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SHEAVES FOR AN INVOLUTIVE QUANTALOID

by W. Dale GARRAWAY

Abstract

Dans cet article, nous explorons d'abord les ensembles à valeur dans $\mathcal{Q}$, un quantaloide avec involution. Nous définissons alors la catégorie des préfaisceaux de $\mathcal{Q}$ comme des foncteurs à valeur dans les ensembles à partir desquels nous définissons les faisceaux en termes de la propriété d'amalgamation unique de familles compatibles. À partir de ceci, nous montrons que la catégorie des ensembles à valeur dans $\mathcal{Q}$ est équivalente à la catégorie des faisceaux quand $\mathcal{Q}$ est "pseudo-rightsided".

1 Introduction

In 1984 Higgs[9] showed that, for a complete Heyting algebra $\mathcal{H}$, the category of $\mathcal{H}$-valued sets is equivalent to the category of sheaves on $\mathcal{H}$. His category of $\mathcal{H}$-valued sets is equivalent to the category of left adjoints in the symmetric idempotent splitting completion of the category of matrices on $\mathcal{H}$. It is well known that the category of $\mathcal{DH}$-valued sets is equivalent to the presheaf category $\mathbf{SET}^{\mathcal{H}^{op}}$, where $\mathcal{DH}$ is the complete Heyting algebra of down sets of $\mathcal{H}$. The following diagram represents these results

$$\begin{tikzcd}
\mathbf{SET}^{\mathcal{H}^{op}} \arrow[r, shift right, "\top"'] \arrow[d, shift right] \arrow[d, shift left, phantom, "\sim"'] \arrow[d, shift left] & SHV(\mathcal{H}) \arrow[l, shift right] \arrow[d, shift right] \arrow[d, shift left, phantom, "\sim"'] \arrow[d, shift left] \\
\mathcal{DH}-Set \arrow[r, shift right, "\top"'] & \mathcal{H}-Set \arrow[l, shift right]
\end{tikzcd}$$

Around the same time Mulvey, building on the Gelfand-Naimark theorem, explored the lattice of closed linear subspaces of a C^*-algebra and called the general form a quantale[11]. A quantale is a one object supremum enriched category and is a generalisation of complete Heyting algebras. The results led to the study of sheaves on a quantale $\mathcal{Q}$ in terms of $\mathcal{Q}$-valued sets. There have been different definitions of $\mathcal{Q}$-valued sets, but all begin with matrices of $\mathcal{Q}$ as their starting point.

A couple of years later, extending the work of Freyd and Carboni-Walters, Pitts worked with supremum enriched categories (quantaloids) and showed that a subcategory of the category of quantaloids, called bounded complete distributive categories of relations (bcDCR), is equivalent to the opposite category of grothendieck toposes. In fact the equivalence is given by the functors $MAP \dashv REL : Gtop^{op} \to bcDCR$ where REL takes a Grothendieck topos to the category of relations and MAP sends a bcDCR to the category of left adjoints[15]. The completion of a bounded distributive category of relations is given by taking the category of matrices and splitting the symmetric idempotents. This gives the completion with respect to coproducts and the splitting of symmetric idempotents. Any complete Heyting algebra is a bounded distributive category of relations and the category of left adjoints in the completion is the category of $\mathcal{H}$-valued sets of Higgs. This result points the way to defining the category of sheaves on a quantaloid $\mathcal{Q}$ as the category of left adjoints in the idempotent splitting completion of the category of matrices on $\mathcal{Q}$. This category has been studied by Van den Bosche[17] and most recently by Gylys[7].

In this paper we define the categories of Presheaves, $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$, for $\mathcal{Q}$ an involutive quantaloid and by extending the equivalence of Higgs we define the category of sheaves for $\mathcal{Q}$ a *pseudo-rightsided* quantaloid. We begin with an exploration of the category of $\mathcal{Q}$-valued sets and the notions of strictness and completeness of a $\mathcal{Q}$-valued set. By utilizing the relationship between the one object supremum enriched view of a Heyting algebra and the traditional multiobject view we construct the category of presheaves for $\mathcal{Q}$. Using the work of Merovitz[10] as a template we generalise the equivalence of Higgs to pseudo-rightsided quantaloids and show that the category of $\mathcal{Q}$-valued sets is a reflective subcategory of the category of presheaves. This result guides us to the definition of the category of sheaves for $\mathcal{Q}$ in terms of the unique amalgamation of matching families. It now follows that every Grothendieck topos can be formulated in this way.

2 $\mathcal{Q}$-valued sets for quantaloids

We begin with an exploration of *quantaloids* and $\mathcal{Q}$-valued sets for $\mathcal{Q}$ a quantaloid, but before that we introduce the concept of semicategories.

Definition 2.1 *A* semicategory $\mathcal{C}$ *consists of*
- *A set of* objects $|\mathcal{C}|$
- *For each pair of objects A, B, a set of* morphisms $\mathcal{C}(A, B)$.
- *For each triple of objects A, B & C, a function, called* composition,
$$\circ_{ABC} : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$
which satisfies the associativity axiom. $\diamondsuit$

We represent the composite $\circ_{ABC}(f, g)$ by $f \circ g$ or just fg. A semicategory is then a category without the identity axioms.

Example 2.1 *A one object semicategory is a semigroup, where we take the morphisms as the elements of the semigroup.* $\diamondsuit$

Example 2.2 *We can construct a one object semicategory out of a C^*-algebra $\mathcal{A}$ by taking as the morphisms the elements in $\mathcal{A}$ and the composition of morphisms is determined by the multiplication in $\mathcal{A}$. This is a category if and only if $\mathcal{A}$ is unital.* $\diamondsuit$

Definition 2.2 *Let $\mathcal{V}$ be a monoidal category. A* $\mathcal{V}$-semicategory *consists of*
- *A set of objects $|\mathcal{C}|$.*
- *For each pair of objects A, B, an object $\mathcal{C}(A, B)$ of $\mathcal{V}$.*
- *For each triple of objects A, B & C, a $\mathcal{V}$-morphism*
$$\&_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$
which satisfies the associativity axiom. $\diamondsuit$

We represent the composite $\&(p, q)$ by either $p\&q$ or just pq. This differs from $\mathcal{V}$-categories by the removal of the identity axioms.

Definition 2.3 *If $\mathcal{C}$ and D are $\mathcal{V}$-semicategories, then a $\mathcal{V}$*-semifuntor *$F : \mathcal{C} \to \mathcal{D}$ consists of*

- *A function $F : |\mathcal{C}| \to |\mathcal{D}|$.*
- *For each pair of objects A, B a $\mathcal{V}$-morphism*

$$F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$$

such that for any triple of objects, A, B & C, the following square commutes.

$$\begin{tikzcd}
\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \arrow[rr, "\&"] \arrow[d, "F \otimes F"'] & & \mathcal{C}(A,C) \arrow[d, "F"] \\
\mathcal{D}(F(B),F(C)) \otimes \mathcal{D}(F(A),F(B)) \arrow[rr, "\&"'] & & \mathcal{D}(F(A),F(C))
\end{tikzcd}$$

$\diamondsuit$

Our particular interest will be with the monoidal category **SUP** of complete supremum lattices and suprema preserving functions.

Definition 2.4 *A* quantaloid *$\mathcal{Q}$ is a supremum enriched semicategory.* $\diamondsuit$

The composition of morphisms in a quantaloid thus satisfies the following equations

$$p \,\&\left(\bigvee_i q_i\right) = \bigvee_i (p \,\&\, q_i) \text{ and } \left(\bigvee_i p_i\right) \&q = \bigvee_i (p_i \,\&\, q)$$

for all morphisms p and q of $\mathcal{Q}$ and for all families of morphisms p_i and q_i. In a quantaloid $\mathcal{Q}$ each morphism set $\mathcal{Q}(A, B)$ has a distinguished morphism, $\top_{AB}$, which is the supremum of all morphisms in $\mathcal{Q}(A, B)$. A one object quantaloid is frequently called a quantale. If the quantaloid $\mathcal{Q}$ is in fact a supremum enriched category, then we will say that $\mathcal{Q}$ is a *unital quantaloid.*

Example 2.3 *A complete Heyting algebra $\mathcal{H}$ is a unital quantale with an identity element $\top$. In particular the lattice of open sets of any topology is a quantale.* $\diamondsuit$

Example 2.4 *The powerset of a semigroup $\mathcal{G}$ is a quantale with the composition of two subsets X, Y equal $XY = \{xy \mid x \in X \text{ and } y \in Y\}$. If $\mathcal{G}$ is a group, then it is unital with the subset $\{e\}$ acting as the identity.* $\diamondsuit$

Example 2.5 *Mulvey[11] showed how to construct a quantale, called $MAX(\mathcal{A})$, out of a C^*-algebra $\mathcal{A}$, where the morphisms are taken to be the closed linear subspaces of $\mathcal{A}$. The composition of two linear subspaces X and Y is given by setting $XY = closure\{xy \mid x \in X \text{ and } y \in Y\}$. If $\mathcal{A}$ has an identity element, then $MAX(\mathcal{A})$ is unital with the identity the subspace generated by the unit.* $\diamondsuit$

Example 2.6 *The category with sets as objects and relations as morphisms is a quantaloid. In general for $\mathcal{E}$ a Grothendieck topos, the category of relations on $\mathcal{E}$ is a quantaloid.* $\diamondsuit$

Example 2.7 *A distributive category of relations (DCR) is a unital quantaloid that is cartesian (see [15]) and every object is discrete (see[15],[4]). It is complete (cDCR) if it has all coproducts and all symmetric idempotents split (for the latter we invoke the fact that every DCR is involutive). A DCR is bounded (denoted bDCR) if there is a small set of objects G that has the property that*

$$1_A = \bigvee\{pq \mid codomain(q) = domain(p) \in G \text{ and } pq \leq 1_A\}.$$

A one object bDCR is a complete Heyting algebra and for $\mathcal{E}$ a Grothendieck topos the category of relations of $\mathcal{E}$ is a bounded and complete DCR (denoted bcDCR). $\diamondsuit$

Recall that the Gelfand-Naimark theorem tells us that the category of locally compact Hausdorff spaces is equivalent to the category of commutative C^*-algebras and that the existence of a unit in the C^*-algebra is encoded in the equivalence with compact Hausdorff spaces. It has thus been interpreted that the study of non-commutative C^*-algebras is the study of non-commutative topologies. When trying to generalise to all C^*-algebras one should work with semicategories. Mulvey[11] interpreted the quantale $MAX(\mathcal{A})$ as an appropriate notion of non-commutative topology.

Definition 2.5 *Let $\mathcal{Q}$ be a unital quantaloid. A morphism $q : A \longrightarrow B$ is a* map *if there exists an arrow $q^{\#} : B \longrightarrow A$ such that*

$$\begin{aligned} 1_A &\leq q^{\#}q \\ qq^{\#} &\leq 1_B \end{aligned}$$

where, the arrows 1_A and 1_B respectively represent the identity arrows on the objects A, and B . ◇

Denote the relationship between q and $q^{\#}$ by $q \dashv q^{\#}$.

Definition 2.6 *Let $\mathcal{Q}$ be a unital quantaloid. The category $MAP(\mathcal{Q})$, of maps for $\mathcal{Q}$, has objects $|\mathcal{Q}|$, the objects of $\mathcal{Q}$, and arrows, the maps in $\mathcal{Q}$.* ◇

Example 2.8 *Pitts[15] showed that the functor*

$$MAP : bcDCR \longrightarrow GTOP^{op},$$

which sends a bcDCR to its category of maps, is an equivalence of categories. For the particular case when $\mathcal{Q}$ is the quantaloid of relations the maps are just the functions. ◇

In $MAP(\mathcal{Q})$ we have the following well-known results.

Theorem 2.1 *If $\mathcal{Q}$ is a supremum enriched category and $A \xrightarrow{q} B$ is a map in q, then the following hold*

1. $q = qq^{\#}q$
2. *q is a monomorphism if and only if $1_A = q^{\#}q$.*
3. *q is an epimorphism if and only if $1_B = qq^{\#}$*

4. q is an isomorphism if and only if it is both a monomorphism and an epimorphism

Proof:

1) $q = q1_A \leq qq^{\#}q \leq 1_B q = q$.

2) If $1_A = q^{\#}q$ and $qp_1 = qp_2$ for morphisms p_1 and p_2, then

$$q^{\#}qp_1 = p_1 = p_2 = q^{\#}qp_2.$$

If q is a monomorphism, then since $q1_A = qq^{\#}q$, we have $1_A = q^{\#}q$. By duality we get 3) and 4). ∎

Definition 2.7 *A quantaloid $\mathcal{Q}$ is* involutive *if there is a* **SUP**-*semi-functor of the form* $()^* : \mathcal{Q}^{op} \to \mathcal{Q}$, *that is the identity on objects and satisfies* $()^{*^{op}}()^* = \mathbf{1}$. ◇

If $\mathcal{Q}$ is involutive, we thus have for morphisms p, q and q_i

$$p^{**} = p, \quad (p\&q)^* = q^*\&p^* \text{ and } \left(\bigvee_i q_i\right)^* = \bigvee_i q_i^*.$$

Involutive quantales were introduced by Mulvey to aid his research on C^*-algebras. See, for example, his work with Pelletier in [13].

Example 2.9 *By definition C^*-algebras come equipped with an involution, which passes down to the associated quantale $MAX(\mathcal{A})$. For X, a linear subspace of $\mathcal{A}$, define X^* to be the linear subspace $\{x^* \mid x \in X\}$.* ◇

Example 2.10 *A complete Heyting algebra is involutive with the involution the identity morphism. More generally, any quantale where the morphisms commute has the identity morphism as an involution.* ◇

Example 2.11 *The power set of a group $\mathcal{G}$ is involutive with the involution determined by the inverse. If X is a subset of $\mathcal{G}$, then $X^* = \{x^{-1} \mid x \in X\}$.* ◇

Example 2.12 *The quantaloid of Relations has an involution given by taking the inverse relation. In general every DCR has a canonical involution, which is definable from the given structure (see, for example, [4] or [15]).* ◇

Definition 2.8 *Let $\mathcal{Q}$ be an involutive quantaloid. $\mathcal{Q}$ satisfies* Freyd's law of modularity *[6] if every triple of arrows $r : A \to B$, $s : B \to C$ and $t : A \to C$ satisfies $sr \wedge t \leq s(r \wedge s^*t)$.* ◇

Example 2.13 *All Heyting algebras satisfy Freyd's law of modularity. This is trivial since in this case we have $s \wedge r \wedge t \leq s \wedge (r \wedge s \wedge t)$.* ◇

Example 2.14 *The powerset of a group $\mathcal{G}$ satisfies Freyd's law of modularity. Suppose that g is in $XY \wedge Z$ then g is in Z and XY. So there exists $h \in X$ and $k \in Y$ such that $g = hk$. It now follows that $k \in X^*Z$ thus g is in $X(Y \wedge X^*Z)$.* ◇

Example 2.15 *The quantaloid of Relations satisfies Freyd's law of modularity and, in particular, every DCR does as well.* ◇

We will say that a map $q : A \to B$ is *symmetric* if $q \dashv q^*$ and denote the semicategory of symmetric maps by $MAP^*(\mathcal{Q})$. The following result gives us a situation in which all maps are symmetric.

Theorem 2.2 *Let $\mathcal{Q}$ be an involutive unital quantaloid. If $\mathcal{Q}$ satisfies Freyd's law of modularity and if q is a map, with $q \dashv p$, then $p = q^*$ (Note: for simplicity we have set $q^\# = p$).*

Proof: First
$$qp = qpp^*p \leq p^*p \quad \text{and} \quad p^*q^* = p^*q^*qq^* \leq qq^*.$$

Using these inequalities we now apply the law of modularity and show that $p \leq q^*$.

$$\begin{aligned} p &= 1p \\ &\leq q^*p^*p \wedge p \\ &\leq q^*(p^*p \wedge qp) \\ &= q^*qp \\ &\leq q^*. \end{aligned}$$

In a similar way we show that $q^* \leq p$ thus $p = q^*$. ∎

The next definition is a sufficient condition for the main results that follow. It would be interesting to know if it is also a necessary condition. It is suspected not to be the case.

Definition 2.9 *For $\mathcal{Q}$ an involutive quantaloid, $\mathcal{Q}$ is* pseudo-rightsided *if for every arrow $q : A \to B$, $qq^*q \leq q$ implies that $qq^*q = q$.* ♢

Observe that for a C^*-algebra $\mathcal{A}$, if a closed linear subspace A of $\mathcal{A}$ satisfies $AA^*A \leq A$, then it follows that AA^* and A^*A are sub C^*-algebras of $\mathcal{A}$. Thus there is an approximate unit made up of elements of A and A^* and so it seems likely that if $AA^*A \leq A$, then we should pick up all of A (we have not been able to show that this is true).

Lemma 2.3 *If the quantaloid $\mathcal{Q}$ satisfies Freyd's law of modularity and if for every morphism $q : A \to B$, $q \leq q\top_{BB}$, then $\mathcal{Q}$ is pseudo-rightsided (Recall that $\top_{BB}$ is the top arrow in $\mathcal{Q}(B, B)$).*

Proof: For every morphism $q : A \to B$ we have

$$q = q\top_{BB} \wedge q \leq q(\top_{BB} \wedge q^*q) = qq^*q$$ ∎

If the quantaloid $\mathcal{Q}$ is a category, then $q \leq q\top_{BB}$ for every object B. This result tells us that every bcDCR is pseudo-rightsided and so the results that follow carry over to every Grothendieck topos.

3 $\mathcal{Q}$-valued sets

Expanding on the work of Higgs, Pitts, Gylys and others we construct the category of $\mathcal{Q}$-valued sets and then explore some subcategories. In the particular case where $\mathcal{Q}$ is pseudo-rightsided we will see that all of these subcategories are equivalent. The exploration of these categories isolates the relevant details that come into play when we explore the category of sheaves on $\mathcal{Q}$. We begin with the category of matrices on a quantaloid.

Definition 3.1 *For $\mathcal{Q}$ an involutive quantaloid, Mat(Q), the category of* matrices *consists of*

- *Objects:* *Pairs (X, ρ_X) where X is a set and $\rho_X : X \to |\mathcal{Q}|$*
- *Arrows:* $(X, \rho_X) \xrightarrow{M} (Y, \rho_Y)$ *such that* $M : Y \times X \to arrows(\mathcal{Q})$ *with* $M(y, x) \in \mathcal{Q}(\rho_Y(x)\rho_X(y))$, *for all* $(y, x) \in Y \times X$

$\diamondsuit$

Note the usual definition of a matrix would have $M : (Y, \rho_Y) \to (X, \rho_X)$. Also note that $M(y, x)$ is a $\mathcal{Q}$-morphism from $\rho_X(x)$ to $\rho_Y(y)$

It is easy to show that the semicategory of matrices on a quantaloid is a quantaloid with the supremum defined pointwise. If the quantaloid is unital, then the semicategory of matrices is unital, with the unit given by the diagonal matrix Δ_X. It is also the completion of $\mathcal{Q}$ as a quantaloid with respect to all coproducts (see, for example, [5]). If $\mathcal{Q}$ is involutive, then $MAT(\mathcal{Q})$ is involutive with the involution, $()^\circ$: $MAT(\mathcal{Q})^{op} \to MAT(\mathcal{Q})$, on a matrix $N : X \to Y$ defined by $N^\circ(y, x) = N(x, y)^*$, for a matrix N.

Recall that for a lattice $\mathcal{L}$ the supremum is left adjoint to the down functor, $\mathcal{D} : \mathcal{L} \to \mathcal{D}(\mathcal{L})$. This extends to matrices on a unital quantaloid in the following way. For $M : X \to Y$, a matrix on $\mathcal{Q}$, define the functor $\mathcal{D}M : X \to Y$ to be the matrix on $\mathcal{D}\mathcal{Q}$ given by $\mathcal{D}M(y, x) = (M(y, x))^{\downarrow}$. In this case $\mathcal{D}$ does not preserve the composition of morphisms, since it is being asked to relate unions to supremums. Thus $\mathcal{D}$ is a lax functor.

Now given $N : Z \to W$, a matrix on $\mathcal{D}\mathcal{Q}$, define $\bigvee N : Z \to W$ to be the matrix on $\mathcal{Q}$ where $(\bigvee N)(w, z) = \bigvee\{q \mid q \in N(w, z)\}$. In this case it is easy to see that $\bigvee : MAT(\mathcal{D}\mathcal{Q}) \to MAT(\mathcal{Q})$ is a functor.

Since the supremum distributes over the composition of arrows we can show that the composite $\bigvee \circ \mathcal{D}$ is the identity functor on $MAT(\mathcal{Q})$. Let Δ_X be the identity matrix on the set X in the category of matrices, $MAT(\mathcal{D}\mathcal{Q})$. By setting $\varepsilon_X = \Delta_X$ we obtain a *lax transformation*[5] $\varepsilon : 1_{MAT(\mathcal{D}\mathcal{Q})} \to \mathcal{D} \circ \bigvee$. So, in a sense, we can say that $\bigvee$ is lax left adjoint to $\mathcal{D}$.

Definition 3.2 *For $\mathcal{Q}$ an involutive quantaloid, the semicategory of* modules *of $\mathcal{Q}$ consists of the following*

- *Objects: endo arrows $q : A \to A$ that satisfy $qq \leq q$ and $q^* = q$.*

- *Arrows: a morphism $p : q_1 \to q_2$ is a $\mathcal{Q}$-morphism that satisfies*

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q_1"'] \arrow[dr, "p"] \arrow[d, phantom, shift left=3ex, "\leq"] & B \arrow[d, "q_2"] \arrow[dl, phantom, shift right=1ex, "\geq"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

◊

Denote the semicategory of modules on $\mathcal{Q}$ by $MOD(\mathcal{Q})$. Since modules are defined by an inequality the lax adjunction from above, $\bigvee \dashv_{lax} \mathcal{D} : MAT(\mathcal{DQ}) \to MAT(\mathcal{Q})$, easily extends to give a lax adjunction $\bigvee \dashv_{lax} \mathcal{D} : MOD(MAT(\mathcal{DQ})) \to MOD(MAT(\mathcal{Q}))$.

In the literature symmetry($q = q^*$) is usually not required for a module (see for example Betti[3]). They are also frequently called bimodules (see for example Rosenthal[16]).

Definition 3.3 *For $\mathcal{Q}$ an involutive quantaloid, the* symmetric idempotent splitting completion *of $\mathcal{Q}$ consists of the following*

- *Objects • endo arrows $q : A \to A$ that satisfy $qq = q$ and $q^* = q$.*

- *Arrows • a morphism $p : q_1 \to q_2$ is a $\mathcal{Q}$ morphism that satisfies*

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q_1"'] \arrow[dr, "p"] \arrow[d, phantom, shift left=3ex, "="] & B \arrow[d, "q_2"] \arrow[dl, phantom, shift right=1ex, "="] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

We will denote the symmetric idempotent splitting completion of $\mathcal{Q}$ by $KAR^*(\mathcal{Q})$. The idempotent splitting completion is also know as the Karoubian envelope, hence the notation. ◊

The lax adjunction that exists for matrices and modules does not extend to the idempotent splitting completion. Since $\mathcal{D}$ does not preserve the composition, it does not preserve the objects.

Definition 3.4 *Let $\mathcal{Q}$ an involutive quantaloid. The category, $\mathcal{Q}$-Set, of $\mathcal{Q}$*-valued sets *consists of*

- *Objects:* *Triples (X, ρ_X, δ_X), where (X, ρ_X) is a matrix object and $\delta_X : (X, \rho_X) \to (X, \rho_X)$ is a matrix arrow satisfying*

$$\delta_X \delta_X = \delta_X \text{ and } \delta_X = \delta_X^\circ.$$

- *Arrows:* *$(X, \rho_X, \delta_X) \xrightarrow{R} (Y, \rho_Y, \delta_Y)$ is a matrix $(X, \rho_X) \xrightarrow{R} (Y, \rho_Y)$ satisfying*

$$\begin{gathered} \delta_X \le R^\circ R, \\ RR^\circ \le \delta_Y, \text{ and} \\ \delta_Y R = R = R\delta_X \end{gathered}$$

$\diamondsuit$

It is easy to see that the category $MAP^*(KAR^*(MAT(\mathcal{Q})))$ is the category of $\mathcal{Q}$-valued sets. When the context is clear we will drop the subscript from δ_X and ρ_X.

Example 3.1 *For $\mathcal{Q}$ an involutive quantaloid and $q : A \to A$ a symmetric idempotent arrow in $\mathcal{Q}$ there is a $\mathcal{Q}$-valued set $[q] = (\{*\}, \rho_q, \delta_q)$ where $\rho(*) = A$ and $\delta_q(*, *) = q$. These objects are known as* singletons. *It is clear that every $\mathcal{Q}$-valued set of the form $(\{*\}, \rho, \delta)$ is equal to $[q]$ for some symmetric idempotent arrow q. For $p : q_1 \to q_2$ an arrow in $MAP^*(KAR^*(\mathcal{Q}))$ there is a $\mathcal{Q}$-Set morphism $\alpha_p : [q_1] \to [q_2]$ where $\alpha_p(*, *) = p$ and that all such morphisms must be of this form. It follows that the full subcategory of $\mathcal{Q}$-Set generated by the singleton $\mathcal{Q}$-valued sets is isomorphic to the category $MAP^*(KAR^*(\mathcal{Q}))$.* $\diamondsuit$

4 Complete $\mathcal{Q}$-valued sets

Integral to the work of Higgs was the notion of singletons. For his result Higgs made use of the fact that for a complete Heyting algebra an $\mathcal{H}$-valued set (X, δ_X) can be recovered from the morphisms of the form $[q] \xrightarrow{\alpha} (X, \delta_X)$ via $\bigvee_\alpha \alpha\alpha^\circ = \delta_X$, where the supremum is taken over all morphisms with domain $[q]$ and codomain (X, δ_X).

Definition 4.1 *For (X, ρ_X, δ_X) a $\mathcal{Q}$-valued set a* Singleton morphism *is a morphism $[q] \xrightarrow{\alpha} (X, \rho_X, \delta_X)$.* $\diamondsuit$

We will usually denote $\alpha(x, *)$ by $\alpha(x)$. A singleton morphism is then an X-indexed family of arrows with common domain that satisfies for all $x, x' \in X$

$$\bigvee_{x \in X} \alpha(x)^* \;\&\; \alpha(x) \geq q, \qquad \alpha(x) \;\&\; \alpha(x')^* \;\leq\; \delta(x, x'),$$
$$\bigvee_{x} \alpha(x) \;\&\; \delta(x, x') = \alpha(x')$$
$$\text{and} \qquad \alpha(x) \;\&\; q = \alpha(x)$$

Definition 4.2 *For (X, ρ_X, δ_X) a $\mathcal{Q}$-valued set we say that $x \in X$ is* strict *if for every $x' \in X$ we have*

$$\delta(x, x) \;\&\; \delta(x, x') = \delta(x, x') \quad \textit{and} \quad \delta(x', x) \;\&\; \delta(x, x) = \delta(x', x)$$

$\diamondsuit$

If every $x \in X$ is strict, then we say that (X, ρ_X, δ_X) is strict. Denote the full subcategory of $\mathcal{Q}$-*Set* generated by the strict $\mathcal{Q}$-valued sets by *Strict*$(\mathcal{Q})$.

Example 4.1 *If x is a strict element of (X, ρ_X, δ_X), then there is a singleton morphism $[x] \xrightarrow{\alpha_x} (X, \rho_X, \delta_X)$ given by $\alpha_x = \delta(-, x)$, where $[x]$ is the singleton $\mathcal{Q}$-valued set associated to the symmetric idempotent $\delta(x, x)$.* $\diamondsuit$

Lemma 4.1 *If $\mathcal{Q}$ is a pseudo-rightsided quantaloid, then Strict$(\mathcal{Q})$ is equal to $\mathcal{Q}$-Set.*

Proof: If (X, ρ_X, δ_X) is a $\mathcal{Q}$-valued set, then by pseudo-rightsidedness we have

$$\begin{aligned} \delta(x, x') &= \delta(x, x') \;\&\; \delta(x, x')^* \;\&\; \delta(x, x') && \text{by pseudo-rightsidedness} \\ &\leq \delta(x, x) \;\&\; \delta(x, x') && \text{since } \delta\delta\delta \leq \delta\delta \\ &\leq \delta(x, x') && \text{since } \delta\delta \leq \delta \end{aligned}$$

Thus $\delta(x,x)$ & $\delta(x,x') = \delta(x,x')$. Similarly for $\delta(x',x)$ & $\delta(x,x) = \delta(x',x)$. ∎

Thus for every distributive category of relations $\mathcal{Q}$, every $\mathcal{Q}$-valued set is strict. In particular, when $\mathcal{E}$ is a Grothendieck topos the $\mathcal{Q}$-valued sets of the quantaloid $REL(\mathcal{E})$ are all strict.

Definition 4.3 *A $\mathcal{Q}$-valued set (X, ρ_X, δ_X) is* atomic *if*

$$\delta_X = \bigvee_{\gamma} \gamma\gamma^\circ$$

where the supremum is taken over the singleton morphisms with codomain (X, ρ_X, δ_X). ◇

Denote the full subcategory of $\mathcal{Q}$-*Set* generated by the atomic $\mathcal{Q}$-valued sets by *Atomic*$(\mathcal{Q})$. We will show that the category *Atomic*$(\mathcal{Q})$ is equivalent to the category *Strict*$(\mathcal{Q})$. In many of the subsequent proofs it is the atomic property that is needed as opposed to strictness.

Lemma 4.2 *If (X, ρ_X, δ_X) is strict, then it is atomic.*

Proof: This follows because we have $\gamma\gamma^\circ \leq \delta_X$ for all singleton morphisms γ and by strictness we have

$$\alpha_x(x)\alpha_x^\circ(x') = \delta(x,x) \;\&\; \delta(x,x') = \delta(x,x')$$ ∎

We now build an equivalence $\iota \cong \Lambda : Atomic(\mathcal{Q}) \to Strict(\mathcal{Q})$ which makes use of the singleton morphisms. We will later show that Λ factors through a functor category, $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$, when $\mathcal{Q}$ is pseudo-rightsided. We then use this to define sheaves for $\mathcal{Q}$.

If (X, ρ_X, δ_X) is a $\mathcal{Q}$-valued set, then there is a $\mathcal{Q}$-valued set

$$\Lambda(X, \rho_X, \delta_X) = (\bar{X}, \bar{\rho}, \bar{\delta})$$

where

- $\bar{X} = \{\alpha : [q] \to (X, \rho_X, \delta_X) \mid \alpha \text{ is a singleton morphism }\}$

- $\bar{\rho}$ is defined by $\bar{\rho}(\alpha : [q] \longrightarrow (X, \rho_X, \delta_X)) = A$ where $A \xrightarrow{q} A$.
- $\bar{\delta}(\alpha, \beta) = \alpha^\circ \beta$.

To show that $\bar{\delta}$ is an idempotent observe that $\alpha^\circ \gamma\gamma^\circ \beta \leq \alpha^\circ \beta$. When γ equals α we have $\alpha^\circ \alpha\alpha^\circ \beta = \alpha^\circ \beta$ thus $\bar{\delta}$ is an idempotent. $(\bar{X}, \bar{\rho}, \bar{\delta})$ is strict since

$$\bar{\delta}(\alpha, \alpha) \ \& \ \bar{\delta}(\alpha, \beta) = \alpha^\circ \alpha\alpha^\circ \beta = \alpha^\circ \beta = \bar{\delta}(\alpha, \beta)$$

for all singletons α and β. Since $(\bar{X}, \bar{\rho}, \bar{\delta})$ is strict it is atomic.

For $(X, \rho_X, \delta_X) \xrightarrow{R} (Y, \rho_Y, \delta_Y)$ a morphism of $\mathcal{Q}$-valued sets there is a morphism $\Lambda(R) = (\bar{X}, \bar{\rho}, \bar{\delta}) \xrightarrow{\bar{R}} (\bar{Y}, \bar{\rho}, \bar{\delta})$ given by $\bar{R}(\alpha, \beta) = \alpha^\circ R\beta$. We immediately have the following

Lemma 4.3 *$\mathcal{Q}$-Set $\xrightarrow{\Lambda}$ Strict($\mathcal{Q}$) is a lax functor.*

Proof: For composable morphisms R and S observe that

$$\begin{aligned} \bar{R}\bar{S}(\alpha, \beta) &= \bigvee_\gamma \{\alpha^\circ R\gamma\gamma^\circ S\beta\} \\ &\leq \alpha^\circ RS\beta \\ &= \overline{RS}(\alpha, \beta) \end{aligned}$$ ∎

We have equality when *codomain*(S) is atomic. Thus Λ is a functor from *Atomic*$(\mathcal{Q})$ to *Strict*$(\mathcal{Q})$.

Lemma 4.4 *If (X, ρ_X, δ_X) is atomic, then (X, ρ_X, δ_X) is isomorphic to $(\bar{X}, \bar{\rho}, \bar{\delta})$*

Proof: Define the isomorphism $(X, \rho_X, \delta_X) \xrightarrow{\varepsilon} (\bar{X}, \bar{\rho}, \bar{\delta})$ by setting $\varepsilon(\alpha, x) = \alpha^\circ(x)$. It is easy to check that ε is a morphism and to see that it is an isomorphism. Observe that

$$\begin{aligned} \varepsilon^\circ \varepsilon(x_1, x_2) &= \bigvee_\gamma \{\varepsilon^\circ(x_1, \gamma) \ \& \ \varepsilon(\gamma, x_2)\} \\ &= \bigvee_\gamma \{\gamma(x_1, *) \ \& \ \gamma^\circ(*, x_2)\} \\ &= \delta(x_1, x_2). \text{ Since } (X, \rho_X, \delta_X) \text{ is atomic} \end{aligned}$$

and

$$\begin{aligned}\varepsilon\varepsilon^\circ(\alpha,\beta) &= \bigvee_x\{\varepsilon(\alpha,x)\ \&\ \varepsilon^\circ(x,\beta)\}\\ &= \bigvee_x\{\alpha^\circ(*,x)\ \&\ \beta(x,*)\}\\ &= \alpha^\circ\beta(*,*)\\ &= \bar{\delta}(\alpha,\beta)\end{aligned}$$

∎

For $\iota : Strict(\mathcal{Q}) \hookrightarrow \mathcal{Q}\text{-}Set$, the inclusion functor, we thus have $\Lambda\iota \cong \mathbf{1}_{Strict(\mathcal{Q})}$. There is a natural isomorphism $\eta : \iota\Lambda \longrightarrow \mathbf{1}_{\mathcal{Q}-Set}$ given by setting $\eta_X(x,\alpha)$ equal to $\varepsilon^\circ(x,\alpha)$, which is just $\alpha(x)$, for each $\mathcal{Q}$-valued set (X,ρ_X,δ_X). Thus Λ is right adjoint to the inclusion.

We now have the following equivalence of categories.

Theorem 4.5 *If $\mathcal{Q}$ is an involutive quantaloid, then $Strict(\mathcal{Q})$ is equivalent to $Atomic(\mathcal{Q})$*

Proof: The equivalence is witnessed by the functors

$$Strict(\mathcal{Q}) \underset{\iota}{\overset{\Lambda}{\underset{\cong}{\leftrightarrows}}} Atomic(\mathcal{Q})$$

∎

Definition 4.4 *A $\mathcal{Q}$-valued set (X,ρ_X,δ_X) is* complete *if it is strict and every singleton is of the form $\alpha_x = \delta(-,x)$.* ◇

Denote the full subcategory of $\mathcal{Q}\text{-}Set$ generated by the complete $\mathcal{Q}$-valued sets by $Comp(\mathcal{Q})$.

Lemma 4.6 *If (X,ρ_X,δ_X) a $\mathcal{Q}$-valued set, then $(\bar{X},\bar{\rho},\bar{\delta})$ is Complete*

Proof: We want to show that for a singleton $[q] \xrightarrow{A} (\bar{X},\bar{\rho},\bar{\delta})$, A has the form $\bar{\delta}(-,\alpha)$ for some singleton $\alpha : [q] \longrightarrow (X,\rho_X,\delta_X)$. To this end examine

$$\begin{aligned} A(\beta,*) &= \bar{\delta}A(\beta,*) \\ &= \bigvee_{\gamma}\{\bar{\delta}(\beta,\gamma) \ \& \ A(\gamma,*)\} \\ &= \bigvee_{\gamma}\{\beta^{\circ}\gamma \ \& \ A(\gamma,*)\} \\ &= \bigvee_{\gamma}\left\{\bigvee_{x}\{\beta^{\circ}(*,x) \ \& \ \gamma(x,*)\} \ \& \ A(\gamma,*)\right\} \\ &= \bigvee_{x}\left\{\beta^{\circ}(*,x) \ \& \ \bigvee_{\gamma}\{\gamma(x,*) \ \& \ A(\gamma,*)\}\right\}. \end{aligned}$$

Our goal is to show that $A(\beta,*) = \bar{\delta}(\beta,\alpha)$, where the latter is equal to $\bigvee_{x}\{\beta^{\circ}(*,x) \ \& \ \alpha(x,*)\}$. Comparing with the last line of the equation above we define α to be the matrix

$$\alpha(x,*) = \bigvee_{\gamma}\{\gamma(x,*) \ \& \ A(\gamma,*)\}.$$

It is now easy to show that α is a singleton morphism on (X,ρ_X,δ_X) from which it follows that $A = \bar{\delta}(-,\alpha)$ (we denote the latter by A_α). ∎

The following is an immediate consequence of this.

Theorem 4.7 *If $\mathcal{Q}$ is an involutive quantaloid, then the categories $Strict(\mathcal{Q})$ and $Comp(\mathcal{Q})$ are equivalent.*

Proof: The equivalence is given by

$$Strict(\mathcal{Q}) \underset{\iota}{\overset{\Lambda}{\rightleftarrows}} Comp(\mathcal{Q}) \qquad \cong$$ ∎

5 Subobjects

The category of $\mathcal{Q}$-valued sets has a terminal object, $\mathbf{1} = (|\mathcal{Q}|,\rho,\top)$, where $\top(A,B) = \top_{BA}$ is the top arrow from B to A (Note: we assume that $\top_{BC}\&\top_{AB} = \top_{AC}$). For any object, (X,ρ_X,δ_X), there is a morphism $(X,\rho_X,\delta_X) \xrightarrow{\alpha} (|\mathcal{Q}|,\rho,\top)$, given by

$$\alpha(A,x) = \bigvee_{x'}\{\top(A,x') \ \& \ \delta(x',x)\}$$

It is easy to see that this is in fact a morphism of $\mathcal{Q}$-valued sets. If S and R are morphisms between (X, ρ_X, δ_X) and $(|\mathcal{Q}|, \rho, \top)$, then we have

$$S = S\delta \leq SR^\circ R \leq \top R = R$$

Thus $S = R$ and so $(|\mathcal{Q}|, \rho, \top)$ is terminal.

Each morphism of $\mathcal{Q}$-valued sets factors as an epimorphism followed by a monomorphism in two canonical ways via

$$\begin{tikzcd} & (Y,\rho,RR^\circ) \arrow[dr, "RR^\circ"] & \\ (X,\rho_X,\delta_X) \arrow[ur, "R"] \arrow[rr, "R"'] & & (Y,\rho_Y,\delta_Y) \end{tikzcd} \qquad \begin{tikzcd} & (X,\rho,R^\circ R) \arrow[dr, "R"] & \\ (X,\rho_X,\delta_X) \arrow[ur, "R^\circ R"] \arrow[rr, "R"'] & & (Y,\rho_Y,\delta_Y) \end{tikzcd}$$

It follows that a subobject, $(X, \rho_X, \delta_X) \hookrightarrow (Y, \rho_Y, \delta_Y)$, can be formulated via the factorizations as follows

$$\begin{tikzcd} & (Y,\rho,RR^\circ) \arrow[dr, "RR^\circ"] & \\ (X,\rho_X,\delta_X) \arrow[ur, "R", "\sim"'] \arrow[rr, tail, "R"'] & & (Y,\rho_Y,\delta_Y) \end{tikzcd} \qquad \begin{tikzcd} & (X,\rho,R^\circ R) \arrow[dr, "R"] & \\ (X,\rho_X,\delta_X) \arrow[ur, "R^\circ R", "\sim"'] \arrow[rr, tail, "R"'] & & (Y,\rho_Y,\delta_Y) \end{tikzcd}$$

Using the first factorization, a subobject of (Y, ρ_Y, δ_Y) is isomorphic to one of the form (Y, ρ, δ'), where $\delta' \leq \delta$ and $\delta'\delta = \delta = \delta\delta'$. When $\mathcal{Q}$ is an involutive quantale the subobjects of the terminal object are singletons $[q] \overset{\alpha_q}{\hookrightarrow} (\{*\}, \top)$, where q is a two-sided symmetric idempotent. That is $q\&\top = q = \top\&q$ and $qq = q = q^*$. Thus the lattice of subobjects of the terminal object, $sub(\mathbf{1})$, is the lattice of two-sided symmetric idempotents.

6 Presheaves

When defining the category of presheaves for an involutive quantaloid we use the fact that for $\mathcal{H}$, a complete Heyting algebra, the regular interpretation of $\mathcal{H}$ as a partial order can be recovered from the one-object quantale interpretation, $\mathcal{H}_Q$, via

$$\mathcal{H} \cong MAP^*(KAR^*(\mathcal{H}_Q))$$

Observe that each morphism in $MAP^*(KAR^*(\mathcal{H}_Q)$ is a monomorphism. Recall that for a quantaloid $\mathcal{Q}$, $MAP^*(KAR^*(\mathcal{Q}))$ is equivalent to the full subcategory of $\mathcal{Q}$-SET generated by the singletons. We will denote the category of monomorphisms in $MAP^*(KAR^*(\mathcal{Q}))$ by $\overline{\mathcal{Q}}$.

Let $sub(\top)$ be the lattice of subobjects of the top element of $\mathcal{Q}$. For $\mathcal{Q}$ a quantale, $sub(\top)$ is a sub-lattice of $\overline{\mathcal{Q}}$. In particular we may have an arrow $q_1 \overset{p}{\hookrightarrow} q_2$ in $\overline{\mathcal{Q}}$ while q_2 may not be a subobject of $\top$. From previous comments $[q_2]$ is a subobject of $[\top]$ if and only if q_2 is a two-sided symmetric idempotent. If $\mathcal{H}$ is a complete Heyting algebra we do have $\overline{\mathcal{H}_Q}$ equal to $sub(\top)$ equal to $\mathcal{H}$.

Definition 6.1 *Let $\mathcal{Q}$ be a quantaloid. The category of* presheaves *on $\mathcal{Q}$ is the category* $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$. ◇

Given a presheaf F, an arrow $q_1 \xrightarrow{p} q_2$ and $x \in F(q_2)$ denote $F(p)(x) \in F(q_1)$ by $x_{|_p}$. We say that x is restricted to p.

We now build an adjunction $\Phi \dashv \widehat{\Psi} : \mathbf{SET}^{\overline{\mathcal{Q}}^{op}} \to Comp(\mathcal{Q})$ from which we will define the category of sheaves on $\mathcal{Q}$

Let (X, ρ_X, δ_X) be a complete $\mathcal{Q}$-valued set and define a presheaf $\Phi((X, \rho_X, \delta_X)) = F_\delta$ as follows

- On Objects $A \xrightarrow{q} A$: $\quad F_\delta(q) = \{[q] \xrightarrow{\alpha} (X, \rho_X, \delta_X)\}$
- On Arrows $q \overset{p}{\hookrightarrow} q'$: $\quad F_\delta(p)(\alpha) = \alpha\alpha_p = \alpha_{|_p} \quad$ (where $[q] \xrightarrow{\alpha_p} [q']$)

For a morphism of $\mathcal{Q}$-valued sets, $(X, \rho_X, \delta_X) \xrightarrow{R} (Y, \rho_Y, \delta_Y)$, define a transformation $\Phi(R) = F_{\delta_X} \xrightarrow{\tau_R} F_{\delta_Y}$ by setting $\tau_R(q)(\alpha) = R\alpha$. It is easy to show that for $\mathcal{Q}$ a quantaloid Φ is a functor (In fact it is functor from $\mathcal{Q}$-Set to $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$).

Define the functor $\widehat{\Psi} : \mathbf{SET}^{\overline{\mathcal{Q}}^{op}} \to Comp(\mathcal{Q})$ first by defining a functor $\Psi : \mathbf{SET}^{\overline{\mathcal{Q}}^{op}} \to \mathcal{DQ}$-$Set$. Recall that $\mathcal{DQ}$ is the quantaloid whose objects are as in $\mathcal{Q}$ and a morphism from A to B is a down set in $\mathcal{Q}(A, B)$. Given a presheaf $\overline{\mathcal{Q}}^{op} \xrightarrow{F} \mathbf{SET}$ define a $\mathcal{DQ}$-valued set $\Psi(F) = (X_F, \rho_F, \delta_F)$ by

- $X_F = \coprod_q F(q)$

- $\rho_F(x) = A$ such that $x \in F(q)$ and $A \xrightarrow{q} A$
- $\delta_F(x, x') = \{p_1 \& p_2^* \mid x_{|p_1} = x'_{|p_2}\}^{\downarrow}$

Let $F \xrightarrow{\tau} G$ be a transformation. Then a morphism of $\mathcal{Q}$-valued sets, $\Psi_\tau = (X_F, \rho_F, \delta_F) \xrightarrow{R_\tau} (Y_G, \rho_G, \delta_G)$, is defined by

$$R_\tau(y, x) = \{p_1 \& p_2^* \mid y_{|p_1} = \tau_{x_{|p_2}}\}^{\downarrow}$$

where, for $x \in F(q)$, τ_x is the element $\tau_q(x)$ of $G(q)$. Note that $R_\tau(y, x)$ is equal to $\delta_G(y, \tau_x)$

Theorem 6.1 $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}} \xrightarrow{\Psi} \mathcal{D}\mathcal{Q}$*-Set is a functor, If* $\mathcal{Q}$ *is a pseudo-rightsided quantaloid*

Proof: We first need to show that for a presheaf F, δ_F is an idempotent matrix. We have that $\delta_F \leq \delta_F \delta_F$ since if $p_1 p_2^* \in \delta_F(x, y)$, then $p_1 p_1^* p_1 p_2^*$ which equals $p_1 p_2^*$ is contained in $\delta_F \delta_F(x, y)$.

Now assume that $p_1 p_2^* p_3 p_4^* \in \delta_F \delta_F(x, z)$.

if $p_1 : r_1 \hookrightarrow q_1$, $p_2 : r_1 \hookrightarrow q_2$, $p_3 : r_2 \hookrightarrow q_2$, $p_4 : r_2 \hookrightarrow q_3$ are the arrows in $\overline{\mathcal{Q}}$ that witness $p_1 p_2^* p_3 p_4^* \in \delta_F \delta_F(x, z)$, then we have the following diagram in $\overline{\mathcal{Q}}$.

$$\begin{tikzcd}
q_1 & & q_2 & & q_3 \\
& r_1 \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & & r_2 \arrow[ul, "p_3"'] \arrow[ur, "p_4"'] & \\
& & p_3^* p_2 p_2^* p_3 \arrow[ul, "p_2^* p_3"] \arrow[ur, "p_3^* p_2 p_2^* p_3"'] & &
\end{tikzcd}$$

To show that $p_3^* p_2 p_2^* p_3$ is an idempotent observe that we have

$$\begin{aligned} p_3^* p_2 (p_3^* p_2)^* p_3^* p_2 &= p_3^* p_2 p_2^* p_3 p_3^* p_2 \\ &\leq p_3^* p_2. \end{aligned}$$

Since $\mathcal{Q}$ is pseudo-rightsided the above becomes an equality. Thus

$$p_3^*p_2(p_3^*p_2)^*p_3^*p_2(p_3^*p_2)^* = p_3^*p_2p_2^*p_3.$$

We clearly have that $p_3^*p_2p_2^*p_3$ is a monomorphism from $p_3^*p_2p_2^*p_3$ to r_2 To show that $p_2^*p_3$ is a morphism from $p_3^*p_2p_2^*p_3$ to r_1 the only difficult part is to show that $p_2^*p_3p_3^*p_2p_2^*p_3$ is equal to $p_2^*p_3$ which the pseudo-rightsidedness of $\mathcal{Q}$ gives us. .

Since the top section of the diagram witnesses $p_1p_2^*p_3p_4^* \in \delta_F\delta_F(x,z)$ we obtain $x_{|_{p_1p_2^*p_3}} = z_{|_{p_4p_3^*p_2p_2^*p_3}}$. Thus $(p_1p_2^*p_3)\&(p_4p_3^*p_2p_2^*p_3)^* = p_1p_2^*p_3p_4^*$ is an element of $\delta_F(x,y)$. It now follows that δ_F is idempotent. In fact δ_F is strict and hence atomic. If $p_1p_2^* \in \delta_F(x_1,x_2)$, then $p_1p_2^* = p_1p_1^*p_1p_2^*$ is an element of $\delta_F(x_1,x_1)$ & $\delta_F(x_1,x_2)$. From which it follows that

$$\delta_F(x_1,x_1)\ \&\ \delta_F(x_1,x_2) = \delta_F(x_1,x_2),$$

which tells us that δ_F is strict, and thus isomorphic to a complete $\mathcal{Q}$-valued set.

Now to show that for $F \xrightarrow{\tau} G$ a transformation of presheaves, R_τ is a morphism in $\mathcal{DQ}$-Set. Since $R_\tau(y,x) = \delta_G(y,\tau_x)$ we immediately have $\delta_G R_\tau = R_\tau$. To show that $R_\tau\delta_F = R_\tau$ we want to show that $\bigcup_y\{\delta_G(x,\tau_y)\&\delta_F(y,z)\} = \delta_G(x,\tau_z)$. First we clearly have

$$\bigcup_y\{\delta_G(x,\tau_y)\&\delta_F(y,z)\} \supseteq \delta_G(x,\tau_z)\&\delta_F(z,z).$$

If $p_1p_2^* \in \delta_G(x,\tau_z)$, then $p_1p_2^* = p_1p_2^*p_2p_2^* \in \delta_G(x,\tau_z)\&\delta_F(z,z)$ thus

$$\bigcup_y\{\delta_G(x,\tau_y)\&\delta_F(y,z)\} \supseteq \delta_G(x.\tau_z)$$

Now assume that $x \in G(q_1)$, $y \in F(q_2)$, $z \in F(q_3)$ and that $x_{|_{p_1}} = \tau_{y|_{p_2}}$ and $y_{|_{p_3}} = z_{|_{p_4}}$. Pictorially we have

$$
\begin{tikzcd}
G(q_1) \arrow[dr, "G(p_1)"'] & & G(q_2) \arrow[dl, "G(p_2)"] \arrow[dr, "G(p_3)"] & & F(q_2) \arrow[ll, "\tau_{q_2}"'] \arrow[dr, "F(p_3)"] & & F(q_3) \arrow[dl, "F(p_4)"] \\
& G(r_1) \arrow[dr, "G(p_2^* p_3)"'] & & G(r_2) \arrow[dl, "G(p_3^* p_2 p_2^* p_3)"] & & F(r_2) \arrow[ll, "\tau_{r_2}"'] & \\
& & G(p_3^* p_2 p_2^* p_3) & & & &
\end{tikzcd}
$$

We see that $x_{|_{p_1 p_2^* p_3}} = \tau_{z|_{p_4 p_3^* p_2 p_2^* p_3}}$ and so $p_1 p_2^* p_3 p_4^*$ is an element of $\delta_G(x, \tau_z)$. This tells us that $\bigcup_y \{\delta_G(x, \tau_y) \& \delta_F(y, z)\} \subseteq \delta_G(x, \tau_z)$, hence the desired equality and thus R_τ is a morphism in $KAR(MAT(\mathcal{Q}))$. Similarly R_τ° is a morphism.

To show that $R_\tau \dashv R_\tau^\circ$ we begin by showing that $\delta_F(x, y) \subseteq \delta_G(\tau_x, \tau_y)$. Assume that $x \in F(q_1)$ and $y \in F(q_2)$ and that $x_{|_{p_1}} = y_{|_{p_2}}$. Pictorially

$$
\begin{tikzcd}
& & F(q_2) \arrow[rrr, "\tau(q_2)"] \arrow[ddl, "F(p_2)"] & & & G(q_2) \arrow[ddl, "G(p_2)"] \\
F(q_1) \arrow[rrr, "\tau(q_1)"] \arrow[dr, "F(p_1)"'] & & & G(q_1) \arrow[dr, "G(p_1)"'] & & \\
& F(r) \arrow[rrr, "\tau(r)"] & & & G(r) &
\end{tikzcd}
$$

By naturality of τ, $\tau_{x_{|_{p_1}}} = \tau_{y_{|_{p_2}}}$ and so we have $\delta_F(x, y) \subseteq \delta_G(\tau_x, \tau_y)$. This tells us that

$$\delta_F(x, y) \subseteq \delta_G(\tau_x, \tau_y) = \delta_G \delta_G(\tau_x, \tau_y) = R_\tau^\circ R_\tau(x, y).$$

For the other inequality we have

$$
\begin{aligned}
R_\tau R_\tau^\circ(x, y) &= \bigcup_z \{R_\tau(x, z) \ \& \ R_\tau^\circ(z, y)\} \\
&= \bigcup_z \{\delta_G(x, \tau_z) \ \& \ \delta_G(\tau_z, y)\} \\
&\subseteq \delta_G(x, y)
\end{aligned}
$$

Thus R_τ is a $\mathcal{D}\mathcal{Q}$-Set morphism.

Finally to show that composition of morphisms is preserved we have for a second transformation $G \xrightarrow{\sigma} H$

$$R_\sigma R_\tau(x,y) = R_\sigma \delta_G(x,\tau_y) = R_\sigma(x,\tau_y) = \delta_H(x,\sigma_{\tau_y}) = R_{\sigma\tau}(x,y) \quad \blacksquare$$

Composing Ψ with the functor $\bigvee : \mathcal{D}\mathcal{Q}\text{-}Set \to \mathcal{Q}\text{-}Set$ gives us a functor $\widehat{\Psi} : \mathbf{SET}^{\overline{\mathcal{Q}}^{op}} \to \mathcal{Q}\text{-}Set$.

Recall on page 15 we defined the functor $\Lambda : Atomic(\mathcal{Q}) \to Strict(\mathcal{Q})$, which takes an atomic $\mathcal{Q}$-valued set and sends it to the $\mathcal{Q}$-valued set of singletons.

Lemma 6.2 *if $\mathcal{Q}$ is a pseudo-rightsided quantaloid, then the functor Λ is the composite $\widehat{\Psi}\Phi : \mathcal{Q}\text{-}Set \to \mathcal{Q}\text{-}Set$.*

Proof: Since $\mathcal{Q}$ is pseudo-rightsided we need only work with complete $\mathcal{Q}$-valued sets. Assume $\alpha\alpha_{p_1} = \beta\alpha_{p_2}$. Then we have

$$\begin{aligned} \alpha_{p_1}\alpha^\circ_{p_2} &= \delta_{q_1}\alpha_{p_1}\alpha^\circ_{p_2} \\ &\leq \alpha^\circ\alpha\alpha_{p_1}\alpha^\circ_{p_2} \\ &= \alpha^\circ\beta\alpha_{p_2}\alpha^\circ_{p_2} \\ &\leq \alpha^\circ\beta\delta_{q_2} \\ &= \alpha^\circ\beta. \end{aligned}$$

This tells us that $\widehat{\Psi}\Phi_\delta(\alpha,\beta) \leq \alpha^\circ\beta$.

Since we are working with complete $\mathcal{Q}$-valued sets the only singletons are morphisms of the form α_x.

We first examine the case where α and β are monomorphisms. It is easy to show that diagrammatically we have the following

$$\begin{tikzcd} & (X,\rho_X,\delta_X) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ {[q_1]} \arrow[ur] \arrow[dr, "\alpha^\circ\beta"'] & & {[q_2]} \arrow[ul] \arrow[dl] \\ & {[\beta^\circ\alpha\alpha^\circ\beta(*,*)]} \arrow[ul] \arrow[ur, "\beta^\circ\alpha\alpha^\circ\beta"'] & \end{tikzcd}$$

To show that the diagram commutes observe that

$$\begin{aligned}
\alpha\alpha^\circ\beta &= \alpha\alpha^\circ\beta\beta^\circ\alpha\alpha^\circ\alpha\alpha^\circ\beta \quad \text{by pseudo-rightsidedness}\\
&\leq \beta\beta^\circ\alpha\alpha^\circ\beta\\
&\leq \alpha\alpha^\circ\beta
\end{aligned}$$

Thus $\alpha^\circ\beta\beta^\circ\alpha\alpha^\circ\beta$, which is equal to $\alpha^\circ\beta$ by pseudo-rightsidedness, is less than or equal $\widehat{\Psi}\Phi_\delta(\alpha,\beta)$, hence we have $\widehat{\Psi}\Phi(\alpha,\beta)$ equal to $\Lambda(\alpha,\beta)$.

Examine the case where we have morphisms $\alpha_x : [q] \longrightarrow (X,\rho_X,\delta_X)$ and $\alpha_x : [x] \hookrightarrow (X,\rho_X,\delta_X)$ (it need not be the case that $q = \delta(x,x)$).

$$\begin{tikzcd}
 & (X,\rho_X,\delta_X) & \\
{[q]} \arrow[ur, "\alpha_x"] & & {[x]} \arrow[ul, "\alpha_x"'] \\
 & {[q]} \arrow[ul, "q"] \arrow[ur, "\delta(x,x)"'] &
\end{tikzcd}$$

This tells us that $\widehat{\Psi}\Phi(\alpha_x,\alpha_x) = \Lambda(\alpha_x,\alpha_x)$, since

$$\widehat{\Psi}\Phi(\alpha_x,\alpha_x) = q \;\&\; \delta(x,x) = \delta(x,x) = \alpha_x^\circ\alpha_x$$

.

Finally we bring the two preceding parts together. Let α_x and $\alpha_{x'}$ be two arbitrary singletons. We have

$$\begin{aligned}
\widehat{\Psi}\Phi(\alpha_x,\alpha_{x'}) &= \bigvee_{\gamma,\beta}\{\widehat{\Psi}\Phi(\alpha_x,\gamma) \;\&\; \widehat{\Psi}\Phi(\gamma,\beta) \;\&\; \widehat{\Psi}\Phi(\beta,\alpha_{x'})\}\\
&\geq \widehat{\Psi}\Phi(\alpha_x,\alpha_x) \;\&\; \widehat{\Psi}\Phi(\alpha_x,\alpha_{x'}) \;\&\; \widehat{\Psi}\Phi(\alpha_{x'},\alpha_{x'})\}
\end{aligned}$$

where the middle morphisms have the form $[x] \overset{\alpha_x}{\rightarrowtail} (X,\rho_X,\delta_X)$

$$\begin{aligned}
&= \alpha_x^\circ\alpha_x\alpha_x^\circ\alpha_{x'}\alpha_{x'}^\circ\alpha_{x'} \quad \text{by the previous cases}\\
&= \alpha_x^\circ\alpha_{x'}
\end{aligned}$$

∎

Theorem 6.3 *If $\mathcal{Q}$ is a pseudo-rightsided quantaloid, then $\mathcal{Q}$-Set is a reflective subcategory of* $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$.

Proof: We take the natural isomorphism ε, of Lemma 4.4, to be the counit of our adjunction. The unit, $\eta : \mathbf{1} \to \Phi\widehat{\Psi}$, is defined on a presheaf F by $\eta_F(q)(x) = \alpha_x$, where $\alpha_x(x', *) = \delta_F(x', x)$ (the Yoneda embedding). To see that this is natural observe that for any morphism τ, the first composite in the defining square is $\Phi\widehat{\Psi}(\eta(x)) = R_\tau \alpha_y$, from which we have

$$R_\tau \alpha_y(x, *) = \bigvee_z \{\delta_G(x, \tau_z) \ \& \ \delta_F(z, y)\} = \delta_G(x, \tau_y) = \alpha_{\tau_y}(x, *)$$

The second composite is simply

$$\eta(\tau_y) = \alpha_{\tau_y}$$

To show that the two triangle equalities hold, first examine the composite $\Phi_\delta \xrightarrow{\eta\Phi} \Phi\widehat{\Psi}\Phi_\delta \xrightarrow{\Phi\varepsilon} \Phi_\delta$. $\eta_{\Phi_\delta}(\beta : [q] \to (X, \rho_X, \delta_X))$ is the arrow $R_\beta(\gamma, *) = \widehat{\Psi}\Phi_\delta(\gamma, \beta) = \gamma^\circ\beta$. So we have that $\Phi_\varepsilon \eta_{\Phi_\delta}(\beta)$ is equal to $\varepsilon_\delta R_\beta$.

$$\begin{aligned}
\varepsilon_\delta R_\beta(x, *) &= \bigvee_\gamma \{\varepsilon_\delta(x, \gamma) \ \& \ R_\beta(\gamma, *)\} \\
&= \bigvee_\gamma \{\gamma(x, *) \ \& \ \gamma^\circ\beta(*, *)\} \\
&= \bigvee_\gamma \{\gamma\gamma^\circ\beta(x, *)\} \\
&= \delta_X \beta(x, *) \quad \text{atomic} \\
&= \beta(x, *).
\end{aligned}$$

Thus $\Phi_\delta \xrightarrow{\eta\Phi} \Phi\widehat{\Psi}\Phi_\delta \xrightarrow{\Phi\varepsilon} \Phi_\delta$ is the identity transformation on Φ. Now examine the other triangle $\widehat{\Psi}_F \xrightarrow{\widehat{\Psi}_{\eta_F}} \widehat{\Psi}\Phi\widehat{\Psi}_F \xrightarrow{\varepsilon_{\widehat{\Psi}_F}} \widehat{\Psi}_F$. Note that we have $\varepsilon_{\widehat{\Psi}_F}(x, \alpha) = \alpha(x, *)$, where $\alpha : [q] \to \widehat{\Psi}_F$, and

$$\begin{aligned}
\widehat{\Psi}_{\eta_F}(\beta, y) &= \bigvee \{p_1 p_2^* : \alpha_{p_1}\beta = \alpha_{p_2}\alpha_y\} \\
&= \widehat{\Psi}\Phi\widehat{\Psi}_F(\beta, \alpha_y) \\
&= \beta^\circ\alpha_y(*, *).
\end{aligned}$$

So the composite becomes

$$\begin{aligned}
\varepsilon_{\widehat{\Psi}_F}\widehat{\Psi}_{\eta_F}(x,y) &= \bigvee_{\alpha}\{\varepsilon_{\widehat{\Psi}_F}(x,\alpha)\ \&\ \widehat{\Psi}_{\eta_F}(\alpha,y)\} \\
&= \bigvee_{\alpha}\{\alpha(x,*)\ \&\ \alpha^{\circ}\alpha_y(*,*)\} \\
&= \bigvee_{\alpha}\left\{\alpha(x,*)\ \&\ \bigvee_{z}\{\alpha^{\circ}(*,z)\ \&\ \alpha_y(z,*)\}\right\} \\
&= \bigvee_{z}\left\{\bigvee_{\alpha}\{\alpha(x,*)\ \&\ \alpha^{\circ}(*,z)\}\ \&\ \alpha_y(z,*)\right\} \\
&= \bigvee_{z}\{\widehat{\Psi}_F(x,z)\ \&\ \widehat{\Psi}(z,y)\} \qquad \text{atomic} \\
&= \widehat{\Psi}_F(x,y).
\end{aligned}$$

Thus the triangle equalities hold and so $\mathcal{Q}$-*Set* is a reflective subcategory of $\mathbf{SET}^{\overline{\mathcal{Q}}^{op}}$. ∎

7 Sheaves

As we have previously seen, we know that when $\mathcal{E}$ is a Grothendieck topos, ($REL(\mathcal{E})$-*Set*), the $\mathcal{Q}$-*Set* construction on the category of relations of $\mathcal{E}$, yields a category that is isomorphic to $\mathcal{E}$. Also $REL(\mathcal{E})$ is a bounded complete distributive category of relations and, as such, we obtain an equivalence between the categories of $REL(\mathcal{E})$-valued sets and complete $REL(\mathcal{E})$-valued sets. By examining when the unit of the adjunction we constructed in the previous section is an isomorphism we arrive at the following definition of covers, matching families and amalgamations for the presheaf category $SET^{\overline{\mathcal{Q}}^{op}}$.

Let $\mathcal{H}$ be a complete Heyting algebra and h an element of $\mathcal{H}$. Traditionally a cover of h consist of a family of elements, $\langle k_i \in \mathcal{H}\rangle$, that satisfies $\vee k_i = h$.

Definition 7.1 *For $q : A \longrightarrow A$, a symmetric idempotent in $\mathcal{Q}$,* a cover *of q is a family of arrows in $KAR^*(\mathcal{Q})$*

$$\langle p_i : q \longrightarrow q_i \rangle,$$

such that

$$\bigvee_i (p_i^* \& p_i) \geq q.$$

◇

This matches the traditional notion of a cover for $\mathcal{H}$ a complete Heyting algebra. There is a morphism $k_i : h \longrightarrow h_i$, in the category $KAR^*(\mathcal{H})$, if and only if $k_i \leq h \wedge h_i$. So a cover satisfies the equation $\vee(k_i^* \wedge k_i) = \vee k_i = h$. Thus the k_i form a cover. On the other hand if the k_i form a traditional cover of h, then the morphisms $k_i : h \longrightarrow k_i$ constitute a cover.

Let F be a presheaf of a complete Heyting algebra, $\mathcal{H}$, and let $\langle k_i : h \longrightarrow h_i \rangle$ be a cover of h. Traditionally, a matching family for the cover consists of a family of elements $\langle x_i \in F(k_i) \rangle$, that must satisfy $(x_i)_{|(k_i \cap k_j)} = (x_j)_{|(k_i \cap k_j)}$. In our case this needs to be loosened slightly.

Definition 7.2 *Given a cover $\langle p_i : q \longrightarrow q_i \rangle$, a* matching family *for a presheaf F, is a family $\langle x_i \in F(q_i) \rangle$ such that*

1. $p_i p_j^* \;\leq\; \bigvee\{p_1 p_2^* : x_{i_{|p_1}} = x_{j_{|p_2}}\}$ *for all i, j*
2. *If $x_{i_{|p_1}} = x_{j_{|p_2}}$, then $p_1 p_2^* p_i \;\leq\; p_j$ for all i, j.*

where $p_1 : r \longrightarrow q_i$ and $p_2 : r \longrightarrow q_j$ are arrows in $\overline{\mathcal{Q}}$. ◇

It is easy to see that a traditional matching family of a complete Heyting algebra $\mathcal{H}$ satisfies conditions 1 and 2 above.

Now a matching family for a cover, $\langle k_i : h \longrightarrow h_i \rangle$, is a family of elements $\langle x_i \in F(h_i) \rangle$. It need not be the case that $x_{i|_{k_i \wedge k_j}} = x_{j|_{k_i \wedge k_j}}$, but we do have, by condition 1, $\vee\{k \mid x_{i|_k} = x_{j|_k}\} = k_i \wedge k_j$. For each $k_i \wedge k_j$, we obtain a cover of $k_i \wedge k_j$ (the k's) and a matching family in the traditional sense (the $x_{i|_k}$). Condition 2 is a technical condition needed at a latter stage which is trivially true in the Heyting case.

Definition 7.3 *Given a matching family $\langle x_i \in F(q_i) \rangle$ of the cover $\langle p_i : q \longrightarrow q_i \rangle$ an* amalgamation *is a $y \in F(q)$ such that for every $i \in I$;*

$$p_i \;=\; \bigvee\{p_1p_2^* : x_{i_{|p_1}} \;=\; y_{|p_2}\}.$$

$\diamondsuit$

Definition 7.4 *A presheaf F of a quantaloid $\mathcal{Q}$ is a* sheaf *if every matching family has a unique amalgamation. Denote the full subcategory determined by the sheaves of $SET^{\mathcal{Q}^{op}}$ by $SHV(\mathcal{Q})$.* $\diamondsuit$

Let $\mathcal{H}$ be a complete Heyting algebra, F a sheaf, $\langle k_i : h \to h_i\rangle$ a cover, and y an amalgamation of the matching family $\langle x_i \in F(h_i)\rangle$. If $h_i = \vee\{k \mid x_{i|_k} = y_k\}$, then the k's are a cover for h_i and the $x_{i|_k}$ are a matching family for this cover. This immediately implies that $x_i = y_{|_{k_i}}$, since both are amalgamations of this family. Thus F is a sheaf in the traditional sense.

Lemma 7.1 *F is a sheaf if and only if η_F is an isomorphism*

Proof: Assume that F is a sheaf. Then for a morphism $\alpha : [q] \to \widehat{\Psi}_F$ the following family of arrows,

$$\langle \alpha(x) : q \to \delta_F(x,x)\rangle_{x\in X}$$

is a cover of q and $\langle x \in F(\delta_F(x,x))\rangle$ is a matching family since we have

$$\alpha(x) \;\&\; \alpha^\circ(x) \;\le\; \delta_F(x,y) \;=\; \bigvee\{p_1p_2^* : x_{|_{p_1}} = y_{|_{p_2}}\}.$$

If $x_{p_1} = y_{|_{p_2}}$, then we have

$$\begin{aligned} p_1p_2^* \;\&\; \alpha(y) &\le \bigvee\{p_1p_2^* : x_{|_{p_1}} = y_{|_{p_2}}\} \;\&\; \alpha(y) \\ &= \delta_F(x,y) \;\&\; \alpha(y) \\ &\le \alpha(x). \end{aligned}$$

So there is a unique amalgamation y such that

$$\begin{aligned} \alpha(x) &= \bigvee\{p_1p_2^* : x_{|_{p_1}} = y_{|_{p_2}}\} \\ &= \delta_F(x,y) \\ &= \alpha_y(x). \end{aligned}$$

Thus η_F is an isomorphism.

Now assume that η_F is an isomorphism and let $\langle p_i : q \to q_i \rangle_{i \in I}$ be a cover of q and $\langle x_i \in F(q_i) \rangle$ a matching family.

Define $\alpha : [q] \to \widehat{\Psi}_F$ by

$$\alpha(x) = \begin{cases} p_i & \text{if } x = x_i \\ \perp & \text{otherwise} \end{cases}$$

$$\alpha^\circ(x) = \begin{cases} p_i^* & \text{if } x = x_i \\ \perp & \text{otherwise.} \end{cases}$$

The only difficult part in showing that α is a morphism is to show that $\widehat{\Psi}_F \alpha = \alpha$. This follows from the second matching family condition (We would like to dispense with the need for this condition but we have not been able to show that α is a morphism without it). Thus there is a unique $y \in F(q)$ such that

$$p_i = \alpha(x_i) = \alpha_y(x_i) = \bigvee \{ p_1 p_2^* : x_{i_{|p_1}} = y_{|p_2} \}.$$

And so F is a sheaf. ∎

This theorem shows us that a singleton $[q] \xrightarrow{\alpha} (X, \rho_X, \delta_X)$ represents the unique amalgamation of a matching family. The cover is given by the arrows $\alpha(x)$, the matching family by $x \in X$ such that $\alpha(x) \neq \perp$ and the unique amalgamation is the y such that $\alpha = \alpha_y$.

The following result is now automatic.

Theorem 7.2 *For $\mathcal{Q}$ a pseudo-rightsided quantaloid,*

$$\mathcal{Q}\text{-}Set \cong SHV(\mathcal{Q})$$

Proof: We want to show that for (X, ρ_X, δ_X) a $\mathcal{Q}$-SET, Φ_δ is a sheaf. We will show that $\Phi_\delta \xrightarrow{\eta_{\Phi_\delta}} \Phi\widehat{\Psi}\Phi_\delta$ is an isomorphism from which the result follows. We know that $\Phi\widehat{\Psi}(X, \rho_X, \delta_X)$ is complete. If there is a morphism $[q] \xrightarrow{A} (X, \rho_X, \delta_X)$ that equals $\widehat{\Psi}\Phi_\delta(-, \alpha) = \widehat{\Psi}\Phi_\delta(-, \beta)$, then we have $\alpha^\circ \alpha = \alpha^\circ \beta$, which implies that $\alpha \leq \beta$ from which equality follows. Thus η_{Φ_δ} is an isomorphism, hence Φ_δ is a sheaf. ∎

Pictorially we have the following diagram illustrating the relationship between presheaves and complete $\mathcal{Q}$-valued sets when $\mathcal{Q}$ is pseudo-rightsided.

$$
\begin{tikzcd}
MOD(MAT(\mathcal{DQ})) \arrow[rr, shift left, "\bigvee"] \arrow[rr, phantom, "\perp_{lax}"] & & MOD(MAT(\mathcal{Q})) \arrow[ll, shift left, "\mathcal{D}"] \\
\mathcal{DQ}\text{-}Set \arrow[u] \arrow[rr, "\bigvee"] & & Comp(\mathcal{Q}) \cong \mathcal{Q}\text{-}Set \arrow[u] \arrow[dll, shift left, "\Phi"] \arrow[d, shift left, "\Phi"] \\
SET^{\bar{\mathcal{Q}}^{op}} \arrow[u, "\Psi"] \arrow[urr, shift left, "\hat{\Psi}"] \arrow[urr, phantom, "\dashv"] \arrow[rr, shift left, "a"] \arrow[rr, phantom, "\perp"] & & SHV(\mathcal{Q}) \arrow[ll, shift left] \arrow[u, shift left, "\hat{\Psi}"] \arrow[u, phantom, "\sim"]
\end{tikzcd}
$$

The following is an immediate consequence of this and the work of Pitts[15].

Corollary 7.3 *For $\mathcal{Q}$ a bDCR, $SHV(\mathcal{Q})$ is a Grothendieck topos*

Corollary 7.4 *For $\mathcal{E}$ a Grothendieck topos $\mathcal{E} \cong SHV(REL(\mathcal{E}))$*

8 Conclusion

There have been different approaches to the exploration of the correct notion of sheaves for a quantale and, by extension, a quantaloid. Most start from the template of $\mathcal{Q}$-valued sets, which are matrices on $\mathcal{Q}$ satisfying some set of axioms. Mulvey and Nawaz[12] define $\mathcal{Q}$-valued sets for a *right Gelfand* quantale slightly differently than our approach. For them a $\mathcal{Q}$-valued set is a triple (X, ρ_X, δ_X) where $\delta : X \times X \to \mathcal{Q}$ and $\rho : X \to arrows(\mathcal{Q})$ satisfy a set of axioms. Note that ρ in this context does not pick out the single object of $\mathcal{Q}$, but an arrow of $\mathcal{Q}$. A presheaf for Mulvey and Nawaz is a set X and a pair of mappings $L : \mathcal{Q} \times X \to X$ and $R : X \times \mathcal{Q} \to X$ together with a mapping $E : X \to \mathcal{Q}$ that also satisfy a set of conditions. They then proceed to construct an equivalence between *sheaves* and complete $\mathcal{Q}$-valued sets. Gylys in [7, 8] expanded on this approach generalising to quantaloids and involutive quantaloids utilizing *partial maps.*

Another approach to this problem was taken by Borceux and Cruciani in [1]. They examined *right* quantales, introducing a notion of symmetry for $\mathcal{Q}$, without the presence of an involution. They define a $\mathcal{Q}$-valued set as a pair (X, δ_X), where X is a set and $\delta : X \times X \to \mathcal{Q}$. This satisfies $\delta\delta \leq \delta$ and a symmetry condition. A presheaf for them is a pair (v, F) where v is an arrow of $\mathcal{Q}$ and $F : v^{\downarrow} \to SET$. Then, with an appropriate definition of sheaves they construct an equivalence between $\mathcal{Q}$-valued sets and sheaves.

Our approach has been to generalise the work of Higgs. In our context a $\mathcal{Q}$-valued set is an object in the completion of $\mathcal{Q}$ as a quantaloid with respect to coproducts and the splitting of symmetric idempotents. A sheaf is a set-valued functor from the category of maps in the symmetric idempotent splitting completion of $\mathcal{Q}$ to SET satisfying appropriate conditions. Of interest is how far can this approach be extended without the use of symmetry.

References

[1] **F. Borceux and R. Cruciani** Sheaves on a Quantale, *Cahiers. Top. Cat*, **34-3**, 1993. 209-228

[2] **U. Berni-Canani, F. Borceux and R. Succi-Cruciani** A Theory of Quantale Sets, *J. Pure and Applied Algebra* **62**, 1989. 123-136

[3] **R. Betti, A. Carboni, R. Street and R. Walters** Variation Through Enrichment, *J. Pure and Applied Algebra.* **29**, 1983. 109 - 127

[4] **A. Carboni and R. F. C. Walters** Cartesian Bicategories I, *J. Pure and Applied Algebra*, **49**, 1987. 11-32

[5] **W. Dale Garraway** Generalized Supremum Enriched Categories and Their Sheaves, *PhD. Thesis, Dalhousie University* 2002

[6] **P. Freyd** Categories Allegories, *North Holland Pub. Co.* 1990

[7] **R. P. Gylys** Sheaves on Quantaloids, *Lithuanian Math. J.*, **40-2**, 2000. 133-171

[8] **R. P. Gylys** Sheaves on Involutive Quantaloids, Lithuanian Math J. **41-1**, 2001. 44 - 69

[9] **D. Higgs** Injectivity in the Topos of Complete Heyting Algebra Valued Sets, *Can. J. Math*,**36-3**, 1984. 550-568

[10] **S. Merovitz** *$\mathcal{H}$-valued Sets and the Associated Sheaf Functor*, Masters Thesis, Dalhousie University, 1996

[11] **C.J. Mulvey** &, *Rend. Circ. Mat. Palermo Suppl. No. 2* 1986. 99 - 104

[12] **C.J. Mulvey and M. Nawaz** Quantales: Quantal Sets. Non-classical logics and their applications to fuzzy subsets, Theory Decis. Lib. Ser. B Math. Statist. Methods, 32, Kluwer Acad. Publ., Dordrecht, 1995. 159-217

[13] **C. J. Mulvey and J. W. Pelletier** a quantisation of the calculus of relations, Category Theory 1991. Montreal PQ, 1991 345-360, CMS conf. proc. 13, *Amer. Math. Soc.*, Providence RI, 1992.

[14] **M. Nawaz** *Quantales: Quantale Sets*, PhD Thesis, University of Sussex, 1985

[15] **A.M. Pitts** Applications of Sup-Lattice Enriched Category Theory to Sheaf Theory, *Proc. London Math. Soc.(3)* **57**, 1988. 433-480

[16] **K. Rosenthal** The Theory of Quantaloids, *Pitman Research Notes in Math* **No. 348**, 1996

[17] **G. Van den Bossche** Quantaloids and Non-Commutative Ring Representations, *Applied Categorical Structures*, **3**, 1995. 305-320

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