb{K} \xrightarrow{\mathbf{F}} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H}$ be two homomorphisms and $\tau : \mathbf{TF} \rightarrow \mathbf{0}$ a morphism. From the universal property of

the kernel of \mathbf{T} , there exists a homomorphism F' making the following diagram commutative:

$$\begin{array}{ccc}
 & K(\mathbf{T}) & \\
 F' \nearrow & & \searrow j \\
 \mathbb{K} & \xrightarrow{F} & \mathbb{G}
 \end{array}$$

which is given, for any $X \in \mathbb{K}$, by $F'(X) = (F(X), \tau_X)$, for any arrow f in \mathbb{K} , by $F'(f) = F(f)$ and where, for any $X, Y \in \mathbb{K}$, $(\mu_{F'})_{X,Y} = (\mu_F)_{X,Y}$. Then, the triple (F, τ, \mathbf{T}) (or sometimes just the sequence $\mathbb{K} \xrightarrow{F} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H}$ if τ is understood) is said to be *2-exact* if F' is full and essentially surjective. Note that if $(K(\mathbf{T}), j, \epsilon)$ is the kernel of $\mathbf{T} : \mathbb{G} \rightarrow \mathbb{H}$, then the triple $(j, \epsilon, \mathbf{T})$ is 2-exact and there exists (see [25]) an induced exact sequence of groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1(K(\mathbf{T})) & \xrightarrow{\pi_1(j)} & \pi_1(\mathbb{G}) & \xrightarrow{\pi_1(\mathbf{T})} & \pi_1(\mathbb{H}) & (1) \\
 & & & & & \searrow \delta & & \\
 & & \pi_0(K(\mathbf{T})) & \xrightarrow{\pi_0(j)} & \pi_0(\mathbb{G}) & \xrightarrow{\pi_0(\mathbf{T})} & \pi_0(\mathbb{H})
 \end{array}$$

where, for any $u \in \pi_1(\mathbb{H})$, $\delta(u)$ is the connected component of the object $(I, u \cdot \mu_0) \in K(\mathbf{T})$. Moreover, if the functor T is essentially surjective, the above exact sequence is right exact. In general, if (F, τ, \mathbf{T}) is a 2-exact sequence of categorical groups, then $\pi_0(\mathbb{K} \xrightarrow{F} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H})$ and $\pi_1(\mathbb{K} \xrightarrow{F} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H})$ are exact sequences of groups.

Note that the same notion of 2-exactness can be defined for pointed groupoids, pointed functors and pointed natural transformations. In this case, sequence (1) is an exact sequence of groups and pointed sets (the last three terms).

If \mathbb{G} is a categorical group and \mathbb{A} is a braided categorical group, a *\mathbb{G} -action on \mathbb{A}* is a homomorphism of categorical groups $(T, \mu) : \mathbb{G} \rightarrow \mathcal{E}q(\mathbb{A})$, where $\mathcal{E}q(\mathbb{A})$ (see [2, 6]) is the categorical group of monoidal equivalences of the braided categorical group \mathbb{A} . When such a \mathbb{G} -action is given we say that \mathbb{A} is a *\mathbb{G} -module* and, if \mathbb{A} is symmetric, we say that \mathbb{A} is a *symmetric \mathbb{G} -module*.

It is straightforward to see (cf. [6, 12]) that giving a (symmetric) \mathbb{G} -module \mathbb{A} is equivalent to giving a functor

$$ac : \mathbb{G} \times \mathbb{A} \longrightarrow \mathbb{A} \quad , \quad (X, A) \mapsto {}^X A \quad , \quad (f, u) \mapsto {}^f u \quad ,$$

together with natural isomorphisms

$$({}^{X \otimes Y} A) \xrightarrow{\phi_{X,Y,A}} {}^X ({}^Y A) \quad , \quad {}^I A \xrightarrow{\phi_{0,A}} A \quad , \quad {}^X (A \otimes B) \xrightarrow{\psi_{X,A,B}} {}^X A \otimes {}^X B \quad ,$$

satisfying suitable coherence conditions. If $X \in \mathbb{G}$ and u is an arrow in \mathbb{A} , we will write ${}^{id^X} u = {}^X u$ and, if f is an arrow in \mathbb{G} and $A \in \mathbb{A}$, we write ${}^f id_A = {}^f A$.

Note that, for any $X \in \mathbb{G}$, there exists a unique isomorphism $\psi_0 = \psi_{0,X} : {}^X I \longrightarrow I$ such that the equalities $l_{X_A} \cdot (\psi_{0,X} \otimes 1) \cdot \psi_{X,I,A} = {}^X l_A$ and $r_{X_A} \cdot (1 \otimes \psi_{0,X}) \cdot \psi_{X,A,I} = {}^X r_A$ hold and then $\psi_{0,X} \cdot \psi_{0,Y} \cdot \phi_{X,Y,I} = \psi_{0,X \otimes Y}$ also holds. Moreover, once a system of inverses $(A^*, \gamma_A, \vartheta_A)$, $A \in \mathbb{A}$, has been chosen, there are natural isomorphisms $w_{X,A} : {}^X A^* \rightarrow ({}^X A)^*$.

A homomorphism of (symmetric) \mathbb{G} -modules from \mathbb{A} to \mathbb{B} is a homomorphism of braided (symmetric) categorical groups $\mathbb{T} = (T, \mu) : \mathbb{A} \longrightarrow \mathbb{B}$ that is equivariant in the sense that there exists a family of natural isomorphisms

$$\nu = \nu_{X,A} : T({}^X A) \longrightarrow {}^X T(A) \quad ,$$

such that, for any objects $X, Y \in \mathbb{G}$ and $A, B \in \mathbb{A}$, the following conditions hold:

- i) $\phi_{X,Y,T(A)} \cdot \nu_{X \otimes Y, A} = {}^X \nu_{Y,A} \cdot \nu_{X,Y_A} \cdot T(\phi_{X,Y,A}) \quad ,$
- ii) $\phi_{0,T(A)} \cdot \nu_{I,A} = T(\phi_{0,A}) \quad ,$
- iii) $(\nu_{X,A} \otimes \nu_{X,B}) \cdot \mu_{X_A, X_B} \cdot T(\psi_{X,A,B}) = \psi_{X,T(A),T(B)} \cdot {}^X \mu_{A,B} \cdot \nu_{X,A \otimes B} \quad ,$
- iv) ${}^X T(c_{A,B}) \cdot \nu_{X,A \otimes B} = \nu_{X,B \otimes A} \cdot T({}^X c_{A,B}) \quad .$

If $(\mathbb{T}, \nu) : \mathbb{A} \longrightarrow \mathbb{B}$ is a homomorphism of (symmetric) \mathbb{G} -modules, we can consider the kernel $(K(\mathbb{T}), \mathbf{j}, \epsilon)$ of the underlying homomorphism of braided (symmetric) categorical groups \mathbb{T} . Then $K(\mathbb{T})$ is also a (symmetric) \mathbb{G} -module with action given by ${}^X(A, u_A) = ({}^X A, u_{X_A})$, where $u_{X_A} = \psi_{0,X} \cdot {}^X u_A \cdot \nu_{X,A}$, and the homomorphism \mathbf{j} is strictly equivariant.

Given homomorphisms of (symmetric) \mathbb{G} -modules $(\mathbf{T}, \nu), (\mathbf{T}', \nu') : \mathbb{A} \rightarrow \mathbb{B}$, a *morphism* from (\mathbf{T}, ν) to (\mathbf{T}', ν') consists of a morphism $\epsilon : \mathbf{T} \rightarrow \mathbf{T}'$ such that, for all $X \in \mathbb{G}$ and $A \in \mathbb{A}$, the following equality holds:

$${}^X \epsilon_A \cdot \nu_{X,A} = \nu'_{X,A} \cdot \epsilon_{X_A} .$$

In this way, we have the 2-category of (symmetric) \mathbb{G} -modules, which will be denoted by $\mathbb{G}\text{-Mod}$ (respectively by $\mathbb{G}\text{-SMod}$).

We should remark that, throughout the sections below, we will maintain the same notation established in these preliminaries for all canonical and natural isomorphisms introduced here. We will also assume that a system of inverses $(X^*, \gamma_X, \vartheta_X)$ has been chosen for the objects X of any categorical group.

3 Inner derivations

In this section we introduce the categorical-group version of the inner derivation group homomorphism.

Let \mathbb{G} be a categorical group and let \mathbb{A} be a \mathbb{G} -module with braiding c .

A *derivation* from \mathbb{G} into \mathbb{A} (see [14]) is a functor $D : \mathbb{G} \rightarrow \mathbb{A}$ together with a family of natural isomorphisms

$$\beta = \beta_{X,Y} : D(X \otimes Y) \rightarrow D(X) \otimes {}^X D(Y), \quad X, Y \in \mathbb{G},$$

such that, for any objects $X, Y, Z \in \mathbb{G}$, the following coherence condition holds:

$$\begin{aligned} & (1 \otimes \psi_{X, D(Y), Y_{D(Z)}}) \cdot (1 \otimes {}^X \beta_{Y,Z}) \cdot \beta_{X, Y \otimes Z} \cdot D(a_{X,Y,Z}) = \quad (2) \\ & (1 \otimes (1 \otimes \phi_{X,Y,D(Z)})) \cdot a_{D(X), X_{D(Y)}, (X \otimes Y)_{D(Z)}} \cdot (\beta_{X,Y} \otimes 1) \cdot \beta_{X \otimes Y, Z} . \end{aligned}$$

If (D, β) is a derivation, there exists an isomorphism $\bar{\beta}_0 : D(I) \rightarrow I$ determined uniquely by the following two equalities:

$$\begin{aligned} D(r_X) &= r_{D(X)} \cdot (1 \otimes \psi_{0,X}) \cdot (1 \otimes {}^X \bar{\beta}_0) \cdot \beta_{X,I} , \\ D(l_X) &= l_{D(X)} \cdot (1 \otimes \phi_{0,D(X)}) \cdot (\bar{\beta}_0 \otimes 1) \cdot \beta_{I,X} . \end{aligned}$$

The derivation (D, β) is termed *normalized* if the isomorphism $\bar{\beta}_0$ is an identity.

Derivations from \mathbb{G} into \mathbb{A} are the objects of a categorical group $\mathbb{D}er(\mathbb{G}, \mathbb{A})$ (see [14]) in which the tensor product of two derivations (D, β) and (D', β') is defined as the derivation

$$(D, \beta) \otimes (D', \beta') = (D \otimes D', \beta \otimes \beta'), \quad (3)$$

where $D \otimes D' : \mathbb{G} \rightarrow \mathbb{A}$ is the functor given, for any $X \in \mathbb{G}$, by $(D \otimes D')(X) = D(X) \otimes D'(X)$ and, for any arrow f , by $(D \otimes D')(f) = D(f) \otimes D'(f)$, whereas $\beta \otimes \beta' = (\beta \otimes \beta')_{x,y} : (D \otimes D')(X \otimes Y) \rightarrow (D \otimes D')(X) \otimes (D \otimes D')(Y)$ is the family of natural isomorphisms determined, for each pair of objects $X, Y \in \mathbb{G}$, by the following equality:

$$(1 \otimes \psi_{x, D(Y), D'(Y)}) \cdot (\beta \otimes \beta')_{x,y} = (1 \otimes c_{x, D(Y), D'(Y)} \otimes 1) \cdot (\beta_{x,y} \otimes \beta'_{x,y}) .$$

For any object $A \in \mathbb{A}$ there is an *inner derivation* $(D_A, \beta_A) : \mathbb{G} \rightarrow \mathbb{A}$ where, for any $X \in \mathbb{G}$, $D_A(X) = {}^X A \otimes A^*$, for any arrow f in \mathbb{G} , $D_A(f) = {}^f A \otimes 1$ and, for any $X, Y \in \mathbb{G}$, $(\beta_A)_{x,y} : D_A(X \otimes Y) \rightarrow D_A(X) \otimes {}^X D_A(Y)$ is the family of natural isomorphisms determined by the equality:

$$(1 \otimes \psi_{x, Y, A, A^*}) \cdot (\beta_A)_{x,y} \cdot (1 \otimes l_{A^*}) \cdot (1 \otimes \psi_{0,x} \otimes 1) \cdot (1 \otimes {}^x \vartheta_A \otimes 1) = (4)$$

$$(1 \otimes \phi_{x,y,A} \otimes 1) \cdot c_{(X \otimes Y)_A \otimes X_{A^*}, X_{A \otimes A^*}} \cdot (1 \otimes \psi_{x, A^*, A} \otimes 1).$$

Then we can consider the *inner derivation homomorphism* of categorical groups

$$(T, \mu) : \mathbb{A} \longrightarrow \mathbb{D}er(\mathbb{G}, \mathbb{A})$$

which is defined, on objects $A \in \mathbb{A}$, by $T(A) = (D_A, \beta_A)$, on arrows $f : A \rightarrow B$, by $T(f)_x = {}^X f \otimes (f^*)^{-1}$, $X \in \mathbb{G}$, and where, for any $A, B \in \mathbb{A}$, $(\mu)_{A,B}$ is the arrow in $\mathbb{D}er(\mathbb{G}, \mathbb{A})$ determined by the natural transformation whose component at $X \in \mathbb{G}$ is the composite $(1 \otimes c_{x, B \otimes B^*, A^*}) \cdot (\psi_{x, A, B} \otimes v_{A, B})$.

This homomorphism allows to construct the following quotient groupoid of derivations module inner derivations. The objects are the derivations from \mathbb{G} into \mathbb{A} . Given two derivations $(D, \beta), (D', \beta') : \mathbb{G} \rightarrow \mathbb{A}$ let us consider, as pre-arrows, all pairs (A, φ^A) where $A \in \mathbb{A}$ and φ^A is

a morphism of derivations (see [14]) from (D, β) to $(D_A, \beta_A) \otimes (D', \beta')$, that is, φ^A is a natural transformation from D to $D_A \otimes D'$ such that, for any $X, Y \in \mathbb{G}$, the following equality holds:

$$(\varphi_X^A \otimes {}^X\varphi_Y^A) \cdot \beta_{X,Y} = (\beta_A \otimes \beta')_{X,Y} \cdot \varphi_{X \otimes Y}^A. \tag{5}$$

An arrow from (D, β) to (D', β') is then an equivalence class $[A, \varphi^A]$ of pre-arrows where $[A, \varphi^A] = [A', \varphi^{A'}]$ if there exists an arrow in $\mathbb{A} u : A \rightarrow A'$ such that, for any $X \in \mathbb{G}$, the equality $\varphi_X^{A'} = ({}^X u \otimes (u^*)^{-1} \otimes 1) \cdot \varphi_X^A$ holds.

The composition of two arrows $(D, \beta) \xrightarrow{[A, \varphi^A]} (D', \beta') \xrightarrow{[A', \varphi^{A'}]} (D'', \beta'')$ is the class of the pair $(A \otimes A', \varphi^{A \otimes A'})$ where, for any $X \in \mathbb{G}$, $\varphi_{X \otimes X}^{A \otimes A'} : D(X) \rightarrow D_{A \otimes A'}(X) \otimes D''(X)$ is given by the composition $(\psi_{X, A, A'}^{-1} \otimes v_{A, A'}^{-1} \otimes 1) \cdot (1 \otimes c_{A^*, X, A' \otimes A'^*} \otimes 1) \cdot (1 \otimes \varphi_X^{A'}) \cdot \varphi_X^A$. It is straightforward to check that $\varphi^{A \otimes A'}$ satisfies (5) as well as the fact that, if $[A, \varphi^A] = [B, \varphi^B]$ and $[A', \varphi^{A'}] = [B', \varphi^{B'}]$, then $[A \otimes A', \varphi^{A \otimes A'}] = [B \otimes B', \varphi^{B \otimes B'}]$.

The identity on an object (D, β) is the class $[I, \varphi^I]$ where, for any $X \in \mathbb{G}$, $\varphi_X^I = (\psi_{0, X}^{-1} \otimes l_{D(X)}^{-1}) \cdot l_{D(X)}^{-1} : D(X) \rightarrow {}^X I \otimes I \otimes D(X)$.

In this way, we have a category that is actually a groupoid. It is pointed by the trivial derivation, that is, the pair (D_0, β_0) where $D_0 : \mathbb{G} \rightarrow \mathbb{A}$ is the constant functor with value the unit object $I \in \mathbb{A}$ and, for any $X, Y \in \mathbb{G}$, $(\beta_0)_{X,Y} = (1 \otimes \psi_{0, X}^{-1}) \cdot l_I^{-1} : I \rightarrow I \otimes {}^X I$.

4 The categorical group of the \mathbb{G} -invariant objects of a \mathbb{G} -module

In this section we develop, in the higher categorical group level, analogous results to the well-known group theoretical facts about the cohomological character of the subgroup A^G of invariant elements of a G -module A .

Let \mathbb{G} be a categorical group and let \mathbb{A} be a \mathbb{G} -module.

We define the *zero-th cohomology categorical group* of \mathbb{G} with coefficients in the \mathbb{G} -module \mathbb{A} , $\mathcal{H}^0(\mathbb{G}, \mathbb{A})$, as the kernel of the inner derivation homomorphism $(T, \mu) : \mathbb{A} \rightarrow \text{Der}(\mathbb{G}, \mathbb{A})$ introduced in Section 3.

The categorical group $\mathcal{H}^0(\mathbb{G}, \mathbb{A})$ is then a braided categorical group (which is symmetric if \mathbb{A} is a symmetric \mathbb{G} -module) and, using the explicit description of the kernel recalled in Section 2, it is equivalent to the categorical group of \mathbb{G} -invariant objects $\mathbb{A}^{\mathbb{G}}$ constructed below.

A \mathbb{G} -invariant object of \mathbb{A} consists of a pair $(A, (\varphi_A^X)_{X \in \mathbb{G}})$, where $A \in \text{Obj}(\mathbb{A})$ and $\varphi_A^X : {}^X A \rightarrow A$, $X \in \mathbb{G}$, is a family of natural isomorphisms in \mathbb{A} such that, for any $X, Y \in \mathbb{G}$, the following equality holds:

$$\varphi_A^{X \otimes Y} = \varphi_A^X \cdot {}^X \varphi_A^Y \cdot \phi_{X, Y, A}.$$

An arrow from $(A, (\varphi_A^X)_{X \in \mathbb{G}})$ to $(B, (\varphi_B^X)_{X \in \mathbb{G}})$ consists of an arrow $u : A \rightarrow B$ in \mathbb{A} such that, for all $X \in \mathbb{G}$,

$$u \cdot \varphi_A^X = \varphi_B^X \cdot {}^X u.$$

In this way we have a category, $\mathbb{A}^{\mathbb{G}}$, where the composition is given by the composition in \mathbb{A} , which is clearly a groupoid.

Moreover, there is a functor

$$\otimes : \mathbb{A}^{\mathbb{G}} \times \mathbb{A}^{\mathbb{G}} \longrightarrow \mathbb{A}^{\mathbb{G}}$$

that is defined, on objects, by $(A, \varphi_A^X) \otimes (B, \varphi_B^X) = (A \otimes B, \varphi_{A \otimes B}^X)$, where $\varphi_{A \otimes B}^X : {}^X(A \otimes B) \rightarrow A \otimes B$ is the composition $(\varphi_A^X \otimes \varphi_B^X) \cdot \psi_{X, A, B}$ and, on arrows, by the tensor product of arrows in \mathbb{A} .

The above data define a categorical group

$$\mathbb{A}^{\mathbb{G}} = (\mathbb{A}^{\mathbb{G}}, \otimes, \bar{a}, \bar{l}, \bar{l}, \bar{r}),$$

where the unit object \bar{l} is the pair $(I, \psi_{0, X})$ and the associativity, the left unit and the right unit constraints $\bar{a}_{(A, \varphi_A^X), (B, \varphi_B^X), (C, \varphi_C^X)}$, $\bar{l}_{(A, \varphi_A^X)}$ and $\bar{r}_{(A, \varphi_A^X)}$ are given by the respective constraints a, l and r of \mathbb{A} . An inverse for each object $(A, \varphi_A^X) \in \mathbb{A}^{\mathbb{G}}$ is given by the triple $((A^*, \varphi_{A^*}^X), \gamma_A, \vartheta_A)$ where $\varphi_{A^*}^X = (\varphi_A^{X^*})^{-1} \cdot \omega_{X, A}$ and $(A^*, \gamma_A, \vartheta_A)$ is an inverse for $A \in \mathbb{A}$. In addition, $\mathbb{A}^{\mathbb{G}}$ is a braided categorical group with braiding given by the braiding c in \mathbb{A} .

Note that the projection functor $\mathbb{A}^{\mathbb{G}} \rightarrow \mathbb{A}$ is an embedding and a homomorphism of braided categorical groups.

Let us remark that the above construction also works for general categorical groups by using the notion of \mathbb{G} -categorical group, that is, a categorical group \mathbb{H} together with a coherent action from \mathbb{G} (or equivalently a homomorphism from \mathbb{G} to the categorical group $\mathcal{E}q(\mathbb{H})$ [2] of the monoidal equivalences of \mathbb{H}).

The examples below justify that $\mathcal{H}^0(\mathbb{G}, \mathbb{A}) \simeq \mathbb{A}^{\mathbb{G}}$ deserves to be called the zero-th cohomology categorical group of \mathbb{G} with coefficients in \mathbb{A} .

Examples 4.1

4.1.1. If G is a group, the discrete category it defines, denoted by $G[0]$, is a strict categorical group where the tensor product is given by the group operation. In the case where A is an abelian group, $A[0]$ is braided (and even symmetric) due to the commutativity of A .

It is easy to check that $\mathcal{E}q(A[0]) = \text{Aut}(A)[0]$ and a $G[0]$ -action on $A[0]$ is actually a G -action, in the usual sense, on A , that is, a G -module structure on A . In this case, $A[0]^{G[0]} = A^G[0]$ (where A^G is the subgroup of A of the G -invariant elements) since an object of $A[0]^{G[0]}$ consists exactly of an element $a \in A$ such that ${}^x a = a$ for any $x \in G$, and all arrows are identities.

If A is an abelian group, the category with only one object it defines, denoted by $A[1]$, is also a strict braided (even symmetric) categorical group where both the composition and the tensor product are given by the group operation. If A is a G -module, $A[1]$ is a $G[0]$ -module and we can consider the categorical group $A[1]^{G[0]}$. An object of this category consists of an element $a_x \in A$ for each $x \in G$ such that, for any $y \in G$, $a_{xy} = a_x + {}^x a_y$, that is, an object is a map $d : G \rightarrow A, x \mapsto a_x$, which is a derivation from G into A . Thus, $\text{Obj}(A[1]^{G[0]}) = \text{Der}(G, A)$, the abelian group of derivations from G into A . An arrow between two objects d and d' consists of an element $b \in A$ such that, for any $x \in G$, $b + d(x) = d'(x) + {}^x b$ (i.e., there is an arrow between two derivations if they differ in an inner derivation) and therefore $\pi_0(A[1]^{G[0]}) = \text{Der}(G, A)/\text{IDer}(G, A)$. On the other hand, the unit object is the trivial derivation $G \xrightarrow{0} A$ and an automorphism of this object is an element $b \in A$ such that, for any $x \in G$, $b = {}^x b$. Thus, $\pi_1(A[1]^{G[0]}) = A^G$.

Note that in $A[1]^{G[0]}$ the composition and the tensor product are given by group operation in A .

4.1.2. It is well known that strict categorical groups or, equivalently, groupoids in the category of groups, are the same as Whitehead crossed modules [23]. Recall that a crossed module of groups is a system $\mathcal{L} = (H, G, \varphi, \delta)$, where $\delta : H \rightarrow G$ is a group homomorphism and $\varphi : G \rightarrow \text{Aut}(H)$ is an action (so that H is a G -group) for which the following conditions are satisfied:

$$\delta(xh) = x\delta(h)x^{-1} \quad , \quad \delta(h)h' = hh'h^{-1} .$$

Given a crossed module \mathcal{L} , the corresponding strict categorical group $\mathbb{G}(\mathcal{L})$ can be described as follows: the objects are the elements of the group G ; an arrow $h : x \rightarrow y$ is an element $h \in H$ with $x = \delta(h)y$; the composition is multiplication in H ; the tensor product is given by

$$(x \xrightarrow{h} y) \otimes (x' \xrightarrow{h'} y') = (xx' \xrightarrow{hh'} yy') .$$

A crossed module \mathcal{L} together with a map $\{-, -\} : G \times G \rightarrow H$ satisfying certain equalities is called a reduced 2-crossed module of groups (see [9]).

Reduced 2-crossed modules $(\mathcal{L}, \{-, -\})$ (also called braided crossed modules in [3]) correspond, through the above equivalence, to strict braided categorical groups $\mathbb{G}(\mathcal{L})$ where the braiding $c = c_{x,y} : xy \rightarrow yx$ is given by $c_{x,y} = \{x, y\}$ (see [4, 11]).

Let us suppose now that $\mathcal{L} = (H \xrightarrow{\delta} G)$ is a crossed module of groups and $\mathcal{A} = (L \xrightarrow{\rho} M, \{-, -\})$ is a braided crossed module of groups. An action of \mathcal{L} on $(\mathcal{A}, \{-, -\})$ (see [14, 15, 24]) is a morphism of crossed modules $\mathcal{L} \rightarrow \text{Act}(\mathcal{A}, \{-, -\})$ where $\text{Act}(\mathcal{A}, \{-, -\})$, called the actor crossed module of $(\mathcal{A}, \{-, -\})$, is the crossed module consisting of the group morphism $\Delta : D(M, L) \rightarrow \text{Aut}(\mathcal{A}, \{-, -\})$ where: $D(M, L)$ is the Whitehead group of regular derivations, that is, the units of the monoid $\text{Der}(M, L)$; $\text{Aut}(\mathcal{A}, \{-, -\})$ is the group of automorphisms of $(\mathcal{A}, \{-, -\})$, that is, pairs of group automorphisms $\phi_0 \in \text{Aut}(M)$ and $\phi_1 \in \text{Aut}(L)$ such that $\phi_0 \cdot \rho = \rho \cdot \phi_1$, $\phi_1(m'l) = \phi_0(m)\phi_1(l)$ and $\phi_1(\{m, m'\}) = \{\phi_0(m), \phi_0(m')\}$); Δ is given by $\Delta(d) = (\sigma_d, \theta_d)$,

where $\sigma_d(m) = \rho(d(m))m$ and $\theta_d(l) = d(\rho(l))l$; and the action of $Aut(\mathcal{A}, \{-, -\})$ on $D(M, L)$ is given by $^{(\phi_0, \phi_1)}d = \phi_1 \cdot d \cdot \phi_0^{-1}$. When an action of \mathcal{L} on $(\mathcal{A}, \{-, -\})$

$$\begin{array}{ccc} H & \xrightarrow{\delta} & G \\ \epsilon \downarrow & & \downarrow \phi \\ D(M, L) & \xrightarrow{\Delta} & Aut(\mathcal{A}, \{-, -\}) \end{array}$$

is given, there are induced actions, via ϕ , of G on M , denoted by $x \mapsto {}^x m$, and of G on L , denoted as $x \mapsto {}^x l$. There is also an action of the semidirect product $H \rtimes G$ on the semidirect product $L \rtimes M$ given by $^{(h,x)}(l, m) = (\epsilon_h({}^x(\rho(l)m)) {}^x l, {}^x m)$.

The actor crossed module $Act(\mathcal{A}, \{-, -\})$ is precisely (see [2]) the crossed module associated to the categorical group $Aut(\mathbb{G}(\mathcal{A}))$, which is the categorical subgroup of $\mathcal{E}q(\mathbb{G}(\mathcal{A}))$ whose objects are the equivalences (T, μ) of the braided categorical group $\mathbb{G}(\mathcal{A})$ that are strict and where T is an isomorphism. Then, any action of a crossed module \mathcal{L} on a braided crossed module $(\mathcal{A}, \{-, -\})$ determines an action of $\mathbb{G}(\mathcal{L})$ on the braided categorical group $\mathbb{G}(\mathcal{A})$ which is given, on objects, by the group action of G on M and, on arrows, by the group action of $H \rtimes G$ on $L \rtimes M$. Then, $\mathbb{G}(\mathcal{A})$ is a $\mathbb{G}(\mathcal{L})$ -module and we can consider the categorical group $\mathbb{G}(\mathcal{A})^{\mathbb{G}(\mathcal{L})}$ whose objects are pairs $(m, (l_x)_{x \in G})$, where $m \in M$ and $l_x \in L$ satisfies ${}^x m = \rho(l_x)m$ and for any $x, y \in G$, $l_{xy} = {}^x l_y l_x$, (i.e., $l : G \rightarrow L$ is a derivation) and, for any $h \in H$ and $x \in G$, the equality $l_{\delta(h)x} = \epsilon_h({}^x m) l_x$ holds. An arrow from $(m, (l_x)_{x \in G})$ to $(m', (l'_x)_{x \in G})$ is an element $u \in L$ such that $m = \rho(u)m'$ and such that, for any $h \in H$ and $x \in G$, the equality $l_{\delta(h)x} u = \epsilon_h({}^x(\rho(u)m')) {}^x u l'_x$ holds. Thus, since the unit object is the pair $(1, (1_x)_{x \in G})$ with $1_x = 1 \in L$ for all $x \in G$, an automorphism of the unit object is an element $u \in L$ such that $\rho(u) = 1$ and such that $u = {}^x u$ for any $x \in G$, and therefore $\pi_1(\mathbb{G}(\mathcal{A})^{\mathbb{G}(\mathcal{L})}) = (Ker \rho)^G$.

In the particular case where the action of \mathcal{L} on $(\mathcal{A}, \{-, -\})$ is the trivial action, an object of $\mathbb{G}(\mathcal{A})^{\mathbb{G}(\mathcal{L})}$ is a pair (m, f) where $m \in M$ and $f : G \rightarrow Ker(\rho)$ is a group homomorphism, and there is an arrow from (m, f) to (m', f') if there exists an element $u \in L$ such that $m = \rho(u)m'$ and $f = f'$. Thus, in this case, $\pi_0(\mathbb{G}(\mathcal{A})^{\mathbb{G}(\mathcal{L})}) = Coker \rho \oplus$

$Hom(G, Ker \rho)$.

Note that in $\mathbb{G}(\mathcal{A})^{\mathbb{G}(\mathcal{L})}$ the composition is given by group operation in L , whereas the tensor product is given by

$$[(m, (l_x)) \xrightarrow{u} (n, (d_x))] \otimes [(m', (l'_x)) \xrightarrow{v} (n', (d'_x))] = [(mm', (l_x \ m'_x)) \xrightarrow{u^nv} (nn', (d_x \ n \ d'_x))]$$

4.1.3. If S is a commutative ring, there is a symmetric categorical group $Pic(S) = (\mathcal{P}ic(S), \otimes_S, a, S, l, r)$ (cf. [8, 28]) where $\mathcal{P}ic(S)$ is the category of invertible S -modules (i.e., finitely generated projective S -modules of constant rank 1), \otimes_S is the tensor product of S -modules, the unit object is the S -module S , the associativity and unit constraints are the usual ones for the tensor product of modules,

$$(P \otimes_S Q) \otimes_S T \xrightarrow{\alpha_{P,Q,T}} P \otimes_S (Q \otimes_S T), \quad P \otimes_S S \xrightarrow{\tau_P} P \xleftarrow{l_P} S \otimes_S P,$$

an inverse for any object P is given by the dual module $P^* = Hom_S(P, S)$ and the symmetry $c_{P,Q} : P \otimes_S Q \rightarrow Q \otimes_S P$ is the usual isomorphism. Note that $\pi_0(\mathcal{P}ic(S)) = Pic(S)$, the Picard group of S , and for any invertible S -module P , $Aut_{\mathcal{P}ic(S)}(P) \cong U(S)$ the group of units of S . In particular, $\pi_1(\mathcal{P}ic(S)) \cong U(S)$.

Let G be a group operating on a commutative ring S by ring automorphisms. Then, $\mathcal{P}ic(S)$ is a symmetric $G[0]$ -module with action $G[0] \times \mathcal{P}ic(S) \rightarrow \mathcal{P}ic(S)$, $(\sigma, P) \mapsto {}^\sigma P$, where ${}^\sigma P$ is the same abelian group as P with action from S by $s \cdot x = \sigma^{-1}(s)x$, $s \in S$, $x \in P$. Note that this action is strict in the sense that the natural isomorphisms $\phi_{\sigma,\tau,P} : {}^{\sigma\tau}P \rightarrow {}^\sigma({}^\tau P)$, $\phi_{0,P} : {}^1P \rightarrow P$ and $\psi_{\sigma,P,Q} : {}^\sigma(P \otimes_S Q) \rightarrow {}^\sigma P \otimes_S {}^\sigma Q$ are identities. Then we can consider the symmetric categorical group $\mathcal{P}ic(S)^{G[0]}$ and it is plain to see that $\pi_0(\mathcal{P}ic(S)^{G[0]}) = Pic(S)^G$ while $\pi_1(\mathcal{P}ic(S)^{G[0]}) = U(S)^G$.

The functoriality, with respect to the \mathbb{G} -module \mathbb{A} , of the categorical group of derivations $Der(\mathbb{G}, \mathbb{A})$ (see [14]) together with the equivalence $\mathcal{H}^0(\mathbb{G}, \mathbb{A}) \simeq \mathbb{A}^G$, assures that the construction of the categorical group of invariant objects in a \mathbb{G} -module gives a 2-functor

$$(-)^G : \mathbb{G} - \text{Mod} \longrightarrow BCG, \quad \mathbb{A} \mapsto \mathbb{A}^G$$

defined as follows. For any homomorphism $\mathbf{T} = (T, \mu) : \mathbb{A} \rightarrow \mathbb{B}$, $\mathbf{T}^{\mathbb{G}} = (T^{\mathbb{G}}, \mu^{\mathbb{G}}) : \mathbb{A}^{\mathbb{G}} \rightarrow \mathbb{B}^{\mathbb{G}}$ is given, on objects, by $T^{\mathbb{G}}(A, \varphi_A^X) = (TA, \varphi_{TA}^X)$, with $\varphi_{TA}^X : {}^X(TA) \rightarrow TA$ the composition $T(\varphi_A^X) \cdot \nu_{X,A}^{-1}$ and, on arrows $u : (A, \varphi_A^X) \rightarrow (B, \varphi_B^X)$, by $T^{\mathbb{G}}(u) = T(u)$; as for

$$\mu_{(A, \varphi_A^X), (B, \varphi_B^X)}^{\mathbb{G}} : T^{\mathbb{G}}((A, \varphi_A^X) \otimes (B, \varphi_B^X)) \rightarrow T^{\mathbb{G}}(A, \varphi_A^X) \otimes T^{\mathbb{G}}(B, \varphi_B^X)$$

it is given by $\mu_{A,B} : T(A \otimes B) \rightarrow T(A) \otimes T(B)$. If $\epsilon : \mathbf{T} = (T, \mu) \rightarrow \mathbf{T}' = (T', \mu')$ is a morphism between homomorphisms of \mathbb{G} -modules, then $\epsilon^{\mathbb{G}} : T^{\mathbb{G}} \rightarrow T'^{\mathbb{G}}$, given by $\epsilon_{(A, \varphi_A^X)}^{\mathbb{G}} = \epsilon_A$, defines a morphism from $\mathbf{T}^{\mathbb{G}}$ to $\mathbf{T}'^{\mathbb{G}}$. By restricting to \mathbb{G} -SMod we have of course a 2-functor with codomain \mathcal{SCG} .

The next proposition shows that the functor $(-)^{\mathbb{G}}$ is left 2-exact in the sense that it preserves kernels. More precisely:

Proposition 4.2 *Let \mathbb{G} be a categorical group and $\mathbf{T} = (T, \mu) : \mathbb{A} \rightarrow \mathbb{B}$ a homomorphism of \mathbb{G} -modules with kernel $(K(\mathbf{T}), \mathbf{j}, \epsilon)$. Then $K(\mathbf{T})$ is a \mathbb{G} -module and $(K(\mathbf{T})^{\mathbb{G}}, \mathbf{j}^{\mathbb{G}}, \epsilon^{\mathbb{G}})$ is the kernel of the induced homomorphism $\mathbf{T}^{\mathbb{G}} : \mathbb{A}^{\mathbb{G}} \rightarrow \mathbb{B}^{\mathbb{G}}$.*

Proof: We already observed in the preliminaries how $K(\mathbf{T})$ is a \mathbb{G} -module. Now we shall prove that $(K(\mathbf{T})^{\mathbb{G}}, \mathbf{j}^{\mathbb{G}}, \epsilon^{\mathbb{G}})$ satisfies the universal property of the kernel of $\mathbf{T}^{\mathbb{G}}$. Note that $\epsilon^{\mathbb{G}} : \mathbf{T}^{\mathbb{G}} \mathbf{j}^{\mathbb{G}} \rightarrow \mathbf{0}$ is the morphism whose component at the object $((A, TA \xrightarrow{u_A} I), \varphi_{(A, u_A)}^X : {}^X(A, u_A) \rightarrow (A, u_A))$ is the arrow in $\mathbb{B}^{\mathbb{G}}$ given by the arrow $u_A : TA \rightarrow I$ in \mathbb{B} . Now, given a homomorphism $\mathbf{F} = (F, \eta) : \mathbb{K} \rightarrow \mathbb{A}^{\mathbb{G}}$ and a morphism $\tau : \mathbf{T}^{\mathbb{G}} \mathbf{F} \rightarrow \mathbf{0}$, there exists a unique homomorphism $\mathbf{F}' = (F', \eta') : \mathbb{K} \rightarrow K(\mathbf{T})^{\mathbb{G}}$ such that $\mathbf{j}^{\mathbb{G}} \mathbf{F}' = \mathbf{F}$ and $\epsilon^{\mathbb{G}} \mathbf{F}' = \tau$. This homomorphism is defined as follows. If $F(K) = (\tilde{F}(K), \varphi_{\tilde{F}(K)}^X : {}^X \tilde{F}(K) \rightarrow \tilde{F}(K))$, then τ_K is an arrow in $\mathbb{B}^{\mathbb{G}}$ given by the arrow in \mathbb{B} , $\tau_K : T(\tilde{F}(K)) \rightarrow I$, and we let $F'(K) = ((\tilde{F}(K), \tau_K), \varphi_{\tilde{F}(K)}^X)$. For any arrow $h : K_1 \rightarrow K_2$, we let $F'(h) = F(h)$. As for $\eta'_{K_1, K_2} : F'(K_1 \otimes K_2) \rightarrow F'(K_1) \otimes F'(K_2)$, this is the arrow in \mathbb{A} given by η_{K_1, K_2} . It is straightforward now to check that all the required conditions are satisfied. |

Corollary 4.3 *Let \mathbb{G} be a categorical group and $\mathbf{T} : \mathbb{A} \rightarrow \mathbb{B}$ a homomorphism of \mathbb{G} -modules with kernel $(K(\mathbf{T}), \mathbf{j}, \epsilon)$. Then the triple*

$$(\mathbf{j}^{\mathbb{G}}, \epsilon^{\mathbb{G}}, \mathbf{T}^{\mathbb{G}})$$

is 2-exact and there is an induced exact sequence of groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1(K(\mathbf{T})^{\mathbb{G}}) & \xrightarrow{\pi_1(\mathbf{j}^{\mathbb{G}})} & \pi_1(\mathbb{A}^{\mathbb{G}}) & \xrightarrow{\pi_1(\mathbf{T}^{\mathbb{G}})} & \pi_1(\mathbb{B}^{\mathbb{G}}) & (6) \\
 & & & & & \searrow \delta & & \\
 & & \pi_0(K(\mathbf{T})^{\mathbb{G}}) & \xrightarrow{\pi_0(\mathbf{j}^{\mathbb{G}})} & \pi_0(\mathbb{A}^{\mathbb{G}}) & \xrightarrow{\pi_0(\mathbf{T}^{\mathbb{G}})} & \pi_0(\mathbb{B}^{\mathbb{G}}).
 \end{array}$$

Below we show diverse examples of the group exact sequence in the above corollary.

Examples 4.4

4.4.1. Let us consider

$$\phi = (\phi_1, \phi_0) : \mathcal{A} = (L \xrightarrow{\rho} M, \{-, -\}) \longrightarrow \mathcal{B} = (L'' \xrightarrow{\rho''} M'', \{-, -\}),$$

a surjective morphism of reduced 2-crossed modules of groups (i.e., a morphism ϕ with $\phi_0 : M \rightarrow M''$ and $\phi_1 : L \rightarrow L''$ epimorphisms). Let G be a group and suppose that the crossed module $\mathcal{L} = (0 \rightarrow G)$ acts on $(\mathcal{A}, \{-, -\})$ and $(\mathcal{B}, \{-, -\})$ in such a way that ϕ preserves the action. If $L' = Ker(\phi_1)$ and $M' = Ker(\phi_0)$, let $\mathcal{F} = (L' \xrightarrow{\rho'} M', \{-, -\})$ be the 2-reduced crossed module fiber of ϕ (where M' acts on L' by restriction of the action of M on L and $\{-, -\} : M' \times M' \rightarrow L'$ is also induced by restriction). Then \mathcal{L} also acts on $(\mathcal{F}, \{-, -\})$ and $\mathbb{G}(\mathcal{F})$ is equivalent to the kernel of the induced homomorphism of $(\mathbb{G}(\mathcal{L}) = G[0])$ -modules $\mathbb{G}(\mathcal{A}) \rightarrow \mathbb{G}(\mathcal{B})$ where the equivalence $K(\mathbb{G}(\phi)) \rightarrow \mathbb{G}(\mathcal{F})$ is given, on objects, by

$$\{(m, l'') \in M \times L'' / \phi_0(m) = \rho''(l'')\} \mapsto \rho(l)^{-1}m \in M' \text{ with } \phi_1(l) = l'',$$

and, on arrows, by

$$(m, l'') \xrightarrow{t} (n, b'') \mapsto l^{-1}tb \in L' \text{ with } \phi_1(l) = l'' \text{ and } \phi_1(b) = b''.$$

Thus, there is a 2-exact sequence of categorical groups

$$\mathbb{G}(\mathcal{F})^{G[0]} \longrightarrow \mathbb{G}(\mathcal{A})^{G[0]} \longrightarrow \mathbb{G}(\mathcal{B})^{G[0]}$$

and therefore (see Example 4.1.2) sequence (6) particularizes to the exact sequence of groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ker(\rho')^G & \longrightarrow & Ker(\rho)^G & \longrightarrow & Ker(\rho'')^G \\ & & & & & \searrow & \\ & & & & & & \pi_0(\mathbb{G}(\mathcal{B})^{G[0]}) \end{array}$$

$$\pi_0(\mathbb{G}(\mathcal{F})^{G[0]}) \longrightarrow \pi_0(\mathbb{G}(\mathcal{A})^{G[0]}) \longrightarrow \pi_0(\mathbb{G}(\mathcal{B})^{G[0]}).$$

4.4.2. When, in the above Example 4.4.1., the action of \mathcal{L} on $(\mathcal{A}, \{-, -\})$ and $(\mathcal{B}, \{-, -\})$ is trivial, sequence (6) specializes (see Example 4.1.2.) to the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ker\rho' & \longrightarrow & Ker\rho & \longrightarrow & Ker\rho'' \longrightarrow Coker\rho' \oplus Hom(G, Ker\rho') \\ & & & & & \searrow & \\ & & & & & & Coker\rho \oplus Hom(G, Ker\rho) \longrightarrow Coker\rho'' \oplus Hom(G, Ker\rho'') \end{array} \tag{7}$$

which is the sequence obtained by gluing the three last terms of the ker-coker exact sequence $0 \rightarrow Ker\rho' \rightarrow Ker\rho \rightarrow Ker\rho'' \rightarrow Coker\rho' \rightarrow Coker\rho \rightarrow Coker\rho'' \rightarrow 0$ with the exact sequence $0 \rightarrow Hom(G, Ker\rho') \rightarrow Hom(G, Ker\rho) \rightarrow Hom(G, Ker\rho'')$.

4.4.3. If we particularize in Example 4.4.1. to the case in which $\mathcal{A} = (A \rightarrow 0)$ and $\mathcal{B} = (A'' \rightarrow 0)$, where both A and A'' are G -modules and $\phi : A \twoheadrightarrow A''$ is an epimorphism of G -modules, then the fiber crossed module is $\mathcal{F} = (A' \rightarrow 0)$, where $A' = Ker(\phi)$, and sequence (6) gives the well-known exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & A'^G & \longrightarrow & A^G & \longrightarrow & A''^G \\ & & & & & \searrow & \\ & & & & & & \frac{Der(G, A')}{\overline{IDer}(G, A')} \longrightarrow \frac{Der(G, A)}{\overline{IDer}(G, A)} \longrightarrow \frac{Der(G, A'')}{\overline{IDer}(G, A'')} \end{array}$$

4.4.4. If we take, as in Example 4.4.3., \mathcal{A} and \mathcal{B} crossed modules associated to G -modules A and A'' , but $\phi : A \rightarrow A''$ now being any morphism of G -modules with kernel A' , the kernel $K(\phi)$ of the induced homomorphism of categorical groups $\phi : A[1] \rightarrow A''[1]$ is the strict symmetric categorical group associated to the crossed module defined by ϕ and the trivial action of A'' on A . Then the 2-exact sequence

$$K(\phi)^{G[0]} \longrightarrow A[1]^{G[0]} \longrightarrow A''[1]^{G[0]}$$

induces the exact sequence of groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & A'^G & \longrightarrow & A^G & \longrightarrow & A''^G \\ & & & & & \searrow & \\ & & & & & & H^1(E(\phi_*)) \longrightarrow \frac{Der(G,A)}{IDer(G,A)} \longrightarrow \frac{Der(G,A'')}{IDer(G,A'')} \end{array}$$

since it is plain to see that $\pi_1(K(\phi)^{G[0]}) = A'^G$, whereas $\pi_0(K(\phi)^{G[0]}) = H^1(E(\phi_*))$, the first cohomology group of the mapping cone $E(\phi_*)$ of the cochain transformation $\phi_* : Hom_G(B_\bullet, A) \rightarrow Hom_G(B_\bullet, A'')$, where B_\bullet is a free resolution of the trivial G -module \mathbb{Z} .

5 The categorical group $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$

The cohomological character of the quotient abelian group of the group of derivations, from a group G into a G -module A , by the subgroup of inner derivations is raised here to the level of categorical groups.

Let \mathbb{G} be a categorical group and let \mathbb{A} be a symmetric \mathbb{G} -module. In this case $Der(\mathbb{G}, \mathbb{A})$ is a symmetric categorical group.

We define the *first cohomology categorical group* of \mathbb{G} with coefficients in the symmetric \mathbb{G} -module \mathbb{A} , $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$, as the cokernel, in the sense of [19, 29], of the homomorphism of symmetric categorical groups $\mathbb{A} \rightarrow Der(\mathbb{G}, \mathbb{A})$, $A \mapsto (D_A, \beta_A)$, given by inner derivations. Then $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ is a symmetric categorical group that we are going to make explicit below.

The underlying groupoid of $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ is actually the quotient groupoid that we constructed in Section 3. Now, thanks to the symmetry condition on \mathbb{A} (cf. [29, Remark 2.2.1]), the tensor product of derivations (3)

induces a monoidal structure such that $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ becomes a symmetric categorical group. This tensor functor

$$\otimes : \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \times \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{A}),$$

is defined as follows. On objects it is given by the tensor product of derivations (3). As for arrows between derivations, $[A, \varphi^A] : (D, \beta) \rightarrow (F, \alpha)$ and $[A', \varphi^{A'}] : (D', \beta') \rightarrow (F', \alpha')$, we let $[A, \varphi^A] \otimes [A', \varphi^{A'}] = [A \otimes A', \varphi^{A \otimes A'}] : (D \otimes D', \beta \otimes \beta') \rightarrow (F \otimes F', \alpha \otimes \alpha')$ where, for any $X \in \mathbb{G}$, $\varphi_X^{A \otimes A'} = (\psi_{X, A, A'}^{-1} \otimes v_{A, A'}^{-1} \otimes 1) \cdot (1 \otimes c_{A^* \otimes F(X), X, A' \otimes A'^*} \otimes 1) \cdot (\varphi_X^A \otimes \varphi_X^{A'})$. It is straightforward to check that $\varphi^{A \otimes A'}$ satisfies (5) and also that, if $[A, \varphi^A] = [B, \varphi^B]$ and $[A', \varphi^{A'}] = [B', \varphi^{B'}]$, then $[A \otimes A', \varphi^{A \otimes A'}] = [B \otimes B', \varphi^{B \otimes B'}]$.

We then have a symmetric categorical group

$$\mathcal{H}^1(\mathbb{G}, \mathbb{A}) = (\mathcal{H}^1(\mathbb{G}, \mathbb{A}), \otimes, \bar{a}, \bar{I}, \bar{l}, \bar{r}),$$

where $\bar{a} : ((D, \beta) \otimes (D', \beta')) \otimes (D'', \beta'') \rightarrow (D, \beta) \otimes ((D', \beta') \otimes (D'', \beta''))$ is the arrow given by the class of the pair (I, \bar{a}^I) where, for any $X \in \mathbb{G}$, $\bar{a}_X^I = (\psi_{0, X}^{-1} \otimes l_{D(X) \otimes (D'(X) \otimes D''(X))}^{-1}) \cdot l_{D(X) \otimes (D'(X) \otimes D''(X))}^{-1} \cdot a_{D(X), D'(X), D''(X)}$; the unit object \bar{I} is the trivial derivation; the left unit constraint $\bar{l} = \bar{l}_{(D, \beta)} : (D_0, \beta_0) \otimes (D, \beta) \rightarrow (D, \beta)$ is the arrow given by the class of the pair (I, \bar{l}^I) where, for any $X \in \mathbb{G}$, $\bar{l}_X^I = \psi_{0, X}^{-1} \otimes l_{D(X)}^{-1} : I \otimes D(X) \rightarrow {}^X I \otimes I \otimes D(X)$; the right unit constraint $\bar{r} = \bar{r}_{(D, \beta)} : (D, \beta) \otimes (D_0, \beta_0) \rightarrow (D, \beta)$ is the arrow given by the class of the pair (I, \bar{r}^I) where, for any $X \in \mathbb{G}$, $\bar{r}_X^I = (\psi_{0, X}^{-1} \otimes l_{D(X)}^{-1}) \cdot l_{D(X)}^{-1} \cdot r_{D(X)} : D(X) \otimes I \rightarrow {}^X I \otimes I \otimes D(X)$.

An inverse for any object $(D, \beta) \in \mathcal{H}^1(\mathbb{G}, \mathbb{A})$ is given by (D^*, α) , where $D^*(X) = D(X)^*$ and $\alpha_{X, Y} : D(X \otimes Y)^* \rightarrow D(X)^* \otimes {}^X D(Y)^*$ is the arrow in \mathbb{A} determined by the equality

$$(1 \otimes w_{X, D(Y)}) \cdot \alpha_{X, Y} \cdot \beta_{X, Y}^* = v_{X, D(Y), D(X)} \cdot c_{D(Y), D(X)}^* \cdot$$

Note that $\gamma_{(D, \beta)} = [I, \gamma^I] : (D, \beta) \otimes (D^*, \alpha) \rightarrow (D_0, \beta_0)$ where, for any $X \in \mathbb{G}$, $\gamma_X^I = (\psi_{0, X}^{-1} \otimes l_I^{-1}) \cdot l_I^{-1} \cdot \gamma_{D(X)} : D(X) \otimes D(X)^* \rightarrow {}^X I \otimes I \otimes I$ and $\vartheta_{(D, \beta)} = [I, \vartheta^I] : (D^*, \alpha) \otimes (D, \beta) \rightarrow (D_0, \beta_0)$ where, for any $X \in \mathbb{G}$, $\vartheta_X^I = (\psi_{0, X}^{-1} \otimes l_I^{-1}) \cdot l_I^{-1} \cdot \vartheta_{D(X)} : D(X)^* \otimes D(X) \rightarrow {}^X I \otimes I \otimes I$.

The symmetry $\bar{c} = \bar{c}_{(D,\beta),(D',\beta')} : (D, \beta) \otimes (D', \beta') \rightarrow (D', \beta') \otimes (D, \beta)$ is the arrow given by the class of the pair (I, \bar{c}^I) where, for any $X \in \mathbb{G}$, $\bar{c}_X^I = (\psi_{0,X}^{-1} \otimes l_{D'(X) \otimes D(X)}^{-1}) \cdot l_{D'(X) \otimes D(X)}^{-1} \cdot c_{D(X), D'(X)} : D(X) \otimes D'(X) \rightarrow {}^X I \otimes I \otimes D'(X) \otimes D(X)$.

We should remark that the key for constructing the categorical group $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ lies in a more general construction (to be developed in a forthcoming paper) that allows the establishment of a notion of quotient categorical group associated to any crossed module of categorical groups. The latter structure is the natural generalization to categorical groups of the classical Whitehead definition for groups (cf. [2, 13]). In fact, when \mathbb{G} is a categorical group and \mathbb{A} is a symmetric \mathbb{G} -module, the above homomorphism given by inner derivations defines a crossed module of categorical groups.

The above construction determines a 2-functor ,

$$\mathcal{H}^1(\mathbb{G}, -) : \mathbb{G} - \text{SMod} \longrightarrow \text{SCG}, \quad \mathbb{A} \mapsto \mathcal{H}^1(\mathbb{G}, \mathbb{A})$$

defined as follows. For any homomorphism $\mathbb{T} = (T, \mu) : \mathbb{A} \rightarrow \mathbb{B}$, $\mathcal{H}^1(\mathbb{G}, \mathbb{T}) = (T_*, \mu_*) : \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{B})$ is given, on objects, by $T_*(D, \beta) = (TD, \beta_*)$, with $\beta_* = \beta_{*,X,Y} : TD(X \otimes Y) \rightarrow TD(X) \otimes {}^X TD(Y)$ the composition $(1 \otimes \nu_{X,D(Y)}) \cdot \mu_{D(X), X_{D(Y)}} \cdot T(\beta_{X,Y})$ and, on arrows $[A, \varphi^A] : (D, \beta) \rightarrow (D', \beta')$, by $T_*([A, \varphi^A]) = [TA, \varphi^{TA}]$ where, for any $X \in \mathbb{G}$, $\varphi_X^{TA} = (\nu_{X,A} \otimes \lambda_A \otimes 1) \cdot (\mu_{X,A,A^*} \otimes 1) \cdot \mu_{X_{A \otimes A^*, D'(X)}} \cdot T(\varphi_X^A)$; as for $\mu_{*(D,\beta),(D',\beta')} : T_*((D, \beta) \otimes (D', \beta')) \rightarrow T_*(D, \beta) \otimes T_*(D', \beta')$ it is given by the class of the pair (I, φ^I) with $\varphi_X^I : T(D \otimes D')(X) \rightarrow {}^X I \otimes I \otimes (TD \otimes TD')(X)$ the composition $(\psi_{0,X}^{-1} \otimes l_{TD(X) \otimes TD'(X)}^{-1}) \cdot l_{TD(X) \otimes TD'(X)}^{-1} \cdot \mu_{D(X), D'(X)}$. If $\epsilon : (T, \mu) \rightarrow (T', \mu')$ is a morphism between homomorphisms of symmetric \mathbb{G} -modules, then $\epsilon_* : T_* \rightarrow T'_*$ given, for any $(D, \beta) \in \mathcal{H}^1(\mathbb{G}, \mathbb{A})$, by the class of the pair (I, ϵ^I) where, for any $X \in \mathbb{G}$, $\epsilon_X^I = (\psi_{0,X}^{-1} \otimes l_{T'D(X)}^{-1}) \cdot l_{T'D(X)}^{-1} \cdot \epsilon_{D(X)} : TD(X) \rightarrow {}^X I \otimes I \otimes T'D(X)$, defines a morphism from (T_*, μ_*) to (T'_*, μ'_*) .

Examples 5.1

5.1.1. Let G be a group and A a G -module. Then $A[0]$ is a $G[0]$ -module and an object of $\mathcal{H}^1(G[0], A[0])$ consists of a map $d : G \rightarrow A$

such that, for any $x, y \in G$, $d(xy) = d(x) + {}^x d(y)$ (i.e., the objects are derivations from G into A). Moreover, there is an arrow between two derivations d and d' if, and only if, there exists an element $a \in A$ such that $d(x) - d'(x) = {}^x a - a$ and therefore $\pi_0(\mathcal{H}^1(G[0], A[0])) = \text{Der}(G, A)/\text{IDer}(G, A)$. On the other hand, since the unit object is the trivial map $0 : G \rightarrow A$, then $\pi_1(\mathcal{H}^1(G[0], A[0])) = A^G$.

If A is a G -module, then the categorical group $A[1]$ is a $G[0]$ -module and also we can consider the cohomology categorical group $\mathcal{H}^1(G[0], A[1])$. An object of this category consists of a map $\beta : G \times G \rightarrow A$, $(x, y) \mapsto \beta_{x,y}$, such that, for any $x, y, z \in G$, ${}^x \beta_{y,z} + \beta_{x,yz} = \beta_{xy,z} + \beta_{x,y}$, that is, an object is an Eilenberg-MacLane 2-cocycle of G with coefficients in A . This 2-cocycle is normalized (i.e., $\beta_{x,1} = 0 = \beta_{1,x}$) if the derivation is normalized. An arrow between two objects β and β' consists of an equivalence class of maps $d : G \rightarrow A$ such that, for any $x, y \in G$, $\beta'_{x,y} + d(xy) = d(x) + {}^x d(y) + \beta_{x,y}$ (i.e., 1-cochains showing that β and β' are cohomologous 2-cocycles) where $d \sim d'$ if, and only if, there exists an element $b \in A$ such that $d'(x) = {}^x b - b + d(x)$. Thus, $\pi_0(\mathcal{H}^1(G[0], A[1])) = H^2(G, A)$ and, on the other hand, since the unit object is the trivial 2-cocycle, then $\pi_1(\mathcal{H}^1(G[0], A[1])) = \text{Der}(G, A)/\text{IDer}(G, A)$.

5.1.2. Let us suppose that $\mathcal{L} = (H \xrightarrow{\delta} G)$ is a crossed module of groups, $\mathcal{A} = (L \xrightarrow{\rho} M, \{-, -\})$ is a braided crossed module of groups and there is an action of \mathcal{L} on $(\mathcal{A}, \{-, -\})$ (cf. Example 4.1.2.). Then, the associated categorical group $\mathbb{G}(\mathcal{L})$ acts on the braided categorical group $\mathbb{G}(\mathcal{A})$ and we can consider the pointed groupoid $\mathcal{H}^1(\mathcal{L}, \mathcal{A}) = \mathcal{H}^1(\mathbb{G}(\mathcal{L}), \mathbb{G}(\mathcal{A}))$ whose objects are (cf. [14, Example 2, §3]) triplets of maps $d : G \rightarrow M$, $f : H \rightarrow L$ and $l : G^2 \rightarrow L$, $(x, y) \mapsto l_{x,y}$, such that: i) $\rho(f(h))d(x) = d(\delta(h)x)$ for all $x \in G$ and $h \in H$; ii) f is a group morphism; iii) $d(xy) = \rho(l_{x,y})d(x) {}^x d(y)$ for all $x, y \in G$; iv) $l_{x,yz} {}^{d(x)}(l_{y,z}) = l_{xy,z} l_{x,y}$ for all $x, y, z \in G$. An arrow from (d, f, l) to (d', f', l') is an equivalence class of pairs (m, τ) where $m \in M$ and $\tau : G \rightarrow L$, $x \mapsto \tau_x$, is a map such that, if we write $d_m(x) = {}^x m m^{-1}$ for any $x \in G$, the following equalities hold: i) $d(x) = \rho(\tau_x) d_m(x) d'(x)$ for all $x \in G$; ii) $f(h) \tau_x = \tau_{\delta(h)x} \epsilon_h ({}^x m)^{d_m(x)} f'(h)$ for all $x \in G$ and $h \in H$; iii) $l_{x,y} \tau_x {}^{d_m(x)} d'(x) ({}^x \tau_y)^{d_m(x)} \{d'(x), {}^x d_m(y)\} = \tau_{xy} \{{}^x d_m(y), d_m(x)\} {}^{d_m(x)} ({}^x d_m(y)) l'_{x,y}$ for all $x, y \in G$. Two pairs (m, τ) and (m', τ') are in the same class of equivalence if there exists an element

$u \in L$ such that $m = \rho(u)m'$ and $\tau_x \cdot u \cdot {}^x m' (m'^{-1} u^{-1}) = \tau'_x$ for all $x \in G$.

In the particular case that $\mathcal{L} = (0 \xrightarrow{0} G)$ and the action of \mathcal{L} on \mathcal{A} is the trivial one, an object of $\mathcal{H}^1(\mathcal{L}, \mathcal{A})$ is precisely a Dedecker 2-cocycle $(d : G \rightarrow M, l : G^2 \rightarrow L)$ of G with coefficients in \mathcal{A} [10], which is normalized if the derivation is also. An arrow from (d, l) to (d', l') is the class of a pair (m, τ) where $m \in M$ and τ is precisely an equivalence [10] between the Dedecker 2-cocycles (d, l) and (d', l') (i.e., $d(x) = \rho(\tau_x)d'(x)$ and $l_{x,y}\tau_x \cdot d'(x)\tau_y = \tau_{xy}l'_{x,y}$ for all $x, y \in G$) and $[m, \tau] = [m', \tau']$ if there exists an element $u \in L$ such that $m = \rho(u)m'$ and $\tau = \tau'$. Thus, in this particular case, $\pi_0(\mathcal{H}^1(\mathcal{L}, \mathcal{A})) = \mathbb{H}^2(G, \mathcal{A})$ the 2nd non-abelian cohomology set (group in this case) of G with coefficients in the reduced 2-crossed module $(\mathcal{A}, \{-, -\})$ [4]. On the other hand, the unit object is the trivial Dedecker 2-cocycle $d(x) = 1$ and $l_{x,y} = 1$ for all $x, y \in G$, and an automorphism of the unit object is the class of a pair (m, τ) , where $m \in M$ and $\tau : G \rightarrow \text{Ker}(\rho)$ is a group homomorphism, with $[m, \tau] = [m', \tau']$ if $\tau = \tau'$ and there exists $u \in L$ such that $m = \rho(u)m'$. Therefore, $\pi_1(\mathcal{H}^1(\mathcal{L}, \mathcal{A})) = \text{Coker}(\rho) \oplus \text{Hom}(G, \text{Ker}(\rho))$.

Note that, if $\mathcal{A} = (L \xrightarrow{\rho} M, \{-, -\})$ is a stable crossed module [9] (i.e., the condition $\{m, m'\} \{m', m\} = 1$ holds for all $m, m' \in M$), then $\mathbb{G}(\mathcal{A})$ is a strict symmetric categorical group and therefore $\mathcal{H}^1(\mathcal{L}, \mathcal{A})$ is a symmetric categorical group. In this case, whenever $\mathcal{L} = (0 \xrightarrow{0} G)$, we have that $\pi_0(\mathcal{H}^1(\mathcal{L}, \mathcal{A}))$ coincides with Ulbrich's cohomology [28] at dimension 2 of G with coefficients in the Picard category $\mathbb{G}(\mathcal{A})$.

5.1.3. Let G be a group operating on a commutative ring S by ring automorphisms. Then, $\text{Pic}(S)$ is a symmetric $G[0]$ -module (see Example 4.1.3.) and we can consider the symmetric categorical group $\mathcal{H}^1(G[0], \text{Pic}(S))$. An object of this categorical group consists of a family of invertible S -modules $\{P_\sigma\}_{\sigma \in G}$ together with S -isomorphisms $\beta_{\sigma,\tau} : P_{\sigma\tau} \rightarrow P_\sigma \otimes {}^\sigma P_\tau$ such that, for any $\sigma, \tau, \gamma \in G$, $(1 \otimes {}^\sigma \beta_{\tau,\gamma}) \cdot \beta_{\sigma,\tau\gamma} = (\beta_{\sigma,\tau} \otimes 1) \cdot \beta_{\sigma\tau,\gamma}$. An arrow from $(P_\sigma, \beta_{\sigma,\tau})_{\sigma,\tau \in G}$ to $(Q_\sigma, \alpha_{\sigma,\tau})_{\sigma,\tau \in G}$ is an equivalence class of pairs consisting of an invertible S -bimodule T and a family of S -bimodule isomorphisms $\varphi_\sigma^T : P_\sigma \rightarrow {}^\sigma T \otimes T^* \otimes Q_\sigma$ such that, omitting canonical isomorphisms, $(1 \otimes \alpha_{\sigma,\tau}) \cdot \varphi_{\sigma\tau}^T = (\varphi_\sigma^T \otimes {}^\sigma \varphi_\tau^T) \cdot \beta_{\sigma\tau}$; two such pairs (T, φ_σ^T) and $(T', \varphi_\sigma^{T'})$ are equivalent if there exists an S -isomorphism $v : T \rightarrow T'$ satisfying $({}^\sigma v \otimes (v^*)^{-1} \otimes 1) \cdot \varphi_\sigma^T = \varphi_\sigma^{T'}$.

In the case where G is finite and S a Galois extension of $R = {}^G S$, it is straightforward to check that $\pi_i(\mathcal{H}^1(G[0], \mathcal{P}ic(S)))$, $i = 0, 1$, coincides with the cohomology group $H^i(G, S)$, $i = 2, 1$, defined by Hatyory in [16] and so, taking into account the results of [16, Section 5] (see also [14, Example 4, §3]), $\pi_0(\mathcal{H}^1(G[0], \mathcal{P}ic(S))) \cong Br(S/R)$, where the latter is the Brauer group of S/R -Azumaya algebras, whereas $\pi_1(\mathcal{H}^1(G[0], \mathcal{P}ic(S))) \cong Pic(R)$ the Picard group of R .

6 A six-term 2-exact sequence

Our aim in this section is to show a six-term exact sequence of symmetric categorical groups, connecting the cohomology categorical groups \mathcal{H}^0 and \mathcal{H}^1 , which is associated to any short exact sequence of symmetric \mathbb{G} -modules. This notion is detailed in the following definition (cf. [25, Definition 3.2.1]):

Definition 6.1 *Let \mathbb{G} be a categorical group and $\mathbb{A}' \xrightarrow{\mathbb{T}'} \mathbb{A} \xrightarrow{\mathbb{T}} \mathbb{A}''$ a sequence of symmetric \mathbb{G} -modules and homomorphisms of symmetric \mathbb{G} -modules. This sequence is called a short exact sequence of symmetric \mathbb{G} -modules if \mathbb{T} is essentially surjective and there is a morphism $\epsilon : \mathbb{T}\mathbb{T}' \rightarrow \mathbf{0}$ such that the induced homomorphism from \mathbb{A}' to $K(\mathbb{T})$ is an equivalence.*

Then we have:

Theorem 6.2 *Let $(\mathbb{A}' \xrightarrow{\mathbb{T}'} \mathbb{A} \xrightarrow{\mathbb{T}} \mathbb{A}'', \epsilon : \mathbb{T}\mathbb{T}' \rightarrow \mathbf{0})$ be a short exact sequence of symmetric \mathbb{G} -modules. Then there is a natural induced 2-exact sequence of symmetric categorical groups*

$$\mathbb{A}'^{\mathbb{G}} \xrightarrow{\mathbb{T}'^{\mathbb{G}}} \mathbb{A}^{\mathbb{G}} \xrightarrow{\mathbb{T}^{\mathbb{G}}} \mathbb{A}''^{\mathbb{G}} \xrightarrow{\Delta} \mathcal{H}^1(\mathbb{G}, \mathbb{A}') \xrightarrow{\mathbb{T}'_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \xrightarrow{\mathbb{T}_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}''). \tag{8}$$

Proof: We can suppose, without loss of generality, that $(\mathbb{A}', \mathbb{T}', \epsilon)$ is exactly the kernel of \mathbb{T} . The 2-functoriality of $(-)^{\mathbb{G}}$ and $\mathcal{H}^1(\mathbb{G}, -)$, already remarked on the previous sections, gives the definition of the homomorphisms $\mathbb{T}'^{\mathbb{G}}$, $\mathbb{T}^{\mathbb{G}}$, \mathbb{T}'_* and \mathbb{T}_* . Next we define the connecting homomorphism $\Delta : \mathbb{A}''^{\mathbb{G}} \rightarrow \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$. The underlying functor is defined

on objects $(A'', \varphi_{A''}^X) \in \mathbb{A}''^{\mathbb{G}}$ as follows. Since \mathbf{T} is essentially surjective, we can select an object $A \in \mathbb{A}$ and an isomorphism $\varpi_{A''} : T(A) \rightarrow A''$ and then, considering the inner derivation $(D_A, \beta_A) : \mathbb{G} \rightarrow \mathbb{A}$, we let $\Delta(A'', \varphi_{A''}^X) = (\bar{D}_A, \bar{\beta}_A)$ the derivation from \mathbb{G} to \mathbb{A}' given for any $X \in \mathbb{G}$, by $\bar{D}_A(X) = (D_A(X), u_{D_A(X)})$, with $u_{D_A(X)} = \gamma_{A''} \cdot (\varphi_{A''}^X \otimes 1) \cdot ({}^X \varpi_{A''} \otimes (\varpi_{A''}^*)^{-1}) \cdot (\nu_{X,A} \otimes \lambda_A) \cdot \mu_{X_{A,A^*}} : T(D_A(X)) \rightarrow I$, for any $f \in \mathbb{G}$, by $\bar{D}_A(f) = D_A(f) = {}^f A \otimes 1$ and where $(\bar{\beta}_A)_{X,Y} : \bar{D}_A(X \otimes Y) \rightarrow \bar{D}_A(X) \otimes {}^X \bar{D}_A(Y)$, $X, Y \in \mathbb{G}$, is given by the natural isomorphisms $(\beta_A)_{X,Y}$ determined by the equality (4). To check that $(\bar{D}_A, \bar{\beta}_A) \in \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ is straightforward. For another choice of an object $\tilde{A} \in \mathbb{A}$ and an isomorphism $\varpi'_{\tilde{A}} : T(\tilde{A}) \rightarrow A''$, the corresponding derivation $(\bar{D}_{\tilde{A}}, \bar{\beta}_{\tilde{A}}) \in \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ is isomorphic to $(\bar{D}_A, \bar{\beta}_A)$. To do so, note that if $(A, u_A) \in \mathbb{A}'$, then the inner derivation $(D_{(A, u_A)}, \beta_{(A, u_A)}) : \mathbb{G} \rightarrow \mathbb{A}'$ is given, for any $X \in \mathbb{G}$, by $D_{(A, u_A)}(X) = (D_A(X), r_I \cdot [(\psi_{0,X} \cdot {}^X u_A \cdot \nu_{X,A}) \otimes ((u_A^*)^{-1} \cdot \lambda_A)]) \cdot \mu_{X_{A,A^*}}$ and $(\beta_{(A, u_A)})_{X,Y}$. Then the isomorphism $(\bar{D}_A, \bar{\beta}_A) \rightarrow (\bar{D}_{\tilde{A}}, \bar{\beta}_{\tilde{A}})$ is given by the class of the pair $((A \otimes \tilde{A}^*, u_{A \otimes \tilde{A}^*}), \varphi^{A \otimes \tilde{A}^*})$, where $u_{A \otimes \tilde{A}^*} = \gamma_{A''} \cdot (\varpi_{A''} \otimes (\varpi'_{\tilde{A}})^{-1}) \cdot (1 \otimes \lambda_{\tilde{A}}) \cdot \mu_{A, \tilde{A}^*} : T(A \otimes \tilde{A}^*) \rightarrow I$ and $\varphi^{A \otimes \tilde{A}^*} : (\bar{D}_A, \bar{\beta}_A) \rightarrow (D_{(A \otimes \tilde{A}^*, u_{A \otimes \tilde{A}^*})}, \beta_{(A \otimes \tilde{A}^*, u_{A \otimes \tilde{A}^*})}) \otimes (\bar{D}_{\tilde{A}}, \bar{\beta}_{\tilde{A}})$ is the morphism of derivations determined, for any $X \in \mathbb{G}$, by the natural isomorphism $\varphi_X^{A \otimes \tilde{A}^*} : D_A(X) \rightarrow D_{A \otimes \tilde{A}^*}(X) \otimes D_{\tilde{A}}(X)$, given by composition only of canonical isomorphisms, namely,

$$\begin{aligned}
 \varphi_X^{A \otimes \tilde{A}^*} &= (1 \otimes v_{A, \tilde{A}^*}^{-1} \otimes 1) \cdot (1 \otimes t_{\tilde{A}} \otimes 1) \cdot (\psi_{X, A, \tilde{A}^*}^{-1} \otimes c_{A^*, \tilde{A}} \otimes 1) \cdot (1 \otimes c_{A^*, X_{\tilde{A}^*}} \otimes c_{X_{\tilde{A} \otimes \tilde{A}^*, \tilde{A}}}) \\
 &\quad (9) \\
 &\quad (1 \otimes \psi_{X, \tilde{A}^*, \tilde{A}} \otimes 1) \cdot (1 \otimes {}^X \vartheta_{\tilde{A}}^{-1} \otimes \vartheta_{\tilde{A}}^{-1}) \cdot (1 \otimes \psi_{0, X}^{-1} \otimes 1) \cdot (1 \otimes r_I^{-1}) \cdot r_{X_{A \otimes \tilde{A}^*}}^{-1}.
 \end{aligned}$$

If $v : (A'', \varphi_{A''}^X) \rightarrow (B'', \varphi_{B''}^X)$ is an arrow in $\mathbb{A}''^{\mathbb{G}}$, we select objects $A, B \in \mathbb{A}$ and isomorphisms $\varpi_{A''} : T(A) \rightarrow A''$, $\varpi_{B''} : T(B) \rightarrow B''$ and then $\Delta(v) : (\bar{D}_A, \bar{\beta}_A) \rightarrow (\bar{D}_B, \bar{\beta}_B)$ is the arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ given by the class of the pair $((A \otimes B^*, u_{A \otimes B^*}), \varphi^{A \otimes B^*})$, where $u_{A \otimes B^*} = \gamma_{B''} \cdot (v \otimes 1) \cdot (\varpi_{A''} \otimes (\varpi_{B''}^*)^{-1}) \cdot (1 \otimes \lambda_{B^*}) \cdot \mu_{A, B^*} : T(A \otimes B^*) \rightarrow I$ and $\varphi^{A \otimes B^*} : (\bar{D}_A, \bar{\beta}_A) \rightarrow (D_{(A \otimes B^*, u_{A \otimes B^*})}, \beta_{(A \otimes B^*, u_{A \otimes B^*})}) \otimes (\bar{D}_B, \bar{\beta}_B)$ is the morphism of derivations determined, for any $X \in \mathbb{G}$, by the same composition of canonical isomorphisms of (9).

To check that Δ is a functor is simply routine.

Moreover, Δ defines a homomorphism from $\mathbb{A}^{\mathbb{G}}$ to $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ where, for any objects $(A'', \varphi^{A''}), (B'', \varphi^{B''}) \in \mathbb{A}^{\mathbb{G}}$, the natural isomorphisms

$$\Delta((A'', \varphi^{A''}) \otimes (B'', \varphi^{B''})) \longrightarrow \Delta(A'', \varphi^{A''}) \otimes \Delta(B'', \varphi^{B''})$$

are given, once we have chosen isomorphisms $\varpi_{A''} : T(A) \rightarrow A'', \varpi_{B''} : T(B) \rightarrow B''$ and we have $\varpi_{A'' \otimes B''} = (\varpi_{A''} \otimes \varpi_{B''}) \cdot \mu_{A, B} : T(A \otimes B) \rightarrow A'' \otimes B''$, by the class of the pair $((I, \mu_0), \varphi^I)$ where, for any $X \in \mathbb{G}$, $\varphi_X^I : D_{A \otimes B}(X) \rightarrow D_A(X) \otimes D_B(X)$ is the following composition of isomorphisms $(\psi_{0, X}^{-1} \otimes l_{X_{A \otimes A^* \otimes X_{B \otimes B^*}}}) \cdot l_{X_{A \otimes A^* \otimes X_{B \otimes B^*}}}^{-1} \cdot (1 \otimes c_{X_{B \otimes B^*, A^*}}) \cdot (\psi_{X, A, B} \otimes v_{A, B})$. To see that all the required coherence conditions hold is straightforward.

Now we are ready to start the proof of the 2-exactness in each point of the sequence.

The 2-exactness at $\mathbb{A}^{\mathbb{G}}$, that is, that the triple $(\mathbb{T}'^{\mathbb{G}}, \epsilon^{\mathbb{G}}, \mathbb{T}^{\mathbb{G}})$ is 2-exact has already shown in Proposition 4.2.

To check the 2-exactness at $\mathbb{A}^{\mathbb{G}}$, let us observe first that there is a morphism $\tau : \Delta \mathbb{T}^{\mathbb{G}} \rightarrow \mathbf{0}$ given by the natural transformation, whose component at $(A, \varphi_A^X) \in \mathbb{A}^{\mathbb{G}}$ is the arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ from $\Delta T^{\mathbb{G}}(A, \varphi_A^X) = \Delta(TA, \varphi_{TA}^X) = (\bar{D}_A, \bar{\beta}_A)$ to (D_0, β_0) given by the equivalence class of the pair $((I, \mu_0), \varphi^I)$ where $\varphi^I : (\bar{D}_A, \bar{\beta}_A) \rightarrow (\bar{D}_{(I, \mu_0)}, \bar{\beta}_{(I, \mu_0)}) \otimes (D_0, \beta_0)$ is the morphism of derivations determined, for any $X \in \mathbb{G}$, by the natural isomorphism $\varphi_X^I = (\psi_{0, X}^{-1} \otimes r_I^{-1}) \cdot r_I^{-1} \cdot \gamma_A \cdot (\varphi_A^X \otimes 1) : {}^X A \otimes A^* \rightarrow {}^X I \otimes I \otimes I$.

Then, if $K(\Delta)$ is the kernel of Δ we must verify whether the comparison homomorphism $H : \mathbb{A}^{\mathbb{G}} \rightarrow K(\Delta)$, which is given for any $(A, \varphi_A^X) \in \mathbb{A}$ by $H(A, \varphi_A^X) = ((TA, T\varphi_A^X \cdot \nu_{X, A}^{-1}), \tau_{(A, \varphi_A^X)})$, is full and essentially surjective.

For any object $((A'', \varphi_{A''}^X), [(B, u_B), \varphi^{(B, u_B)}]) \in K(\Delta)$ where, for any $X \in \mathbb{G}$, $\varphi_X^{(B, u_B)} : D_A(X) \rightarrow D_B(X) \otimes I$ with $\varpi_{A''} : T(A) \rightarrow A''$, we consider the object $(A \otimes B^*, \varphi_{A \otimes B^*}^X) \in \mathbb{A}^{\mathbb{G}}$, where $\varphi_{A \otimes B^*}^X = r_{A \otimes B^*} \cdot (1 \otimes \psi_{0, X}) \cdot (1 \otimes {}^X \gamma_B) \cdot (1 \otimes \psi_{X, B, B^*}^{-1}) \cdot (1 \otimes c_{X_{B, B^*}} \otimes 1) \cdot (c_{X_{B \otimes B^*, A}} \otimes 1) \cdot (1 \otimes l_A \otimes 1) \cdot (\varphi_X^{(B, u_B)} \otimes 1) \cdot (1 \otimes \vartheta_A^{-1} \otimes 1) \cdot (r_{X_A}^{-1} \otimes 1) \cdot \psi_{X, A, B^*} : {}^X(A \otimes B^*) \rightarrow A \otimes B^*$. Then there is an isomorphism in $K(\Delta)$ from

$H(A \otimes B^*, \varphi_{A \otimes B^*}^X) = ((T(A \otimes B^*), T\varphi_{A \otimes B^*}^X \cdot \nu_{X, A \otimes B^*}^{-1}), \tau_{(A \otimes B^*, \varphi_{A \otimes B^*}^X)})$ to $((A'', \varphi_{A''}^X), [(B, u_B), \varphi^{(B, u_B)}])$ given by the following isomorphism in \mathbb{A}'' , $\varpi_{A''} \cdot r_{T(A)} \cdot (1 \otimes (u_B^*)^{-1}) \cdot (1 \otimes \lambda_B) \cdot \mu_{A, B^*} : T(A \otimes B^*) \rightarrow A''$ and therefore H is essentially surjective.

To see that H is full, let (A, φ_A^X) and (B, φ_B^X) be any two objects of \mathbb{A}^G and suppose that there is an arrow in $K(\Delta)$, $V : H(A, \varphi_A^X) \rightarrow H(B, \varphi_B^X)$. This means that there is an arrow $\chi : TA \rightarrow TB$ in \mathbb{A}'' such that

$$\chi \cdot T\varphi_A^X \cdot \nu_{X, A}^{-1} = T\varphi_B^X \cdot \nu_{X, B}^{-1} \cdot \chi \quad (10)$$

and $\tau_{(B, \varphi_B^X)} \cdot \Delta(V) = \tau_{(A, \varphi_A^X)}$. The latter is the equality of arrows in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$

$$[(I, \mu_0), \varphi^I] \cdot [(A \otimes B^*, u_{A \otimes B^*}), \varphi^{A \otimes B^*}] = [(I, \mu_0), \varphi^I]$$

and taking into account how is the composition in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$, this equality means that there is an arrow $\xi : A \otimes B^* \otimes I \rightarrow I$ in \mathbb{A} such that $\mu_0 \cdot T(\xi) = r_I \cdot ((\gamma_{T(B)} \cdot (\chi \otimes 1) \cdot (1 \otimes \lambda_{B^*}) \cdot \mu_{A, B^*}) \otimes \mu_0) \cdot \mu_{A \otimes B^*, I}$ and, for any $X \in \mathbb{G}$, $\varphi_X^I = ({}^X\xi \otimes (\xi^*)^{-1} \otimes 1) \cdot \varphi_X^{A \otimes B^* \otimes I}$. The arrow ξ determines the arrow in \mathbb{A} $\zeta = l_B \cdot (\xi \otimes 1) \cdot (1 \otimes l_B^{-1}) \cdot (1 \otimes \vartheta_B^{-1}) \cdot r_A^{-1} : A \rightarrow B$ and this one determines the arrow in \mathbb{A}^G , $U : (A, \varphi_A^X) \rightarrow (B, \varphi_B^X)$ since it is straightforward to see that for all $X \in \mathbb{G}$, ${}^X\zeta \cdot \varphi_A^X = \varphi_B^X \cdot \zeta$. Then, since $H(U) = T(\zeta)$, equality (10) ensures that $H(U) = V$ and so H is full.

To check the 2-exactness at $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$, we first observe that there is a morphism $\tau' : \mathbf{T}'_* \Delta \rightarrow \mathbf{0}$ given by the natural transformation, whose component at $(A'', \varphi_{A''}^X) \in \mathbb{A}''^G$, is the arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ from $T'_* \Delta(A'', \varphi_{A''}^X) = T'_*(\bar{D}_A, \bar{\beta}_A) = (T'\bar{D}_A, (\bar{\beta}_A)_*)$ to (D_0, β_0) , where $A \in \mathbb{A}$ and $\varpi_{A''} : T(A) \rightarrow A''$, given by the equivalence class of the pair (A, φ^A) with $\varphi_X^A = r_{D_A(X)}^{-1}$ for all $X \in \mathbb{G}$.

Then, if $\bar{K}(\mathbf{T}'_*)$ is the kernel of \mathbf{T}'_* we have to show that the comparison homomorphism $H : \mathbb{A}''^G \rightarrow \bar{K}(\mathbf{T}'_*)$ is full and essentially surjective. Recall that, for any $(A'', \varphi_{A''}^X) \in \mathbb{A}''^G$, $H(A'', \varphi_{A''}^X) = (\Delta(A'', \varphi_{A''}^X), \tau'_{(A'', \varphi_{A''}^X)}) = ((\bar{D}_A, \bar{\beta}_A), \tau'_{(A'', \varphi_{A''}^X)})$ with $A \in \mathbb{A}$ such that there is an isomorphism $\varpi_{A''} : T(A) \rightarrow A''$. Suppose $((D, \beta), u_{(D, \beta)}) \in \bar{K}(\mathbf{T}'_*)$, so that (D, β) belongs to $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ (i.e., for all $X \in \mathbb{G}$, $D(X) =$

(B_X, u_{B_X}) with $B_X \in \mathbb{A}$ and $u_{B_X} : T(B_X) \rightarrow I$ and $u_{(D, \beta)} : T_*(D, \beta) = (T^*D, \beta_*) \rightarrow (D_0, \beta_0)$ is an arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$, that is, $u_{(D, \beta)} = [A, \varphi^A]$ with $A \in \mathbb{A}$, and, for all $X \in \mathbb{G}$, $\varphi_X^A : B_X \rightarrow D_A(X) \otimes I$. Then, if we consider the object $(TA, \varphi_{TA}^X) \in \mathbb{A}''^{\mathbb{G}}$, where $\varphi_{TA}^X = l_{TA} \cdot (u_{B_X} \otimes 1) \cdot (T((\varphi_X^A)^{-1}) \otimes 1) \cdot \mu_{X_{A \otimes A^* \otimes I, A}} \cdot T(1 \otimes l_A^{-1}) \cdot T(1 \otimes \vartheta_A^{-1}) \cdot T(r_{X_A}^{-1}) \cdot \nu_{X, A}^{-1} : {}^X TA \rightarrow TA$, there is an isomorphism in $K(\mathbf{T}'_*)$ from $((\bar{D}_A, \bar{\beta}_A), \tau'_{(TA, \varphi_{TA}^X)})$ to $((D, \beta), u_{(D, \beta)})$ given by the isomorphism in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ from $(\bar{D}_A, \bar{\beta}_A)$ to (D, β) determined by the class of the pair $((I, \mu_0), \varphi^{(I, \mu_0)})$ where, for any $X \in \mathbb{G}$, $\varphi_X^{(I, \mu_0)} = (\psi_{0, X}^{-1} \otimes l_{B_X}^{-1}) \cdot l_{B_X}^{-1} \cdot (\varphi_X^A)^{-1} \cdot r_{X_{A \otimes A^*}}^{-1} : {}^X A \otimes A^* \rightarrow {}^X I \otimes I^* \otimes B_X$. Thus H is essentially surjective.

To see that H is full, let $(A'', \varphi_{A''}^X)$ and $(B'', \varphi_{B''}^X)$ be any two objects of $\mathbb{A}''^{\mathbb{G}}$ and suppose that there is an arrow in $K(\mathbf{T}'_*)$, $W : H(A'', \varphi_{A''}^X) = ((\bar{D}_A, \bar{\beta}_A), \tau'_{(A'', \varphi_{A''}^X)}) \rightarrow H(B'', \varphi_{B''}^X) = ((\bar{D}_B, \bar{\beta}_B), \tau'_{(B'', \varphi_{B''}^X)})$ with $T(A) \xrightarrow{\varpi_{A''}} A''$ and $T(B) \xrightarrow{\varpi_{B''}} B''$. This means that there is an arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$, $[(P, u_P), \varphi^{(P, u_P)}] : (\bar{D}_A, \bar{\beta}_A) \rightarrow (\bar{D}_B, \bar{\beta}_B)$ such that

$$\tau'_{(B'', \varphi_{B''}^X)} \cdot T_*([(P, u_P), \varphi^{(P, u_P)}]) = \tau'_{(A'', \varphi_{A''}^X)} \quad (11)$$

Then $P \in \mathbb{A}$, $u_P : T(P) \rightarrow I$ and, for any $X \in \mathbb{G}$, $\varphi_X^{(P, u_P)} : (D_A(X), u_{D_A(X)}) \rightarrow (D_{(P, u_P)}(X), \beta_{(P, u_P)}) \otimes (D_B(X), u_{D_B(X)})$, that is, $\varphi_X^{(P, u_P)} : D_A(X) \rightarrow D_P(X) \otimes D_B(X)$. Moreover, (11) is the equality of arrows in $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$, $[B, \varphi^B] \cdot [P, \varphi^P] = [A, \varphi^A]$, that is, $[B \otimes P, \varphi^{B \otimes P}] = [A, \varphi^A]$, which means that there is an arrow in \mathbb{A} , $\xi : P \otimes B \rightarrow A$ such that, for all $X \in \mathbb{G}$, $\varphi_X^A = ({}^X \xi \otimes (\xi^*)^{-1} \otimes 1) \cdot \varphi_X^{P \otimes B}$. Then, $T(\xi) : T(P \otimes B) \rightarrow T(A)$ determines the arrow in \mathbb{A}'' , $\zeta = l_{B''}^{-1} \cdot (1 \otimes \varpi_{B''}) \cdot (u_P \otimes 1) \cdot \mu_{P, B} \cdot T(\xi)^{-1} \cdot \varpi_{A''}^{-1} : A'' \rightarrow B''$ and this is an arrow in $\mathbb{A}''^{\mathbb{G}}$ from $(A'', \varphi_{A''}^X)$ to $(B'', \varphi_{B''}^X)$ since it is straightforward to see that $\zeta \cdot \varphi_{A''}^X = \varphi_{B''}^X \cdot {}^X \zeta$. Moreover, $H(\zeta) = [(A \otimes B^*, u_{A \otimes B^*}), \varphi^{A \otimes B^*}] = [(P, u_P), \varphi^{(P, u_P)}] = W$ thanks to the arrow $\tilde{\xi} = (\xi \otimes 1) \cdot (1 \otimes \gamma_B^{-1}) \cdot r_P^{-1} : P \rightarrow A \otimes B^*$ and so H is full.

Finally, we prove the 2-exactness at $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$. We observe first that there is a morphism $\tau'' : \mathbf{T}_* \mathbf{T}'_* \rightarrow \mathbf{0}$ given by the natural transformation

whose component at $(D', \beta') \in \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ is the arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}'')$ from $T_*T'_*(D', \beta') = (TT'D', \beta'_*)$ to (D_0, β_0) given by the equivalence class of the pair (I, φ^I) where I is the unit object of \mathbb{A}'' and $\varphi^I : (TT'D', \beta'_*) \rightarrow (D_I, \beta_I) \otimes (D_0, \beta_0)$ is the morphism of derivations determined, for any $X \in \mathbb{G}$, and when $D'(X) = (A_X, u_{A_X}) \in \mathbb{A}'$, by the natural isomorphism $\varphi_X^I = (\psi_{0,X}^{-1} \otimes r_I^{-1}) \cdot r_I^{-1} \cdot u_{A_X} : TA_X \rightarrow {}^X I \otimes I \otimes I$.

Let $H : \mathcal{H}^1(\mathbb{G}, \mathbb{A}') \rightarrow K(\mathbb{T}_*)$ be the comparison homomorphism to the kernel of \mathbb{T}_* , which is given for any $(D', \beta') \in \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ by $H(D', \beta') = ((T'D', \beta'_*), \tau''_{(T'D', \beta'_*)})$, and let $((D, \beta), u_{(D, \beta)})$ be any object of $K(\mathbb{T}_*)$. Thus, (D, β) belongs to $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ and $u_{(D, \beta)} : T_*(D, \beta) = (TD, \beta_*) \rightarrow (D_0, \beta_0)$ is an arrow in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}'')$, that is, $u_{(D, \beta)} = [A'', \varphi^{A''}]$ with $A'' \in \mathbb{A}''$ and, for all $X \in \mathbb{G}$, $\varphi_X^{A''} : TD(X) \rightarrow D_{A''}(X) \otimes I$. Then, since \mathbb{T} is essentially surjective, there is an object $A \in \mathbb{A}$ and an isomorphism $\varpi_{A''} : T(A) \rightarrow A''$ and we can consider the derivation $(D', \beta') \in \mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ where, for any $X \in \mathbb{G}$, $D'(X) = (A \otimes {}^X A^* \otimes D(X), u_{A \otimes {}^X A^* \otimes D(X)})$ with $u_{A \otimes {}^X A^* \otimes D(X)} = \gamma_{A''} \cdot (r_{A''} \otimes r_{A''}) \cdot (1 \otimes \psi_{0,X} \otimes 1) \cdot (1 \otimes {}^X \vartheta_{A''} \otimes 1) \cdot (1 \otimes \psi_{X, A'', A''}^{-1} \otimes 1) \cdot (\varpi_{A''} \otimes {}^X (\varpi_{A''}^{-1}) \otimes 1) \cdot (1 \otimes {}^X \lambda_A \otimes 1) \cdot (1 \otimes \nu_{X, A^*} \otimes 1) \cdot (\mu_{A, X, A^*} \otimes \varphi_X^{A''}) \cdot \mu_{A \otimes {}^X A^*, D(X)} : T(A \otimes {}^X A^* \otimes D(X)) \rightarrow I$ and, for any arrow f in \mathbb{G} , $D'(f) = 1 \otimes {}^f A^* \otimes D(f)$. As for β' , it is given, for any $X, Y \in \mathbb{G}$, by $\beta'_{X,Y} = (1 \otimes \psi_{X, A \otimes {}^Y A^*, D(Y)}^{-1}) \cdot (1 \otimes \psi_{X, A, {}^Y A^*}^{-1} \otimes 1) \cdot (1 \otimes c_{X, A \otimes {}^X ({}^Y A^*), D(X)} \otimes 1) \cdot (1 \otimes \psi_{X, A^*, A} \otimes 1) \cdot (1 \otimes {}^X \vartheta_A^{-1} \otimes 1) \cdot (1 \otimes \psi_{0,X}^{-1} \otimes 1) \cdot (r_A^{-1} \otimes 1) \cdot (1 \otimes \phi_{X, Y, A^*} \otimes \beta_{X,Y})$. To check that (D', β') is actually a derivation from \mathbb{G} into \mathbb{A}'' is straightforward and, moreover, $H(D', \beta')$ is isomorphic to $((D, \beta), u_{(D, \beta)})$ by means of the isomorphism in $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$ from $(T'D', \beta'_*)$ to (D, β) determined by the class of the pair (A^*, φ^{A^*}) where, for any $X \in \mathbb{G}$, $\varphi_X^{A^*} = (1 \otimes t_A \otimes 1) \cdot (c_{A, X, A^*} \otimes 1) : A \otimes {}^X A^* \otimes D(X) \rightarrow {}^X A^* \otimes (A^*)^* \otimes D(X)$. Thus, H is essentially surjective.

To see that H is full, let (D', β') and (F', α') be two objects of $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ with $D'(X) = (A_X, u_{A_X})$ and $F'(X) = (B_X, u_{B_X})$, $X \in \mathbb{G}$, and suppose that there is an arrow W in $K(\mathbb{T}_*)$ from $H(D', \beta') = ((T'D', \beta'_*), \tau''_{(D', \beta'_*)})$ to $H(F', \alpha') = ((T'F', \alpha'_*), \tau''_{(F', \alpha'_*)})$. This means that there is an arrow $[A, \varphi^A] : (T'D', \beta'_*) \rightarrow (T'F', \alpha'_*)$ in $\mathcal{H}^1(\mathbb{G}, \mathbb{A})$, with

$A \in \mathbb{A}$ and, for any $X \in \mathbb{G}$, $\varphi_X^A : A_X \rightarrow {}^X A \otimes A^* \otimes B_X$, such that $\tau_{(D', \beta')}'' = \tau_{(F', \alpha')}'' \cdot T_*([A, \varphi^A])$. This last is the equality of arrows in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}'')$, $[I, \varphi^I] = [I, \varphi^I] \cdot [TA \otimes I, \varphi^{TA \otimes I}]$ which means that there is an arrow $\xi : TA \otimes I \otimes I \rightarrow I$ in \mathbb{A}'' such that, for any $X \in \mathbb{G}$, $\varphi_X^I = ({}^X \xi \otimes (\xi^*)^{-1} \otimes 1) \cdot \varphi_X^{TA \otimes I \otimes I}$. Then, if we consider the pair $((A, u_A), \varphi^{(A, u_A)})$, where $u_A = \xi \cdot r_{TA \otimes I}^{-1} \cdot r_{TA}^{-1} : TA \rightarrow I$ and, for any $X \in \mathbb{G}$, $\varphi_X^{(A, u_A)} : (A_X, u_{A_X}) \rightarrow D_{(A, u_A)} \otimes (B_X, u_{B_X})$ is given by the arrow in \mathbb{A} , $\varphi_X^A : A_X \rightarrow {}^X A \otimes A^* \otimes B_X$, it determines an arrow V in $\mathcal{H}^1(\mathbb{G}, \mathbb{A}')$ from (D', β') to (F', α') , which clearly satisfies that $H(V) = W$ and so H is full. |

Note that the above exact sequence tends to be a special case of a categorical snake lemma which is at present the object of further investigations. Also we remark that, by taking homotopy groups π_i , $i = 0, 1$, in the above exact sequence of categorical groups there are, in addition to the group exact sequence (6), group exact sequences

$$\begin{aligned} \pi_i(\mathbb{A}^{\mathbb{G}} \xrightarrow{\mathbf{T}^{\mathbb{G}}} \mathbb{A}''^{\mathbb{G}} \xrightarrow{\Delta} \mathcal{H}^1(\mathbb{G}, \mathbb{A}')) \\ \pi_i(\mathbb{A}''^{\mathbb{G}} \xrightarrow{\Delta} \mathcal{H}^1(\mathbb{G}, \mathbb{A}') \xrightarrow{\mathbf{T}'_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A})) \\ \pi_i(\mathcal{H}^1(\mathbb{G}, \mathbb{A}') \xrightarrow{\mathbf{T}'_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \xrightarrow{\mathbf{T}_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}'')) \end{aligned}$$

that allow one to obtain the longer group exact sequence established below:

Corollary 6.3 *If $(\mathbb{A}' \xrightarrow{\mathbf{T}'} \mathbb{A} \xrightarrow{\mathbf{T}} \mathbb{A}'', \epsilon : \mathbf{T}\mathbf{T}' \rightarrow 0)$ is a short exact sequence of symmetric \mathbb{G} -modules, then sequence (8) induces the group exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\mathbb{A}'^{\mathbb{G}}) & \xrightarrow{\pi_1(\mathbf{T}'^{\mathbb{G}})} & \pi_1(\mathbb{A}^{\mathbb{G}}) & \xrightarrow{\pi_1(\mathbf{T}^{\mathbb{G}})} & \pi_1(\mathbb{A}''^{\mathbb{G}}) \\ & & & & \searrow \delta & & \\ & & \pi_0(\mathbb{A}'^{\mathbb{G}}) & \xleftarrow{\pi_0(\mathbf{T}'^{\mathbb{G}})} & \pi_0(\mathbb{A}^{\mathbb{G}}) & \xrightarrow{\pi_0(\mathbf{T}^{\mathbb{G}})} & \pi_0(\mathbb{A}''^{\mathbb{G}}) \\ & & & & \searrow \pi_0(\Delta) & & \\ & & \pi_0(\mathcal{H}^1(\mathbb{G}, \mathbb{A}')) & \xleftarrow{\pi_0(\mathbf{T}'_*)} & \pi_0(\mathcal{H}^1(\mathbb{G}, \mathbb{A})) & \xrightarrow{\pi_0(\mathbf{T}_*)} & \pi_0(\mathcal{H}^1(\mathbb{G}, \mathbb{A}'')). \end{array} \tag{12}$$

Remark 6.4 Suppose that $(\mathbb{A}' \xrightarrow{\mathbf{T}'} \mathbb{A} \xrightarrow{\mathbf{T}} \mathbb{A}'', \epsilon : \mathbf{T}\mathbf{T}' \rightarrow \mathbf{0})$ is a short exact sequence of \mathbb{G} -modules in the same sense of Definition 6.1. Then, in the same way as in Theorem 6.2, there is a 2-exact sequence of braided categorical groups and pointed groupoids (the last three terms)

$$\mathbb{A}'^{\mathbb{G}} \xrightarrow{\mathbf{T}'^{\mathbb{G}}} \mathbb{A}^{\mathbb{G}} \xrightarrow{\mathbf{T}^{\mathbb{G}}} \mathbb{A}''^{\mathbb{G}} \xrightarrow{\Delta} \mathcal{H}^1(\mathbb{G}, \mathbb{A}') \xrightarrow{\mathbf{T}'_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \xrightarrow{\mathbf{T}_*} \mathcal{H}^1(\mathbb{G}, \mathbb{A}'').$$

This sequence induces a 9-term exact sequence (12) which is now of groups and pointed sets (the last three terms).

Finally, we show how the above exact sequences particularize in diverse contexts.

Examples 6.5

6.5.1. Suppose, as in Example 4.4.1., that

$$\phi = (\phi_1, \phi_0) : \mathcal{A} = (L \xrightarrow{\rho} M, \{-, -\}) \longrightarrow \mathcal{B} = (L'' \xrightarrow{\rho''} M'', \{-, -\}),$$

is a surjective morphism of reduced 2-crossed modules of groups with fiber $\mathcal{F} = (L' \xrightarrow{\rho'} M', \{-, -\})$ and let G be a group such that the crossed module $\mathcal{L} = (0 \rightarrow G)$ acts on $(\mathcal{A}, \{-, -\})$ and $(\mathcal{B}, \{-, -\})$ in such a way that ϕ preserves the action. Then, since $\mathbb{G}(\mathcal{F})$ is equivalent to the kernel of the induced homomorphism of $(\mathbb{G}(\mathcal{L}) = G[0])$ -modules $\mathbb{G}(\mathcal{A}) \rightarrow \mathbb{G}(\mathcal{B})$ and $\mathbb{G}(\phi)$ is clearly essentially surjective, the sequence $\mathbb{G}(\mathcal{F}) \rightarrow \mathbb{G}(\mathcal{A}) \rightarrow \mathbb{G}(\mathcal{B})$ is a short exact sequence of $G[0]$ -modules and therefore there is, according to Remark 6.4, a 2-exact sequence of categorical groups and pointed groupoids

$$\begin{array}{ccccc} \mathbb{G}(\mathcal{F})^{G[0]} & \longrightarrow & \mathbb{G}(\mathcal{A})^{G[0]} & \longrightarrow & \mathbb{G}(\mathcal{B})^{G[0]} \\ & & & \searrow & \\ & & \mathcal{H}^1(G[0], \mathbb{G}(\mathcal{F})) & \longrightarrow & \mathcal{H}^1(G[0], \mathbb{G}(\mathcal{A})) \longrightarrow \mathcal{H}^1(G[0], \mathbb{G}(\mathcal{B})). \end{array}$$

When the action of \mathcal{L} is trivial and using the results obtained in Example 5.1.2, the above 2-exact sequence induces the following elongation of sequence (7):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Ker\rho' & \longrightarrow & Ker\rho & \longrightarrow & Ker\rho'' \\
 & & & & & \searrow & \\
 Coker\rho' \oplus Hom(G, Ker\rho') & \longrightarrow & Coker\rho \oplus Hom(G, Ker\rho) & \longrightarrow & Coker\rho'' \oplus Hom(G, Ker\rho'') & & \\
 & & & & \searrow \delta & & \\
 \mathbb{H}^2(G, \mathcal{F}) & \longrightarrow & \mathbb{H}^2(G, \mathcal{A}) & \longrightarrow & \mathbb{H}^2(G, \mathcal{B}). & &
 \end{array} \tag{13}$$

We should remark that, taking into account the equivalence between $K(\mathbb{G}(\phi))$ and $\mathbb{G}(\mathcal{F})$ shown in Example 4.4.1., the connecting homomorphism $\delta : Coker(\rho'') \oplus Hom(G, Ker(\rho'')) \rightarrow \mathbb{H}^2(G, \mathcal{F})$ is the one determined by the zero homomorphism $0 : Coker(\rho'') \rightarrow \mathbb{H}^2(G, \mathcal{F})$ and $\xi : Hom(G, Ker(\rho'')) \rightarrow \mathbb{H}^2(G, \mathcal{F})$ given, for any $\alpha : G \rightarrow Ker(\rho'')$, by $\xi(\alpha) = (d' : G \rightarrow M', l' : G^2 \rightarrow L')$, where $d'(x) = \rho(l_x)^{-1}$, with $\phi_1(l_x) = \alpha(x)$, and $l'_{x,y} = 1$ for all $x, y \in G$.

Note that in the case that \mathcal{A} and \mathcal{B} are stable crossed modules, the last three terms in sequence (13) are just Ulbrich's cohomology groups of G with coefficients in the Picard categories, with trivial G -module structure, $\mathbb{G}(\mathcal{F})$, $\mathbb{G}(\mathcal{A})$ and $\mathbb{G}(\mathcal{B})$ respectively.

6.5.2. If we particularize in Example 6.4.1. to the case in which $\mathcal{A} = (A \rightarrow 0)$ and $\mathcal{B} = (A'' \rightarrow 0)$, where both A and A'' are G -modules and $\phi : A \rightarrow A''$ is an epimorphism of G -modules, then the fiber crossed module is $\mathcal{F} = (A' \rightarrow 0)$, where $A' = Ker(\phi)$, and the sequence in that example specializes to the well-known exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A'^G & \longrightarrow & A^G & \longrightarrow & A''G \\
 & & & & & \searrow & \\
 \frac{Der(G, A')}{IDer(G, A')} & \longrightarrow & \frac{Der(G, A)}{IDer(G, A)} & \longrightarrow & \frac{Der(G, A'')}{IDer(G, A'')} & & \\
 & & & & \searrow & & \\
 H^2(G, A') & \longrightarrow & H^2(G, A) & \longrightarrow & H^2(G, A''). & &
 \end{array}$$

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References

- [1] Borceux, F. Handbook of categorical algebra 1,2. Cambridge University Press, **1994**.
- [2] Breen, L. Théorie de Schreier supérieure. Ann. Scient. Éc. Norm. Sup. **1992**, 4^e série, 25, 465-514.
- [3] Brown, R.; Gilbert, N.D. Algebraic models of 3-types and automorphisms structure for crossed modules. Proc. London Math. Soc. **1989**, 59(3), 51-73.
- [4] Bullejos, M.; Cegarra, A.M. A 3-dimensional non-abelian cohomology with applications to homotopy classification of continuous maps. Canadian Journal of Mathematics **1991**, 43 (2), 265-296.
- [5] Carrasco, P.; Cegarra, A.M. Schreier theory for central extensions of categorical groups. Communications in Algebra **1996**, 24 (13), 4059-4112.
- [6] Carrasco, P.; Garzón, A.R.; Miranda, J.G. Schreier theory for singular extensions of categorical groups and homotopy classification. Communications in Algebra **2000**, 28 (5), 2585-2613.
- [7] Carrasco, P.; Garzón, A.R. Obstruction theory for extensions of categorical groups. Applied Categorical Structures (to appear).
- [8] Cegarra, A.M.; Fernández, L. Cohomology of cofibred categorical groups. J. Pure Appl. Algebra **1999**, 143, 107-154.
- [9] Conduché, D. Modules croisés généralisés de longueur 2. J. Pure Appl. Algebra **1984**, 34, 155-178.
- [10] Dedecker, P. Cohomologie non-abélienne. Séminaire de l'Institut Mathématique Lille, **1965**.

- [11] Garzón, A.R.; Miranda, J.G. Homotopy theory for (braided) cat-groups. *Cahiers Top. et Geom. Diff. Cat.* **1997**, *XXXVIII-2*, 99-139.
- [12] Garzón, A.R.; Inassaridze, H. Semidirect products of categorical groups. *Obstruction theory. Homology, Homotopy and its applications* **2001**, *3* (6), 111-138.
- [13] Garzón, A.R.; Del Río, A. The Whitehead categorical group of derivations. *Georgian Math. J.* **2002**, *9* (4), 709-721.
- [14] Garzón, A.R.; Inassaridze, H.; Del Río, A. Derivations of categorical groups. Preprint 2002.
- [15] Gilbert, N.D. Derivations, automorphisms and crossed modules. *Communications in Algebra* **1990**, *18* (8), 2703-2734.
- [16] Hattory, A. On the groups $H^n(S, G)$ and the Brauer group of commutative rings. *Sci. Papers College Gen. Ed. Univ. Tokyo* **1978** *28*, 1-20.
- [17] Hilton P.J.; Stambach U. *A course in Homological Algebra.* Springer Verlag 1970.
- [18] Joyal, A.; Street, R. Braided tensor categories. *Advances in Math.* **1991**, *82*(1), 20-78.
- [19] Kasangian, S.; Vitale, E.M. Factorization systems for symmetric cat-groups. *Theory and Applications of Categories* **2000**, *7*(5), 47-70.
- [20] Kelly, G.M. On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. *J. of Algebra* **1964**, *1*, 397-402.
- [21] Mac Lane, S. Natural associativity and commutativity. *Rice University Studies* **1963**, *49*, 28-46.
- [22] Mac Lane, S. *Homology.* Springer, **1963**.
- [23] Mac Lane, S. *Categories for the working mathematician*, Second edition. Springer-Verlag, **1997**.

- [24] Norrie, K.J. Actions and automorphisms of crossed modules. *Bull. Soc. Math. France* **1990**, *118*, 129-146.
- [25] Rousseau, A. Extensions de Gr-catégories. Université Paris 13, Thèse de doctorat, **2000**.
- [26] Saavedra, N. Catégories tannakiennes. Springer Lecture Notes in Math. **1972**, *265*.
- [27] Sinh, H.X. Gr-catégories. Université Paris VII, Thèse de doctorat, **1975**.
- [28] Ulbrich, K.H. Group Cohomology for Picard Categories. *J. of Algebra* **1984**, *91*, 464-498.
- [29] Vitale, E.M. A Picard-Brauer exact sequence of categorical groups. *J. Pure Appl. Algebra* **2002**, *175*, 383-408.

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