CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

M. MEHDI EBRAHIMI
M. VOJDANI TABATABAEE
M. MAHMOUDI

Metrizability of σ-frames

Cahiers de topologie et géométrie différentielle catégoriques, tome 45, nº 2 (2004), p. 147-156
<http://www.numdam.org/item?id=CTGDC_2004__45_2_147_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLV-2 (2004)

METRIZABILITY OF σ-FRAMES

by M.Mehdi EBRAHIMI, M. Vojdani TABATABAEE and M. MAHMOUDI

RESUME. En imposant les changements nécessaires à la définition de diamètre métrique d'un cadre donnée par Banaschewski et Pultr, les auteurs donnent une définition de σ-cadre, et donc de la catégorie **MσFrm** des σ-cadres métriques et des applications uniformes de σ-cadres. Ils prouvent alors en particulier l'analogue du théorème de métrisabilité sans point de Banaschewski et Pultr. Finalement, ils caractérisent la catégorie **MσFrm** comme étant l'intersection des catégories **MLFrm** des cadres de Lindelöf métriques et **RσFrm** des σ-cadres réguliers.

1 Preliminaries

Here, we recall some definitions from [1], [2], [4], [6], and [12].

1.1 A *frame* (*σ-frame*) is a lattice L which has arbitrary (countable) joins and satisfies the arbitrary (countable) distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$, for all $x \in L$, and arbitrary (countable) subset $S \subseteq L$. A *frame* (*σ-frame*)*map* $h : L \to M$ is a lattice morphism preserving arbitrary (countable) joins. The resulting category is denoted by **Frm** (σ**Frm**).

1.2 A *cover* of a frame (σ-frame) is any (countable) subset $A \subseteq L$ such that $\bigvee A = 1$. The set of all covers of L is denoted by $CovL$. If A, B are covers we say that A *refines* B, and write $A \leq B$, if, for each $a \in A$, there exists $b \in B$ such that $a \leq b$. For a cover A of L and $x \in L$ we put $Ax = \bigvee\{a : a \in A, a \wedge x \neq 0\}$. Let L be a frame (σ-frame) and $\mathcal{U} \subseteq CovL$. We write $x \lhd_{\mathcal{U}} y$ or simply $x \lhd y$ if there exists $A \in \mathcal{U}$ such that $\{a : a \in A, a \wedge x \neq 0\} \subseteq \downarrow y$. We say that $\mathcal{U}$ is an *admissible system* if $x = \bigvee T$, for some (countable) subset T of $\{y : y \lhd x\}$, for each $x \in L$. Note that, if L is a frame, then we have $Ax \leq y$ if and only if $\{a : a \in A, a \wedge x \neq 0\} \subseteq \downarrow y$, for each $A \in CovL$.

1.3 In a bounded lattice, we say that a is rather below b, and write $a \prec b$, if there exists a *separating element* s of L with $a \wedge s = 0$ and $s \vee b = 1$. A frame (σ-frame) L is called *regular* if each of its elements is a (countable) join of elements rather below it. Notice that, in a frame, $a \prec b$ if and only if $a^* \vee b = 1$, where $a^* = \bigvee\{y : y \wedge a = 0\}$. An element x of a frame L is said to be a *Lindelöf element* if whenever $x = \bigvee S$ for some $S \subseteq L$ then, there exists a countable subset T of S such that $x = \bigvee T$. A frame L is said to be a *Lindelöf frame* if 1 is a Lindelöf element. A frame (σ-frame) L is said to be *paracompact* if each cover has a locally finite refinement. In [2] it is shown that any regular σ-frame is paracompact.

1.4 A *basis* of a frame (σ-frame) L is a subset $B \subseteq L$ such that each element of L is a (countable) join of elements of B. For the elements a, b of a frame (σ-frame) L, we say that *a meets b* if $a \wedge b \neq 0$. A subset X of a frame (σ-frame) L is said to be *locally finite* if, there is a cover W of L such that each $w \in W$ meets only finitely many $x \in X$, and it is said to be *discrete* if, each $w \in W$ meets at most one $x \in X$. The above cover W is said to *witness* the local finiteness respectively discreteness of X. A basis is called *σ-locally finite* (*σ-discrete*) if, it is a countable union of locally finite (discrete) sets, and it is called σ-admissible if, it can be written as union of an admissible system of covers.

1.5 Note Let L be a frame. If $\mathcal{U}$ is an admissible system of covers of L, then $\bigcup \mathcal{U}$ is a basis of the frame L [1]. We can show, in the same way, that this is also true for σ-frames. Given $a \in L$, let $a = \bigvee\{y_n : y_n \lhd a\}$. Take $B_n \in \mathcal{U}$ such that $B_n y_n \leq a$. Then, $a = \bigvee\{b : b \in B_n, b \wedge y_n \neq 0, n \in \mathbb{N}\}$.

1.6 Lemma *Any σ-frame with a countable basis is a frame.*

Proof: Let B be a countable basis of a σ-frame L and X be an arbitrary subset of L. It is easy to show that $\bigvee X$ exists and it is equal to $\bigvee\{b \in B, b \le x,\ \text{for some } x \in X\}$.

Also, for each $X \subseteq L$ and $a \in L$, $a \wedge \bigvee X = \bigvee\{a \wedge b : b \in B, b \le x,\ \text{for some } x \in X\} \le \bigvee\{a \wedge x : x \in X\} \le a \wedge \bigvee X$. Hence, L is a frame. □

1.7 Note In [1] it is shown that, if $X \subseteq L$ is locally finite and $x \prec a$, for each $x \in X$, then $\bigvee X \prec a$. By the above lemma, if L is a σ-frame with a countable basis and $X \subseteq L$ is locally finite, then $x \prec a$, for each $x \in X$, implies $\bigvee X \prec a$.

2 Some properties of bases of σ-frames

Here, we prove the following theorem, which is the counter part of the properties of bases of frames, proved in [1].

2.1 Theorem *The following are equivalent for a σ-frame L:*

(1) *L has a countable basis.*

(2) *L has a σ-discrete basis.*

(3) *L has a σ-locally finite basis.*

Moreover, these are equivalent to the following, if L is regular

(4) *L has a σ-admissible basis.*

Proof: (1 ⇒ 2) Let $B = \{b_n : n \in \mathbb{N}\}$ be a countable basis. It is enough to take $B_n = \{b_n\}$, for each $n \in \mathbb{N}$.

(2 ⇒ 3) This follows trivially from the definitions.

(3 ⇒ 1) Let $B = \bigcup_{n\in\mathbb{N}} B_n$, where each B_n is a locally finite set. We show that each B_n is a countable set. Let W be the witnessing cover of B_n. Thus each $w \in W$ meets only finitely many $x \in B_n$. Let

$$\{b : b \in B_n, b \wedge w \neq 0\} = \{b(w,1), ..., b(w,l_w)\},$$

for each $w \in W$. Let $b \in B_n$ and $b \neq 0$. There exists $w_i \in W$ such that $b \wedge w_i \neq 0$ since, if $b \wedge w = 0$, for each $w \in W$, then $b = b \wedge \bigvee W = 0$. This shows that $b \in \{b(w_i,1), ..., b(w_i,l_{w_i})\}$ and so there exists $j \leq l_{w_i}$ such that $b = b(w_i,j)$. Therefore, $B_n = \bigcup\{b(w,j) : w \in W, j \in \{1,...,l_w\}\}$. This gives that B_n is a countable set, since W is a countable set. Hence $B = \bigcup B_n$ is a countable basis of L.

(4 ⇒ 3) Let $\{B_n : n \in \mathbb{N}\}$ be an admissible system of covers of L. Regularity of L implies that L is paracompact, and so each B_n has a locally finite refinement, say A_n. Now, $\{A_n : n \in \mathbb{N}\}$ is an admissible system, and so $\bigcup A_n$ is a basis, by Note 1.5. Thus $A = \bigcup A_n$ is a σ-locally finite basis.

(2 ⇒ 4) Let $B = \bigcup B_n$ be a σ-discrete basis, and B_n be witnessed by W_n. Take $S_w = \{b : b \in B_n, w \wedge b \neq 0\}$. We have $S_w = \emptyset$ or $S_w = \{b^w\}$. For each $x \in L$, put $T = \{b : b \in B_k, b \prec x\}$ and $x_k = \bigvee T$. Since $T \subseteq B_k$, T is a discrete subset of L, and so $\bigvee T \prec x$, by Note 1.7. If $S_w = \{b^w\}$, then ${b^w}_k \prec b^w$ implies that there exists $t(w,k) \in L$ such that ${b^w}_k \wedge t(w,k) = 0$ and $b^w \vee t(w,k) = 1$. Take $\mathcal{U} = \{A_{nk} : n,k \in \mathbb{N}\}$, where

$$A_{nk} = \{w \wedge s : w \in W_n, s \in \{b^w, t(w,k)\} \text{ if } S_w = \{b^w\}, \text{ and } s = 1 \text{ if } S_w = \emptyset\}.$$

It is easy to show that $\mathcal{U} \subseteq CovL$. We claim that $A_{nk}b_k \leq b$, for any $b \in B_n$. Let $x \in A_{nk}$, and $x \wedge b_k \neq 0$. We show that $x = w \wedge b$, for some $w \in W_n$. We have, $x \wedge b_k \neq 0$ implies $w \wedge b_k \neq 0$ and so $w \wedge b \neq 0$. Thus $S_w \neq \emptyset$ and also $s \neq t(w,k)$, since if $x = w \wedge t(w,k)$ then $x \wedge b_k \neq 0$ implies $w \wedge t(w,k) \wedge b_k \neq 0$ and so $t(w,k) \wedge b_k \neq 0$. This contradiction shows that $x = w \wedge b^w$. Now, we have $x \wedge b_k \neq 0$ implies $w \wedge b \neq 0$ which gives $b = b^w$. Therefore $x = w \wedge b \leq b$. Hence $\{x : x \in A_{nk}, x \wedge b_k \neq 0\} \subseteq \downarrow b$ and so $b_k \lhd_{\mathcal{U}} b$. Also, by regularity of L

we have that

$$b = \bigvee\{c \in B : c \prec b\} = \bigvee\{b_k : k \in N\}.$$

Thus, for each $x \in L$, $x = \bigvee\{b \in B : b \leq x\} = \bigvee\{\bigvee\{b_k : k \in N\} : b \leq x\}$, where $b_k \lhd b \leq x$, and hence $x = \bigvee\{b_k : b_k \lhd x\}$. Therefore $\mathcal{U}$ is an admissible system, and thus $\bigcup \mathcal{U}$ is a σ-admissible basis. □

Note that any countable basis B of a regular σ-frame L is in fact σ-admissible. Since, by the above theorem, L has a σ-admissible basis $A = \bigcup A_n$. Now, putting $B_1 = B, B_{n+1} = \{b : b \in B, b \leq a$ for some $a \in A_n\}$, one can show that $B = \bigcup B_n$.

3 Metrization Theorems for σ-frames

In this section, interpreting the definition of a metric diameter on a frame given in [1], we prove the counterparts of the metrization theorems for σ-frames.

3.1 Definition A *metric diameter* on a σ-frame L is a monotone zero-preserving map $d : L \to \overline{R}_+$ such that

(1) for all $a, b, d(a \vee b) \leq d(a) + d(b)$, whenever $a \wedge b \neq 0$,

(2) for each $\varepsilon > 0$, there is a countable subset S of ${D_\varepsilon}^L = \{a : d(a) < \varepsilon\}$ such that $\bigvee S = 1$,

(3) for all $a \in L$, there is a countable subset T of $\{y : y \lhd a\}$ such that $\bigvee T = a$, where $y \lhd a$ means that, there exists $\varepsilon > 0$ such that $\{b : b \in D_\varepsilon, b \wedge y \neq 0\} \subseteq \downarrow a$.

(4) for all $a \in L$, and $\varepsilon > 0, d(a) = \sup\{d(x \vee y) : x, y \in D_\varepsilon \cap \downarrow a\}$, whenever $d(a) \geq \varepsilon$.

A σ-frame that admits a metric diameter is said to be *metrizable.* Also, a σ-frame map $f : L \to M$ between metric σ-frames is said to be *uniform* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that ${D_\delta}^M \leq f[{D_\varepsilon}^L]$.

Metric frames (σ-frames) together with uniform frame (σ-frame) maps form a category, denoted by **MFrm** (**MσFrm**). Metric Lindelöf frames together with uniform frame maps form a category, denoted by **MLFrm**, which is a full subcategory of **MFrm**. See [7], [9], [10], [11] for details.

Having Theorem 2.1, we prove the remaining parts of the following.

3.2 Theorem *For a regular σ-frame, the following statements are equivalent.*

(1) *L is metrizable.*

(2) *L has a σ-locally finite basis. (Nagata-Smirnov)*

(3) *L has a σ-discrete basis. (Bing)*

(4) *L has a σ-admissible basis.*

(5) *L has a countable basis.*

(6) *L has a countable admissible system of covers. (Moore)*

Proof: ($5 \Rightarrow 1$) Let L be a regular σ-frame with a countable basis B. By Lemma 1.6, L is a frame and the regularity of L as a σ-frame implies the regularity of L as a frame. Thus, by Urysohn metrization theorem for frames [1], L is a metrizable frame. Let $d : L \to \overline{R}_+$ be a metric diameter on the frame L.

To show that d is a metric diameter on the σ-frame L, it is enough to check the conditions (2) and (3) of the definition. For each $\varepsilon > 0$, the set $\{b \in B : b \leq x, \text{ for some } x \in D_\varepsilon\}$ is a countable subcover of D_ε. Also, for $x \in L$, $x = \bigvee\{b \in B : b \lhd x\}$. Hence, L is a metric σ-frame.

($1 \Rightarrow 6$) Let L be a metric σ-frame with metric diameter d. For each $n \in \mathbb{N}$, let A_n be a countable subcover of $D_{1/n}$. Take $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$. We show that $\mathcal{U}$ is an admissible system. It is enough to show that $x \lhd_d y$ implies $x \lhd_{\mathcal{U}} y$. Let $x \lhd_d y$. Take $\varepsilon > 0$ such that $\{a : a \in D_\varepsilon, a \wedge x \neq 0\} \subseteq \downarrow y$. We choose $n > 1/\varepsilon$. Then, $\{a : a \in A_n, a \wedge x \neq 0\} \subseteq \downarrow y$, and so $x \lhd_{\mathcal{U}} y$, since $A_n \in \mathcal{U}$. Hence, $\mathcal{U}$ is an admissible system of covers.

($6 \Rightarrow 5$) Let L be a σ-frame and $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$ be a countable admissible system of covers of L. Then, by Note 1.5, $B = \bigcup A_n$ ia a countable basis. □

Compare the following with [1, 4.3].

3.3 Proposition *A regular Lindelöf frame is metrizable if and only if it has a countable basis.*

Proof: By Urysohn metrization theorem [1], a regular frame with countable basis is metrizable. Conversely, let L be a metric Lindelöf frame with a metric diameter d.By Lindelöfness of L, there exists a countable cover A_n such that $A_n \subseteq D_{1/n}$, for each $n \in \mathbb{N}$, since $\bigvee D_{1/n} = 1$. Take $B = \bigcup A_n$. We claim that B is a countable basis of L. Given $x \in L$, we have $x = \bigvee\{y : y \lhd x\}$. Consider $C = \{y : y \lhd x\}$. For each $y \in C$, there exists $\varepsilon_y > 0$ such that $\{a : a \in D_{\varepsilon_y}, a \wedge y \neq 0\} \subseteq \downarrow x$. We choose $n_y > 1/\varepsilon_y$. It is easy to show that

$$x = \bigvee\{a \in A_{n_y} : a \wedge y \neq 0, y \in C\}.$$

Hence, B is a countable basis of L. □

4 Characterization of metric σ-frames

In this section we characterize the category **MσFrm**, of metric σ-frames, as the intersection of the categories **MLFrm**, of metric Lindelöf frames, and **RσFrm**, of regular σ-frames.

4.1 Lemma *Any metric σ-frame is a metric Lindelöf frame.*

Proof: Let L be a metric σ-frame with metric diameter d. By Theorem 3.2, L has a countable basis, say B, and so it is a frame, by Lemma 1.6. Clearly L is a metric frame with metric diameter d. It is enough to show that L is a Lindelöf frame.

Let $\bigvee S = 1$, for some $S \subseteq L$. We have $1 = \bigvee S = \bigvee\{b \in B : b \leq s_b,$ for some $s_b \in S\}$. Thus, $1 = \bigvee\{s_b : b \in B\}$. Hence L is a Lindelöf frame. □

4.2 Note Let L be a σ-frame with a countable basis B. Then, any σ-frame map $f : L \to M$ preserves arbitrary joins. Given $S \subseteq L$, then

$$f(\bigvee S) \leq f(\bigvee\{s : s \leq b, \text{ for some } b \in B\}) \leq \bigvee f(S) \leq f(\bigvee S).$$

4.3 Proposition *The category* **MσFrm** *is a full subcategory of* **MLFrm**.

Proof: By Lemma 4.1, it is enough to show that any uniform σ-frame map between metric σ-frames is a uniform frame map. Let $f : L \to M$ be a uniform σ-frame map. By Theorem 3.2, L has a countable basis and so, by the above note, f is a frame map. Uniformity of f as a σ-frame map implies the uniformity of f as a frame map. Thus, **MσFrm** $\subseteq$ **MLFrm**. Also, any uniform frame map is a uniform σ-frame map. Thus, **MσFrm** is a full subcategory of **MLFrm**. □

4.4 Lemma *Let L be a Lindelöf frame with countable regularity property (each element of L is a countable join of elements rather below it). Then, each element of L is Lindelöf.*

Proof: Let $x = \bigvee S$, for some subset S of L. By countable regularity of L, we have $x = \bigvee\{y_n : y_n \prec x\}$. For each $n \in \mathbb{N}$, we have $1 = y_n^* \vee x = y_n^* \vee \bigvee S = \bigvee\{y_n^* \vee s : s \in S\}$. Lindelöfness of L implies that there exists a countable subset $T_n \subseteq S$ such that $\bigvee\{y_n^* \vee s : s \in T_n\} = 1$ and so $y_n \prec \bigvee T_n$, for each $n \in \mathbb{N}$. Take $T = \bigcup T_n$. Then, $x = \bigvee\{y_n : y_n \prec x\} \leq \bigvee T \leq x$. Therefore, $x = \bigvee T$ and T is a countable subset of S. □

4.5 Lemma **MLFrm** $\cap$ **RσFrm** $\subseteq$ **MσFrm**.

Proof: Let L be a metric Lindelöf frame, with a metric diameter d, as well as a regular σ-frame. We show that d is a metric diameter on the σ-frame L. Lindelöfness of L as a frame gives a countable cover A, for each D_ε. Given $x \in L$, we have $x = \bigvee\{y \in L : y \lhd x\}$.

By Lemma 4.4, x is a Lindelöf element and so there exists a countable subset T of $\{y : y \lhd x\}$ such that $x = \bigvee T$. Thus L is a metric σ-frame. Therefore $Ob(\mathbf{MLFrm}) \cap Ob(\mathbf{R\sigma Frm}) \subseteq Ob(\mathbf{M\sigma Frm}.)$ Also, clearly any uniform frame map is a uniform σ-frame map. Thus $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} \subseteq \mathbf{M\sigma Frm}$. □

4.6 Note Any metric frame (σ-frame) is a regular frame (σ-frame). To see this, it is enough to show that $x \lhd y$ implies $x \prec y$. Let $x \lhd y$. Take $\varepsilon > 0$ such that $\{a : a \in D_\varepsilon, a \wedge x \neq 0\} \subseteq \downarrow y$. Then, there exists a (countable) subset $S \subseteq D_\varepsilon$ such that $\bigvee S = 1$. Take $t = \bigvee\{s : s \in S, s \wedge x = 0\}$. It is easy to show that $x \wedge t = 0$ and $y \vee t = 1$, and so $x \prec y$.

4.7 Theorem *The category* $\mathbf{M\sigma Frm}$ *is exactly the intersection of the categories* $\mathbf{MLFrm}$ *and* $\mathbf{R\sigma Frm}$.

Proof: By Lemma 4.5, we have $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} \subseteq \mathbf{M\sigma Frm}$, and by Proposition 4.3, $\mathbf{M\sigma Frm}$ is a full subcategory of $\mathbf{MLFrm}$. Also, any metric σ-frame is a regular σ-frame, by the above note. Therefore the category $\mathbf{M\sigma Frm}$ is a subcategory of the category $\mathbf{R\sigma Frm}$. Hence $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} = \mathbf{M\sigma Frm}$. □

Acknowledgements We would like to thank Professor Banaschewski for the discussion we had with him during his visit to Iran and his comments.

Our thanks also go to the referee for his/her suggestions making the arrangement and the size of the paper much more reasonable.

The financial support from Shahid Beheshti University of Iran is gratefully acknowledged.

References

1 B. Banaschewski and A. Pultr, *A new look at pointfree metrization theorems*, Comment. Math. Univ. Carolinae 39, 1 (1998) 167-175.

2 B. Banaschewski and C. Gilmour, *Stone-Cech compactification and dimension theory for regular σ-frames*, J. London Math. Soc. 2, 39 (1989) 1-8.

3 M. Mehdi Ebrahimi and M. Vojdani Tabatabaee, *Complete metric σ-frames*, Preprint.

4 M. Mehdi Ebrahimi and M. Mahmoudi, *"Frame"*, Tech. Rep., Shahid Beheshti Univ., 1995.

5 R. Engelking, *"General Topology"*, Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag, Berlin 1989.

6 P.T. Johnstone, *"Stone Spaces"*, Cambridge University Press, Cambridge, 1982.

7 J.R. Isbell, *Atomless parts of spaces,* Math. Scand. 31 (1972), 5-32.

8 J.R. Isbell, *Graduation and dimension in locales, in: Aspects of Topology,* London M5 Lecture Notes, 93 (1985), 195-210.

9 A. Pultr, *Remarks on metrizable locales,* Suppl. Rend. Circ. Mat. Palermo, 6 (1984).

10 A Pultr, *Categories of diametric frames,* Math. Proc. Camb. Phil. Soc. (1989), 105,285.

11 A. Pultr, *Pointless uniformities II (Dia)metrization,* Comment. Math. Univ. Carolinae, 25 (1984), 105-120.

12 S. Vickers, *"Topology Via Logic",* Cambridge Tracts in Theor. Comp. Sci., Number 5, Cambridge University press, Cambridge, 1985.

13 M. Vojdani Tabatabaee and M. Mahmoudi, *Metric σ-Frames versus Metric Lindelöf Spaces,* Preprint.

Address: Department of Mathematics
Shahid Beheshti University
Tehran 19839, Iran
Telefax: (98)(21)(2413135)
e-mail:m-ebrahimi@cc.sbu.ac.ir
m-mahmoudi@cc.sbu.ac.ir
mvojdani2000@yahoo.com