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ON CLONES DETERMINED BY THEIR INITIAL SEGMENTS

To Jiří Adámek on his 60th birthday

by J. SICHLER and V. TRNKOVÁ

Abstract

Les réponses respectives aux questions de savoir quand l'existence d'isomorphismes de clones locaux implique l'existence d'un isomorphisme de clones global diffère pour les clones terminaux, les clones polynomiaux et les clones centralisateurs des algèbres universelles finitaires. Dans chacun de ces trois cas, la réponse est étroitement liée au type de similarité des algèbres considérées.

1 Introduction

When do local isomorphisms of two clones imply their isomorphism?

Recall that an abstract clone with a base object a is a small category k whose object set obj $k = \{a^0, a, a^2, \ldots\}$ consists of all finite powers of its base object a, in which for every $n \in \omega = \{0, 1, 2, \ldots\}$ a unique n-tuple of product projections $\pi_i^{(n)} : a^n \to a$ is specified and enumerated by all $i \in n = \{0, 1, \ldots, n-1\}$. F. W. Lawvere called the abstract clones algebraic theories, and employed them to present his elegant categorical view of varieties of finitary algebras in [6, 7].

For any clone k and $n \in \omega$, let k_n denote the full subcategory of k determined by the set $\{a^0, a, \ldots, a^{n-1}\}$. We call k_n the n-segment of k.

Let k and k' be clones with respective base objects a and a' and respective projections $\pi_i^{(n)} \in k(a^n, a)$ and $(\pi_i^{(n)})' \in k((a')^n, a')$ for all $i \in n \in \omega$. A functor $H : k \to k'$ is a clone homomorphism if

$$H(a^n) = (a')^n \text{ and } H(\pi_i^{(n)}) = (\pi_i^{(n)})' \text{ for all } i \in n \in \omega.$$

An isomorphism of k onto k' is a clone homomorphism that is one-to-one and

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surjective. If there is a clone isomorphism of k onto k' we write $k \cong k'$. Homomorphisms and isomorphisms of clone segments are defined analogously. We say that clones k and k' are locally isomorphic if $k_n \cong k'_n$ for every $n \in \omega$.

P. Hall [5] originally introduced what we call here a clone on a set X as a system $\mathcal{C}$ consisting of finitary operations $f : X^n \to X$ with $n \in \omega$ containing all Cartesian projections $p_i^{(n)} : X^n \to X$ given by $p_i^{(n)}(x_0, \ldots, x_{n-1}) = x_i$ for $i \in n \in \omega$, and closed under the operations S_n^m of superposition defined for any $f_0, \ldots, f_{m-1} : X^n \to X$ and any $g : X^m \to X$ with $m, n \in \omega$ by

$$S_n^m(g; f_0, \ldots, f_{m-1})(x_0, \ldots, x_{n-1})$$
$$= g(f_0(x_0, \ldots, x_{n-1}), \ldots, f_{m-1}(x_0, \ldots, x_{n-1}))$$

for every $(x_0, \ldots, x_{n-1}) \in X^n$.

Any clone $\mathcal{C}$ on a set X is isomorphic to an abstract clone with a base object X, by means of simply extending $\mathcal{C}$ by all maps $f : X^n \to X^m$ such that $p_i^{(m)} \circ f \in \mathcal{C}$ for $i \in m \in \omega$ and then forgetting the actual form of the maps forming the extended category. We denote $f = f_0 \dot{\times} \cdots \dot{\times} f_{m-1}$ the unique member of $\mathcal{C}$ satisfying $p_j^{(m)} \circ (f_0 \dot{\times} \cdots \dot{\times} f_{m-1}) = f_j$ for every $j \in m$.

Conversely, any abstract clone k with a base object a is isomorphic to a clone on the underlying set X of the algebra $A = \mathcal{F}_k(\omega)$ with ω free generators in the variety determined by the abstract clone k. The collection of all term functions of A is then a clone $\mathcal{C}$ on the set X that provides an alternative description of k.

Any algebra $A = (X, \{o_\sigma \mid \sigma \in \Sigma\})$ whose basic operations o_σ are all finitary determines three natural clones on its underlying set X:

- the clone $\mathrm{t}A$ of all its term functions, that is, the least clone on X containing all basic operations o_σ of the algebra A;
- the clone $\mathrm{p}A$ of all its polynomial functions, that is, the least clone on X containing all basic operations o_σ of A and all constant maps $X^n \to X$ for each $n \in \omega$;
- the centralizer clone $\mathrm{c}A$, that is, the least clone on X containing all homomorphisms $A^n \to A$ for each $n \in \omega$.

Clones on a set or, equivalently, clones of all term operations of algebras have become central in algebraic investigations, cf. [3, 4, 11, 13, 14, 15], for instance. Centralizer clones of algebras and of algebraic systems were recently characterized in [21].

In Section 2, we show that locally isomorphic polynomial clones of two algebras of any bounded finitary similarity type must always be isomorphic, and also give an example of two algebras whose finitary type is unbounded and whose polynomial clones are locally isomorphic but not isomorphic. Clones of term operations behave differently: Section 3 shows that locally isomorphic term clones of algebras with at least one binary operation and countably many unary operations need not be isomorphic. And in Section 4 we show that algebras with just two unary operations can have non-isomorphic but locally isomorphic centralizer clones. Aspects of representability of clones from a given class $\mathbb{C}$ as centralizer clones of algebras are considered in Section 5. The concluding Section 6 relates our results to elementary equivalence of clones and their segments.

All category theory notions we use here can be found in [1], of course.

2 Clones of polynomial functions

For any algebra A on a set X, the polynomial clone pA is a clone that includes all constant maps between any two finite powers of X. More precisely, we have $X^0 = \{0\}$ and for every $n \in \omega$ the unique constant map $\tau_n : X^n \to X^0 = \{0\}$ with the value 0, and for every $\mathbf{x} = (x_1, \dots .x_n) \in X^n$ with $n \in \omega$ the map $\xi_{\mathbf{x}}$ given by $\xi_{\mathbf{x}}(0) = \mathbf{x}$. The constant maps $X^m \to X^n$ are then exactly the maps of the form $\xi_{\mathbf{x}} \circ \tau_m$ with $\mathbf{x} \in X^n$. We say that any such clone on the set X has all constants.

All isomorphisms of clones (or clone segments) with all constants have a specific form.

Lemma 2.1. *Let k and k' be clones with all constants on the respective sets X and X' (or clone segments with all constants containing at least the first powers X and X' of these sets). Then $K : k \to k'$ is a clone (segment) isomorphism of k onto k' if and only if there is a bijection $\beta : X \to X'$ such that*

$$K(f) = \beta^n \circ f \circ (\beta^{-1})^m \textit{ for every } f \in k(X^m, X^n),$$

where $\beta^l : X^l \to (X')^l$ denotes the bijection given by $\beta^l(\mathbf{x})(i) = \beta(\mathbf{x}(i))$ for every $i = 1, \dots, l$.

Proof. Let K have the described form. If $f \in k(X^m, X^n)$ and $g \in k(X^n, X^p)$, then $K(f) = \beta^n \circ f \circ (\beta^{-1})^m$ and $K(g) = \beta^p \circ g \circ (\beta^{-1})^n$, so that $K(g) \circ K(f) = \beta^p \circ g \circ f \circ (\beta^{-1})^n = K(g \circ f)$. And for $f = 1_{X^m}$ we obviously have $K(f) = 1_{(X')^m}$, so that K is a functor. For any projection $p_i^{(m)} : X^m \to X$ and any $\mathbf{x} \in X^m$ we have $[\beta \circ p_i^{(m)} \circ (\beta^{-1})^m(\mathbf{x})](i) = [\beta \circ p_i^{(m)}(\beta^{-1} \circ \mathbf{x})](i) = \beta \circ \beta^{-1}(\mathbf{x}(i)) = \mathbf{x}(i)$ for every $i = 1. \dots, m$, so that $K(p_i^{(m)}) : (X')^m \to X'$ is the i-th projection. Therefore K preserves all products. The functor K^{-1} given by $K^{-1}(f') = (\beta^{-1})^n \circ f' \circ \beta^m$ for every $f' \in k'((X')^m, (X')^n)$ is the obvious inverse of K. As a result, the functor K is a clone (segment) isomorphism.

For the converse, let $K : k \to k'$ be a clone (segment) isomorphism. Since k contains X and k' contains X', we have $K(\xi_x) = \xi_{\beta(x)}$ for some bijection $\beta : X \to X'$. For any $\mathbf{x} = (x_1, \ldots, x_m) \in X^m$ we have $\xi_{\mathbf{x}} = \xi_{x_1} \dot{\times} \cdots \dot{\times} \xi_{x_m}$, so that $K(\xi_{\mathbf{x}}) = \xi_{\beta(x_1)} \dot{\times} \cdots \dot{\times} \xi_{\beta(x_m)} = \xi_{\beta^m(\mathbf{x})}$ because K preserves products. If $f \in k(X^m, X^n)$ and $\mathbf{x} \in X^m$, then $f \circ \xi_{\mathbf{x}} = \xi_{f(\mathbf{x})}$ and hence $\xi_{\beta^n(f(\mathbf{x}))} = K(\xi_{f(\mathbf{x})}) = K(f) \circ K(\xi_{\mathbf{x}}) = K(f) \circ \xi_{\beta^m(\mathbf{x})}$. Evaluating this equality at $0 \in X^0$, we get $\beta^n(f(\mathbf{x})) = K(f)(\beta^m(\mathbf{x}))$ for every $\mathbf{x} \in X^m$, from which $\beta^n \circ f = K(f) \circ \beta^m$ follows. Since $(\beta^m)^{-1} = (\beta^{-1})^m$, we conclude that $K(f) = \beta^n \circ f \circ (\beta^{-1})^m$, as claimed. □

Theorem 2.2. *For any two algebras having a bounded finitary similarity type, their clones of polynomial functions are isomorphic whenever they are locally isomorphic.*

Proof. Let $A = (X, \Sigma)$ and $B = (Y, \Sigma')$ be finitary algebras such that their respective sets of k-ary operations satisfy $\Sigma_k = \Sigma'_k = \emptyset$ for all $k > \max\{2, m\}$, and let $K : (\mathrm{p}A)_m \to (\mathrm{p}B)_m$ be an isomorphism of the respective segments of their polynomial operations. These segments thus contain all basic operations of these algebras. By Lemma 2.1, there is a bijection β of X onto Y such that $K(p) = \beta \circ p \circ (\beta^{-1})^k$ for every k-ary polynomial function of A with $k < m$.

Define, for any r-ary polynomial function $p \in \mathrm{p}A$,

$$K^*(p)(y_1, \ldots, y_r) = \beta \circ p(\beta^{-1}(y_1), \ldots, \beta^{-1}(y_r)).$$

Then K^* is one-to-one, and hence the image of $\mathrm{p}A$ under K^* is a clone on the set Y. Since K^* extends the segment isomorphism $K : (\mathrm{p}A)_m \to (\mathrm{p}B)_m$ of the segment $(\mathrm{p}A)_m$ containing all basic term operations of A and because K^* preserves the composition and assigns constants via the bijection β of X onto Y, all of the image of $\mathrm{p}A$ under K^* is contained in $\mathrm{p}B$.

Let $H : (\mathrm{p}B)_m \to (\mathrm{p}A)_m$ be the inverse of K. Then H is associated with the inverse β^{-1} of β, see Lemma 2.1. Let $H^* : \mathrm{p}B \to \mathrm{p}A$ be given by

$$H^*(q)(x_1, \ldots, x_r) = \beta^{-1} \circ q(\beta(x_1), \ldots, \beta(x_r)) \text{ for any } q \in \mathrm{p}B(B^r, B).$$

The clone morphism H^* thus extends H and, by the symmetry of the hypothesis, it maps $\mathrm{p}B$ into $\mathrm{p}A$. Since H^* and K^* are each other's inverses, the clone $\mathrm{p}A$ is isomorphic to the clone $\mathrm{p}B$. □

Observation 2.3. *If A is a finite algebra and if $\mathrm{p}A$ is locally isomorphic to $\mathrm{p}A'$, then A' is finite and $\mathrm{p}A$ is isomorphic to $\mathrm{p}A'$.*

Proof. The underlying set X' of A' is clearly bijective to the underlying set X of

A, so that the set of all bijections of X onto X' is finite. One of the bijections ρ_n with $n \in \omega$ associated with the segment isomorphism $K_n : (\mathrm{p}A)_n \longrightarrow (\mathrm{p}A')_n$ must therefore occur infinitely many times. Let $\rho = \rho_n$ be such a bijection. Then the infinite set $S = \{m \in \omega \mid \rho_m = \rho\}$ is cofinal in ω, and we write $S = \{m_0 < m_1 < \ldots\}$. Since the segment isomorphisms $K_{m_{i+1}}$ and K_{m_i} share their defining bijection ρ, the isomorphism $K_{m_{i+1}}$ extends K_{m_i} for every $i \in \omega$. Setting

$$K = \bigcup \{K_{m_i} \mid i \in \omega\}$$

thus gives rise to a product preserving isofunctor of k onto k'. □

The example below shows that, for infinite algebras, it is essential that the similarity type of at least one of them be bounded.

Example 2.4. *There exist algebras of the same unbounded countable finitary similarity type whose polynomial clones are locally isomorphic but not isomorphic.*

This example is presented in 2.4.1–2.4.7 below.

2.4.1. We define two algebras A and A' with the respective underlying sets

$$X = \{0\} \cup \{a_m, b_m \mid m = 1, 2, \ldots\} \quad \text{and} \quad X' = X \cup \{a_0, b_0\}.$$

For any $m = 1, 2, \ldots$ and $k \in \{1, \ldots m\}$, the basic k-ary operations $\alpha_{m,k}$ of A are defined by

$$\alpha_{m,k}(x_0, \ldots, x_{k-1}) = \begin{cases} b_m & \text{if } (x_0, \ldots, x_{k-1}) = (a_m, \ldots, a_m), \\ 0 & \text{otherwise.} \end{cases}$$

The algebra A' on the set X' will have the operations $\alpha_{m,k}$ defined as above (that is, the operations $\alpha_{m,k}$ of A are extended by 0 to the appropriate powers of the underlying set X' of A') and, in addition, for any $k \geq 1$, a single k-ary operation $\alpha_{0,k}$ given by

$$\alpha_{0,k}(x_0, \ldots, x_{k-1}) = \begin{cases} b_0 & \text{if } (x_0, \ldots, x_{k-1}) = (a_0, \ldots, a_0), \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that any of these operations $\alpha_{m,k}$ are totally symmetric in the sense that $\alpha_{m,k}(x_0, \ldots, x_{k-1}) = \alpha_{m,k}(x_{p(0)}, \ldots, x_{p(k-1)})$ for any permutation p of the set k, and that $\mathrm{Im}(\alpha_{m,k}) = \{0, b_m\}$ for every $m \in \omega$ and every appropriate k.

2.4.2. *The k-ary operations $\alpha_{m,k}$ with $1 \leq k \leq m$ and the operations $\alpha_{0,k}$ with $k \geq 1$ depend on all their k variables.*

Proof. Replacing any single entry of $(a_m, \ldots, a_m)$ by 0 changes the value of $\alpha_{m,k}$ from b_m to 0. □

2.4.3. *The polynomial clone* pA *of the algebra* A *is the set* $\mathcal{S}$ *consisting of all projections, all constants, and of all composites* $\alpha_{m,k} \circ \pi^{[\phi]}$, *where* $\phi : k \to p$ *is an arbitrary mapping (recall that* $\pi^{[\phi]}(\sigma) = \sigma \circ \phi$ *for every* $\sigma \in X^p$*) and* $1 \leq k \leq m$.

Proof. We show that the collection $\mathcal{S}$ is closed under the superposition operations

$$S_r^p(g; f_0, \dots, f_{p-1}) = g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{p-1}).$$

If g is a constant then $g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{p-1})$ is constant, while if $g = \pi_j^{(p)}$ is a projection then $g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{p-1}) = f_j$.

So let us assume that $g = \alpha_{m,k} \circ \pi^{[\phi]}$ for some $\phi : k \to p$. Let $f_0, \dots, f_{p-1}$ be r-ary members of $\mathcal{S}$, and let us denote $h_i = f_{\phi(i)}$ for $i \in k$. Then $S_r^p(g; f_0, \dots, f_{p-1}) = \alpha_{m,k} \circ (h_0 \dot{\times} \cdots \dot{\times} h_{k-1})$.

If $h_i = \alpha_{m',r} \circ \pi^{[\psi]}$ for some i and ψ, and if h_i is not constant, then the only non-zero value of h_i is $b_{m'} \neq a_m$ and hence the only value of $F = \alpha_{m,k} \circ (h_0 \dot{\times} \cdots \dot{\times} h_{k-1})$ is 0. If h_i is constant, then its value must be a_m for otherwise the only value of F is 0. In the remaining case, every h_i is either a projection or a constant with the value a_m and at least one h_i is a projection, for otherwise F would be constant.

Since $\alpha_{m,k}$ is totally symmetric, we may assume that $h_i = \pi_{\psi(i)}^{(r)}$ for $i \in l$ and that $h_l, \dots, h_{k-1}$ are the constants with the value a_m. Thus if $\bar{a}_m$ denotes the r-ary constant with the value a_m then

$$F(x_0, \dots, x_{r-1}) = \alpha_{m,k} \circ (\pi_{\psi(0)}^{(r)} \dot{\times} \cdots \dot{\times} \pi_{\psi(l-1)}^{(r)} \times \bar{a}_m \dot{\times} \cdots \dot{\times} \bar{a}_m)(x_0, \dots, x_{r-1}).$$

The only possible non-zero value of F is b_m and this value occurs exactly when $x_{\psi(t)} = a_m$ for all $t \in l$. Therefore $F = \alpha_{m,l} \circ \pi^{[\psi]}$ for some $1 \leq l < r$ and $\psi : l \to r$.

Altogether, the system $\mathcal{S}$ is closed under all the operations S_r^p. Since $\mathcal{S}$ contains all constants, all projections and all basic operations of A, we have $\mathcal{S} =$ pA, as claimed. □

The proof of 2.4.4 below is a simple extension of the proof of 2.4.3.

2.4.4. *The polynomial clone* pA' *of the algebra* A' *is the set* $\mathcal{S}'$ *consisting of all projections, all constants, and of all composites* $\alpha_{m,k} \circ \pi^{[\phi]}$ *with* $1 \leq k \leq m$ *and all* $\alpha_{0,k} \circ \pi^{[\phi]}$ *with* $k \geq 1$, *where* $\phi : k \to p$ *is an arbitrary mapping.* □

For any $k \geq 1$ and appropriate m, any composite $\beta = \alpha_{m,k} \circ \pi^{[\phi]}$ is a two-valued function, and this is because $\beta(a_m, \dots, a_m) = b_m$ and $\beta(0, \dots, 0) = 0$.

2.4.5. *Any p-ary two-valued polynomial function* β *of* A *depending on all its variables has the form* $\beta = \alpha_{m,p}$ *for some* $1 \leq p \leq m$. *In addition to these, the only*

p-ary two-valued polynomial functions β of A' depending on all their variables have the form $\beta = \alpha_{0,p}$ with $p \geq 1$.

Proof. Let β be a p-ary two-valued polynomial function of A. Then β has the form $\beta = \alpha_{m,k} \circ \pi^{[\phi]}$ for some $\phi : k \longrightarrow p$, by 2.4.3, that is, $\beta(\sigma) = \alpha_{m,k}(\sigma \circ \phi)$ for every $\sigma \in X^p$. If $\phi : k \longrightarrow p$ is not surjective, then $\beta(x_0, \ldots, x_{p-1}) = \alpha_{m,k}(x_{\phi(0)}, \ldots, x_{\phi(k-1)})$ does not depend on any variable x_i with $i \notin \mathrm{Im}(\phi)$. Thus ϕ is surjective.

Suppose that the surjective map ϕ is not injective, and let $\gamma : p \longrightarrow k$ be such that $\phi \circ \gamma = 1_p$ is the identity map. For any sequence $\sigma \in X^p$ define a sequence $\sigma^* \in X^k$ by setting

$$\sigma^*(i) = \begin{cases} \sigma(i) & \text{if } i = \gamma(j) \text{ for some } j \in p, \\ a_m & \text{if } i \notin \mathrm{Im}(\gamma). \end{cases}$$

Thus $\sigma^* \circ \gamma = \sigma$ and hence $\beta(\sigma) = \alpha_{m,k}(\sigma \circ \phi) = \alpha_{m,k}(\sigma^* \circ \gamma \circ \phi)$. Since $\sigma^* \in X^k$ is the constant sequence with the value a_m exactly when $\sigma^* \circ \gamma \circ \phi \in X^p$ is the constant with the value a_m, it follows that $\alpha_{m,k}(\sigma^* \circ \gamma \circ \phi) = \alpha_{m,k}(\sigma^*)$. Furthermore, since $\sigma^* \in X^k$ is the constant with the value a_m exactly when $\sigma \in X^p$ is the constant with the value a_m, we also have $\alpha_{m,k}(\sigma^*) = \alpha_{m,p}(\sigma)$. Altogether, $\beta(\sigma) = \alpha_{m,p}(\sigma)$ for every $\sigma \in X^p$, so that $\beta = \alpha_{m,p}$.

If the surjective map ϕ is also injective, then $\beta = \alpha_{m,k} \circ \pi^{[\phi]}$ is obtained from $\alpha_{m,k}$ by a permutation of its variables. But $\alpha_{m,k}$ is totally symmetric and hence $\beta = \alpha_{m,k}$ in this case. This proves the claim about the algebra A. The proof for A' is similar. □

Now we strengthen the previous claim a little.

(A) Any two-valued polynomial function $\beta : A^p \longrightarrow A$ of A has the form $\beta = \alpha_{m,k} \circ \pi^{[\psi]}$ for some injective $\psi : k \longrightarrow p$ and some $1 \leq k \leq m$. In addition to these, the algebra A' has two-valued p-ary polynomial functions of the form $\beta' = \alpha_{0,k} \circ \pi^{[\psi]}$ with $k \geq 1$ and an injective $\psi : k \longrightarrow p$.

Proof of (A). If $\beta(x_0, \ldots, x_{p-1})$ depends on its variables $x_0, \ldots, x_{k-1}$ and no others, then $\beta(x_0, \ldots, x_{p-1}) = \bar{\beta}(x_0, \ldots, x_{k-1})$ for some k-ary $\bar{\beta}$ depending on all its variables and having the same two values as β. This gives rise to an injective $\psi : k \longrightarrow p$ such that $\beta = \bar{\beta} \circ \pi^{[\psi]}$. But any k-ary two-valued polynomial function $\bar{\beta}$ depending on all its variables has the form $\bar{\beta} = \alpha_{m,k}$ for some $m \geq k$, by 2.4.5, so that $\beta = \alpha_{m,k} \circ \pi^{[\psi]}$ as claimed. The proof for the algebra A' is similar. □

2.4.6. *The clone* pA' *is not isomorphic to* pA.

Proof. For any two-valued polynomial functions β and β' of respective arities k and k' depending on all their variables we define

$$\beta' < \beta \equiv_{\mathrm{df}} k' < k \text{ and } \mathrm{Im}(\beta') = \mathrm{Im}(\beta).$$

Thus $\alpha_{m,k'} < \alpha_{m,k}$ exactly when $k' < k \leq m$ and, according to 2.4.5, no other comparable pairs of two-valued polynomial functions of A depending on all their variables exist. For the algebra A' we have the additional comparable pairs $\alpha_{0,k'} < \alpha_{0,k}$ with $1 \leq k' < k$, see 2.4.5. Therefore all $<$-chains in pA are finite while pA' has an infinite such chain. And the relation $<$ must be preserved by any polynomial clone isomorphism. □

2.4.7. *For every* $N \geq 1$, *the* $(N+1)$*-segments of the clones* pA *and* pA' *are isomorphic.*

Proof. In view of Lemma 2.1, we only need to exhibit a bijection $\gamma : X' \to X$ such that

$$\gamma^{-1} \circ g \circ \gamma^k \in (\mathrm{p}A')_{N+1} \text{ for every } k\text{-ary } g \in (\mathrm{p}A)_{N+1},$$

and such that for the inverse γ^{-1} of γ we have

$$\gamma \circ h \circ (\gamma^{-1})^k \in (\mathrm{p}A)_{N+1} \text{ for every } k\text{-ary } h \in (\mathrm{p}A')_{N+1}.$$

This is clear for any bijection γ in case when g or h is a constant or a projection. In fact, $(\gamma^{-1})^k \circ \pi^{[\psi]} \circ \gamma^p = \pi^{[\psi]}$ for any generalized projection $\pi^{[\psi]}$ determined by the map $\psi : k \to p$ with $k, p \geq 1$. In general, we denote $K(g) = (\gamma^{-1})^k \circ g \circ \gamma^p$.

For the bijection $\phi : \omega \to \omega \setminus \{0\}$ given by

$$\phi(n) = \begin{cases} N+1 & \text{if } n = 0, \\ n & \text{if } 1 \leq n \leq N, \\ n+1 & \text{if } n > N, \end{cases}$$

we define our particular bijection $\gamma : X' \to X$ by setting $\gamma(0) = 0$, and $\gamma(a_n) = a_{\phi(n)}$ and $\gamma(b_n) = b_{\phi(n)}$ for every $n \in \omega$.

Next we show that

$$\text{(e)} \qquad \gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k = \alpha_{\phi^{-1}(m),k} \text{ whenever } 1 \leq k \leq m.$$

For $m = N+1$ and every $k \in \{1, \ldots, N+1\}$ we have

$$\begin{aligned} \gamma^{-1} \circ \alpha_{N+1,k} \circ \gamma^k(a_0, \ldots, a_0) &= \gamma^{-1} \circ \alpha_{N+1,k}(a_{N+1}, \ldots, a_{N+1}) \\ &= \gamma^{-1}(b_{N+1}) = b_0 \end{aligned}$$

and $\gamma^{-1} \circ \alpha_{N+1,k} \circ \gamma^k(\sigma) = 0$ for any $\sigma \neq (a_0, \ldots, a_0)$. Therefore $\gamma^{-1} \circ \alpha_{N+1,k} \circ \gamma^k = \alpha_{0,k}$ in the case of $1 \leq k \leq m = N+1$.

Next suppose that $1 \leq m \leq N$. For every $k \in \{1, \ldots, m\}$ we have

$$\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k(a_m, \ldots, a_m) = \gamma^{-1} \circ \alpha_{m,k}(a_m, \ldots, a_m) = \gamma^{-1}(b_m) = b_m$$

and $\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k(\sigma) = 0$ for any $\sigma \neq (a_m, \ldots, a_m)$. Therefore $\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k = \alpha_{m,k}$ if $1 \leq k \leq m \leq N$.

Finally, suppose that $m > N+1$. Then $m - 1 > N$, and because $\gamma(a_{m-1}) = a_{\phi(m-1)} = a_m$ for any such m, for every $k \in \{1, \ldots, m\}$ we have

$$\begin{aligned}\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k(a_{m-1}, \ldots, a_{m-1}) &= \gamma^{-1} \circ \alpha_{m,k}(a_m, \ldots, a_m)\\ &= \gamma^{-1}(b_m) = b_{m-1}\end{aligned}$$

and $\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k(\sigma) = 0$ for any $\sigma \neq (a_{m-1}, \ldots, a_{m-1})$, so that $\gamma^{-1} \circ \alpha_{m,k} \circ \gamma^k = \alpha_{\phi^{-1}(m),k}$ also in this case. This completes the proof of (e).

Now let $\beta : A^p \to A$ be a two-valued function from the segment $(\mathrm{p}A)_{N+1}$, that is, let $1 \leq p \leq N$. Then $\beta = \alpha_{m,k} \circ \pi^{[\psi]}$ for some $1 \leq k \leq m$ and some injective map $\psi : k \to p$, by 2.4.5(A). But then $1 \leq k \leq N$, so that $\alpha_{m,k} \in (\mathrm{p}A)_{N+1}$ and $\pi^{[\psi]} \in (\mathrm{p}A)_{N+1}$. Since K is a functor preserving all generalized projections $\pi^{[\psi]}$, using (e) we conclude that

$$K(\beta) = K(\alpha_{m,k}) \circ \pi^{[\psi]} = \alpha_{\phi^{-1}(m),k} \circ \pi^{[\psi]} \in (\mathrm{p}A')_{N+1}.$$

Since $K^{-1}(\alpha_{n,k}) = \gamma \circ \alpha_{n,k} \circ (\gamma^{-1})^k = \alpha_{\phi(n),k}$ for every basic operation $\alpha_{n,k}$ of A', a similar argument shows that $K^{-1}(\beta') \in (\mathrm{p}A)_{N+1}$ for every two-valued polynomial function $\beta' : (A')^p \to A'$ of A'. $\square$

Remark 2.5. We do not know whether or not the polynomial clone of an algebra with finitary unbounded similarity type can be locally isomorphic, but not isomorphic to the polynomial clone of an algebra whose finitary type is bounded.

3 Clones of term functions

For algebras of bounded finitary similarity type, their clones of term functions behave unlike clones of polynomial functions: non-isomorphic clones of term functions can be locally isomorphic. The first example of algebras of unbounded finitary similarity type confirming this fact was constructed in [20]. Example 3.2 below offers algebras with one binary and countably many unary operations. In view of the remark below, the binary operation is needed.

Remark 3.1. If k is a unary abstract clone with a base object a, that is, if k is the clone generated by the monoid $k(a,a)$, and if k' is a clone locally isomorphic to k, then k' must be isomorphic to k. This is because every $f \in k(a^n, a)$ has the form $f = g \circ \pi_i^{(n)}$ for some $g \in k(a,a)$, and the existence of a segment isomorphisms implies that the clone k' has the same property. This of course applies to term clones of any unary algebras, and hence shows that the use of a binary operation in Example 3.2 below is necessary.

Example 3.2. *There exist two finitary algebras with one binary and countably many unary operations whose clones of term functions are locally isomorphic but not isomorphic.*

Proof. For an algebra $A = (X, \Sigma)$ on the set X with the set Σ of basic operations, we denote $\Sigma_n \subseteq \Sigma$ the set of all its n-ary basic operations.

In our example, algebras A and A' have sets of operations

$$\Sigma_0 = \Sigma_0' = \{0\}, \qquad \Sigma_1 = \{\sigma_k \mid k = 1, 2, \ldots\} \cup \{\sigma_\infty\},$$
$$\Sigma_2 = \Sigma_2' = \{\beta\}, \qquad \Sigma_1' = \Sigma_1 \setminus \{\sigma_\infty\},$$

and are thus of the same similarity type.

The actual algebras A and A' will be the free algebras on ω generators in two distinct varieties we now describe. To write the defining identities of these varieties, we define some particular terms first. In what follows, we write every $(n+1)$-ary term t as $t(x_0, \ldots, x_n)$ with exactly this sequence of variables.

First we inductively set

$$\beta^1(x_0, x_1) = \beta(x_0, x_1),$$
$$\beta^n(x_0, \ldots, x_n) = \beta(\beta^{n-1}(x_0, \ldots, x_{n-1}), x_n).$$

Then, for any sequence $\rho = (\rho_0, \ldots, \rho_n)$ with $\rho_i \in \{1\} \cup \{\sigma_k \mid k \geq 1\}$ for every $i = 0, \ldots, n$, we denote

$$p_n^\rho(x_0, \ldots, x_n) = \beta^n(\rho_0(x_0), \rho_1(x_1), \ldots, \rho_n(x_n)).$$

All our defining identities are of the form $t \sim 0$ for every term t having a subterm p satisfying one of the following conditions:

(1) p is 0;

(2) $p = \sigma_j\beta(\ldots)$ or $p = \sigma_j\sigma_k(\ldots)$ for any $j, k \geq 1$ (here $\infty > k$ for every integer $k \geq 1$);

(3) $p = \beta(\ldots, \beta(\ldots))$;

(4) $p = p_n^\rho$ with some $\rho_i \in \{\sigma_1, \ldots, \sigma_n\}$ or with at least one repeating variable.

We call any such term t a zero term.

Observation 3.2.1. It is clear that any term with a zero subterm is a zero term itself, and that substituting any term into a zero term produces a zero term.

Thus, for instance,

$$\beta^4(\sigma_5(x_0), x_1, \sigma_6(x_2), \sigma_5(x_3), \sigma_\infty(x_4))$$

is a term of A but not of A', while

$$\beta^4(\sigma_5(x_0), x_1, \sigma_6(x_2), x_0, \sigma_5(x_4)) \text{ and}$$
$$\beta^4(\sigma_5(x_0), x_1, \sigma_6(x_2), x_3, \sigma_1(x_4))$$

are zero terms of both algebras because the first one has a repeating variable x_0 and the second one has the subterm σ_1 with an index that is too small.

Lemma 3.2.2. *The deductive closure of the set of all defining identities consists of all identities $t \approx t'$ such that either t and t' are both zero terms or else $t(x_0, \ldots, x_n) = t'(x_0, \ldots, x_n)$ is the same non-zero term.*

Proof. Let $\approx$ be the least equivalence on the set of all terms t containing all defining identities $t \sim 0$. Then $t \approx t'$ exactly when both t and t' are zero terms or $t = t'$. We need only show that $\approx$ is preserved under the subterm replacement and under the substitution.

For the subterm replacement property, let s be a subterm of t and let t' be obtained from t by the replacement of s by a term s' such that $s' \approx s$. Then either s is a zero term and hence s' is a zero term and therefore both t and t' are zero terms (by Observation 3.2.1), or else $s' = s$ and hence $t' = t$. Hence $\approx$ is preserved under the subterm replacement.

For the substitution property, let $s \approx s'$ and let x be a variable in $s \approx s'$. Let r be any term, and let t and t' result from the substitution of r for every occurrence of the variable x in $s \approx s'$. Then either s and s' are zero terms and hence t and t' are zero terms (by Observation 3.2.1), or else $s = s'$, and then $t = t'$. Thus $t \approx t'$ in either case, so that $\approx$ is preserved under the substitution. $\square$

Observation 3.2.3. For any $n \geq 1$, the $(n+1)$-ary term

$$\gamma_n(x_0, \ldots, x_n) = \beta^n(\sigma_\infty(x_0), \ldots, \sigma_\infty(x_n))$$

of A is a non-zero term. Indeed, noting that every subterm of $\gamma_n = p_n^\rho$ with $\rho = (\sigma_\infty, \ldots, \sigma_\infty)$ has pairwise distinct variables and is of the form γ_m with $1 \leq m \leq n$

or else it is $\sigma_\infty(x_i)$ or a variable x_i, one can readily see that it fails to satisfy any of the conditions (1)–(4).

Lemma 3.2.2 ensures that the term functions of either free algebra are in one-to-one correspondence with the elements of the respective sets consisting of all non-zero terms and one zero term 0 of each arity. We thus retain the notation $t(x_0, \dots, x_n)$ also for the term functions.

3.2.4. We show that $\mathrm{t}A$ is not isomorphic to $\mathrm{t}A'$. Suppose that $H : \mathrm{t}A \to \mathrm{t}A'$ is an isomorphism. For any non-zero unary term operation σ_k of A, the non-zero term operation $H(\sigma_k)$ is also unary, and hence $H(\sigma_\infty) = \sigma_m$ for some $m \in \omega$. For any $n \geq 1$, the clone $\mathrm{t}A$ contains the $(n+1)$-ary non-zero term function $\gamma_n(x_0, \dots, x_n) = \beta^n(\sigma_\infty(x_0), \dots, \sigma_\infty(x_n))$. Therefore $H(\beta^n)$ is $(n+1)$-ary and

$$H(\gamma_n)(y_0, \dots, y_n) = H(\beta^n)(\sigma_m(y_0), \dots, \sigma_m(y_n))$$

is the zero term function in $\mathrm{t}A'$ for every $n \geq m$, see (4). Thus H is not injective, and hence the clone $\mathrm{t}A'$ cannot be isomorphic to $\mathrm{t}A$.

3.2.5. Next we show why, for every $m > 1$, the $(m+1)$-segments of $\mathrm{t}A$ and of $\mathrm{t}A'$ are isomorphic.

Any non-zero term function beginning with β^n must have all its $n+1$ variables pairwise distinct. The highest power in $(m+1)$-segments is the m-th, so that it is enough to consider non-unary term functions that begin with β^n with $n \leq m-1$. We make the assignment $0 \mapsto 0$, $\beta^n \mapsto \beta^n$, $1 \mapsto 1$, and for every $\sigma_k \in \Sigma_1'$

$$\sigma_k \mapsto \begin{cases} \sigma_k & \text{if } k < 2m, \\ \sigma_\infty & \text{if } k = 2m, \\ \sigma_{k-1} & \text{if } k > 2m. \end{cases}$$

In particular, the unary terms of A' are bijectively assigned to the unary terms of A. We use composition to extend this assignment to the $(m+1)$-segment $(\mathrm{t}A)_{m+1}$. Thus, for instance, the assignments for three similar term functions from the 6-segment of A' (i.e., $m = 5$ and hence $2m = 10$) are

$$\begin{aligned} & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_{11}(x_4)) \\ \mapsto\ & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_{10}(x_4)), \\ & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_{10}(x_4)) \\ \mapsto\ & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_\infty((x_4)), \\ & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_9(x_4)) \\ \mapsto\ & \beta^4(\sigma_5(x_0), x_1, \sigma_5(x_2), \sigma_7(x_3), \sigma_9(x_4)); \end{aligned}$$

and $\beta^4(\sigma_{10}(x_0), \dots, \sigma_{10}(x_4)) \mapsto \beta^4(\sigma_\infty(x_0), \dots, \sigma_\infty(x_4))$ – where the first term function is non-zero because $4 < 10$.

It is now apparent that this assignment is an $(m+1)$-segment isomorphism of $(\mathrm{t}A')_{m+1}$ onto $(\mathrm{t}A)_{m+1}$. This establishes the validity of Example 3.2. □

The question of whether or not a similar example exists of algebras with finitely many finitary operations remains unresolved.

4 Centralizer clones

In this section we prove Theorems 4.1 and 4.3 below.

Theorem 4.1. *There exist algebras $C, C' \in Alg(1,1)$ whose centralizer clones* $\mathrm{c}C$ *and* $\mathrm{c}C'$ *are locally isomorphic but not isomorphic.*

Definition. For a clone k and a category $\mathcal{K}$, any full embedding $\Phi : k \to \mathcal{K}$ preserving all finite products is called a representation of the clone k in $\mathcal{K}$. If such a functor Φ exists, we say that k is representable in $\mathcal{K}$. If for every $n \in \omega$ there is a full embedding $\Phi_n : k_n \to \mathcal{K}$ preserving all finite products, we say that k is locally representable in $\mathcal{K}$.

As shown in [16], the centralizer clone $\mathrm{c}B$ of any algebra B of a countable similarity type is representable in $\mathrm{Alg}(1,1)$. Using [20, 21], we also easily obtain Theorem 4.3 below.

Recall that a category $\mathcal{S}$ is strongly connected if all its hom-sets $\mathcal{S}(a,b)$ are non-void. An $\mathcal{S}$-object t is called terminal if the hom-set $\mathcal{S}(a,t)$ is a singleton for every $\mathcal{S}$-object a. We also recall the following result of [20].

Theorem 4.2 [20]. *For any abstract clone k with a base object a, these two conditions are equivalent:*

(i) *if k is locally representable in a category $\mathcal{K}$ then k is representable in $\mathcal{K}$;*

(ii) *for some $n < m$, there is a split epi in $k(a^n, a^m)$.*

If the condition (ii) of Theorem 4.2 fails, that is, if any $k(a^n, a^m)$ contains a split epi only when $n \geq m$, we say that the clone k is loose.

Theorem 4.3. *For any strongly connected countable loose clone k, there exists a full subcategory $\mathcal{C}$ of $Alg(1,1)$ closed under all finite products such that k is locally representable in $\mathcal{C}$ but not representable in $\mathcal{C}$.*

Theorems 4.1 and 4.3 use the following result of [21].

Theorem 4.4 [21]. *For any countable strongly connected category $\mathcal{S}$ with a terminal object there is a full embedding $\Phi_{\mathcal{S}} : \mathcal{S} \to Alg(1,1)$ preserving all finite products existing in k.*

Proof of 4.1 and 4.3. To prove Theorem 4.1, we recall that algebras A and A' defined in Example 3.2 have respective clones $\mathrm{t}A$ and $\mathrm{t}A'$ of term functions which are countable, strongly connected and have a terminal object. As abstract categories $k = \mathrm{t}A$ and $k' = \mathrm{t}A'$, these clones with base objects a and a' have respective full embeddings $\Phi_k : k \to \mathrm{Alg}(1,1)$ and $\Phi_{k'} : k' \to \mathrm{Alg}(1,1)$ that preserve finite products. Setting $C = \Phi_k a$ and $C' = \Phi_{k'} a'$ thus gives algebras whose centralizer clones are locally isomorphic but not isomorphic. This completes the proof of Theorem 4.1.

The proof of Theorem 4.3 is also very simple. According to [20], for any countable strongly connected loose clone k, there is a countable strongly connected category $\mathcal{S}$ closed under all finite products in which k is locally representable but not representable. The full subcategory $\mathcal{C} = \Phi_{\mathcal{S}}(\mathcal{S})$ of $\mathrm{Alg}(1,1)$ isomorphic to $\mathcal{S}$ because of Theorem 4.4 then satisfies the conclusion of Theorem 4.3. The countable strongly connected clone k has a representation elsewhere in $\mathrm{Alg}(1,1)$, of course, by Theorem 4.4. □

5 On $\mathbb{C}$-universality

Definition 5.1. Let $\mathbb{C}$ be a class of clones. We say that a category $\mathcal{K}$ is $\mathbb{C}$-universal (or $\mathbb{C}$-u for short), if every clone $k \in \mathbb{C}$ is representable in $\mathcal{K}$. If every full subcategory of $\mathcal{K}$ closed under finite products other than the one-object subcategory on the terminal object of $\mathcal{K}$ is $\mathbb{C}$-universal, we say that $\mathcal{K}$ is hereditarily $\mathbb{C}$-universal (or h$\mathbb{C}$-u).

A category $\mathcal{K}$ is conditionally $\mathbb{C}$-universal (or c$\mathbb{C}$-u) if every clone $k \in \mathbb{C}$ locally representable in $\mathcal{K}$ is representable in $\mathcal{K}$. And $\mathcal{K}$ is hereditarily conditionally $\mathbb{C}$-universal (or hc$\mathbb{C}$-u) if every full subcategory of $\mathcal{K}$ with more than one object that is closed under finite products is conditionally $\mathbb{C}$-universal.

For a given class $\mathbb{C}$ of clones, the trivial implications between these four properties of a category $\mathcal{K}$ are as follows.

$$\begin{array}{ccc} (\alpha)\,\mathrm{h}\mathbb{C}\text{-u} & \longrightarrow & (\beta)\,\mathbb{C}\text{-u} \\ \downarrow & & \downarrow \\ (\gamma)\,\mathrm{hc}\mathbb{C}\text{-u} & \longrightarrow & (\delta)\,\mathrm{c}\mathbb{C}\text{-u} \end{array}$$

In what follows, we prove these three results:

(A) a characterization of classes $\mathbb{C}$ such that every category $\mathcal{K}$ with finite products satisfies (δ) c$\mathbb{C}$-u, that is, every $\mathcal{K}$ is conditionally $\mathbb{C}$-universal;

(B) a characterization of the classes $\mathbb{C}$ for which there is a category $\mathcal{K}$ which satisfies (α) h$\mathbb{C}$-u;

(C) for the class $\mathbb{C}$ of all strongly connected countable clones, we show that the category Alg$(1,1)$ satisfies (β) $\mathbb{C}$-u but not (γ) hc$\mathbb{C}$-u, and that the category Bool of all Boolean algebras satisfies (γ) but not (β).

We begin with (A), which is fully answered using the result of [20] quoted above as Theorem 4.2. Indeed, a class $\mathbb{C}$ has the property that every category $\mathcal{K}$ with finite products is c$\mathbb{C}$-u iff $\mathbb{C}$ does not contain any loose clone.

Now we turn to (B).

Definition. For a clone k with a base object a and a natural number $M \geq 1$, let Mk denote the clone which is the full subcategory of k with

$$\text{obj}Mk = \{a^0, a^M, a^{2M}, a^{3M}, \ldots\}$$

and product projections specified as the powers

$$[\pi_i^{(n)}]^M : a^{nM} \longrightarrow a^M \text{ for } i \in n \in \omega.$$

Proposition 5.2. *For a class $\mathbb{C}$ of clones, these properties are equivalent:*

(i) *there exists a hereditarily $\mathbb{C}$-universal category $\mathcal{K}$;*

(ii) *for any two clones $k, k' \in \mathbb{C}$ and every natural number M, the clone k can be represented in Mk'.*

Proof. If $\mathbb{C}$ satisfies (ii), then every $k' \in \mathbb{C}$ is a hereditarily $\mathbb{C}$-universal category, and hence (i) holds.

If $\mathbb{C}$ does not satisfy (ii), then there exist $k, k' \in \mathbb{C}$ such that k is not representable in k'. If $\mathcal{K}$ is a hereditarily $\mathbb{C}$-universal category, then $k' \in \mathbb{C}$ can be represented in $\mathcal{K}$. If $\Phi : k' \longrightarrow \mathcal{K}$ is such a representation, then $\Phi(k')$ is a full subcategory of $\mathcal{K}$ which is not $\mathbb{C}$-universal because k is not representable in it. Therefore $\mathcal{K}$ is not hereditarily $\mathbb{C}$-universal. □

Proposition 5.2 thus establishes our answer to (B).

Remark 5.3. All classes known to satisfy (ii) of Proposition 5.2 are very small. Examples are singleton classes consisting of the clone of all continuous maps of the Cantor discontinuum, or of the clone of a Hilbert cube, or of any clone whose base object a is isomorphic to its square a^2.

Question. Are there non-singleton classes $\mathbb{C}$ of clones satisfying (ii) of Proposition 5.2?

We now turn to (C).

Observe that a clone k with a base object a is a strongly connected category exactly when $k(a^0, a) \neq \emptyset$, and that a^0 is the terminal object of k. According to Theorem 4.3 from [21], the category $\mathrm{Alg}(1,1)$ is $\mathbb{C}$-universal for the class $\mathbb{C}$ of all countable strongly connected clones, and hence it satisfies (β) for this class.

To see that $\mathrm{Alg}(1,1)$ fails to satisfy (γ) for this class $\mathbb{C}$, we begin by choosing any countable strongly connected loose clone $k \in \mathbb{C}$ (see 4.2). According to [20], there is a category $\mathcal{K}$ closed under finite products such that k is locally representable but not representable in $\mathcal{K}$. Inspection of the construction of $\mathcal{K}$ in [20] shows that, for our given clone k, the category $\mathcal{K}$ satisfies the hypothesis of the above Theorem from [21]. Thus $\mathcal{K}$ has a representation $\mathcal{K}'$ in $\mathrm{Alg}(1,1)$. Since $\mathcal{K}'$ is a full subcategory of $\mathrm{Alg}(1,1)$ closed under finite products and because k is locally representable but not representable in $\mathcal{K} \cong \mathcal{K}'$, we conclude that $\mathrm{Alg}(1,1)$ is not hereditarily conditionally $\mathbb{C}$-universal, and hence fails to satisfy (γ).

The remainder of (C), that is, the fact that (γ) does not imply (β) is supported also by categories other than the category Bool of Boolean algebras. In [20], the following categories were considered and the references quoted used:

(1) the category Set of sets and mappings and, for any infinite cardinal α, its full subcategory whose objects are sets X with $\mathrm{card} X < \alpha$, see [9];

(2) the category 0Top of all 0-dimensional spaces and all their continuous maps, and its full subcategory BTop of all Boolean spaces, see [9];

(3) the category of all Tychonoff spaces containing an arc, see [9];

(4) the category Bool of Boolean algebras, see [8, 10, 15];

(5) the category Poset of all partially ordered sets, see [12];

(6) the category $\mathrm{DL}_{0,1}$ of all distributive $(0,1)$-lattices, see [12].

Using the quoted results, in [20] it is shown that all these categories are conditionally $\mathbb{C}$-universal for the class $\mathbb{C}$ of all clones. The proof uses the following notions.

Definition 5.4. Let $\mathcal{K}$ be a category. We say that $\mathcal{K}$ is monoid determined if for any $b, c \in \mathrm{obj}\,\mathcal{K}$ and any isomorphism Φ of the monoid $\mathcal{K}(b,b)$ onto the monoid $\mathcal{K}(c,c)$ there exists an isomorphism $\phi \in \mathcal{K}(b,c)$ such that $\Phi f = \phi \circ f \circ \phi^{-1}$ for every $f \in \mathcal{K}(b,b)$. Another property of $\mathcal{K}$ is that $\mathcal{K}$ is weakly monoid determined, meaning that for any sequence $\{b_n \mid n \in \omega\} \subseteq \mathrm{obj}\,\mathcal{K}$ such that there is a monoid isomorphism $\Psi_n : \mathcal{K}(b_n, b_n) \to \mathcal{K}(b_{n+1}, b_{n+1})$, there is a cofinal subset $\omega' \subseteq \omega$ and for any $n < m$ in ω' there is an isomorphism $\phi_{n,m} \in \mathcal{K}(b_n, b_m)$ such that

$$\Phi_{n,m} f = \phi_{n,m} \circ f \circ \phi_{n,m}^{-1}, \text{ where } \Phi_{n,m} = \Psi_{m-1} \circ \ldots \circ \Psi_n.$$

Finally, a category $\mathcal{K}$ with finite products is guileless if every $b \in \operatorname{obj} \mathcal{K}$ is a generator of the full subcategory of $\mathcal{K}$ determined by $\{b^n \mid n \in \omega\} \subseteq \operatorname{obj} \mathcal{K}$.

It is easily seen that these notions are hereditary with respect to full subcategories closed under finite products. Thus each of the categories under (1)–(6) is hereditarily conditionally $\mathbb{C}$-universal for the class $\mathbb{C}$ of all clones. On the other hand, none of them is $\mathbb{C}$-universal for many singleton classes $\mathbb{C}$. This is because no nontrivial group G is isomorphic to the monoid $\mathcal{K}(a, a)$ of any $a \in \operatorname{obj}\mathcal{K}$, so that these categories are not $\mathbb{C}$-universal for any singleton class $\mathbb{C} = \{k\}$ such that $k(a, a) \cong G$ (where a is the base object of the clone k).

6 A note on elementary equivalence

Any abstract clone k with a base object a can be viewed as an ω-sorted algebra whose carrier X_n of the n-th sort is the set $k(a^n, a)$ of all k-morphisms $a^n \to a$, and which has

(c) n distinct nullary operations $\pi_0^{(n)}, \ldots, \pi_{n-1}^{(n)} \in X_n$ of each sort $n \in \omega$, and

(s) for any $m, n \in \omega$, a heterogeneous operation given by

$$S_m^n(h; f_0, \ldots, f_{n-1}) = h \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1}),$$

where $f_0, \ldots, f_{n-1} \in X_m$ and $h \in X_n$.

Hence every abstract clone k determines a unique ω-sorted algebra whose operations are described in (c) and (s), and any such algebra satisfies the equations

(E1) $S_n^n(h; \pi_0^{(n)}, \ldots, \pi_{n-1}^{(n)}) = h$ for every $n \in \omega$,

(E2) $S_m^n(\pi_i^{(n)}; f_0, \ldots, f_{n-1}) = f_i$ for all $m, n \in \omega$ and $i \in n$,

(E3) $S_m^p(h; S_m^n(g_0; f_0, \ldots, f_{n-1}), \ldots, S_m^n(g_{p-1}; f_0, \ldots, f_{n-1}))$
$= S_m^n(S_n^p(h; g_0, \ldots, g_{p-1}); f_0, \ldots, f_{n-1})$ for all $m, n, p \in \omega$.

Conversely, any such ω-sorted algebra determines, up to an isomorphism, an abstract clone k. To see this, we simply name a base object a for k, and formally require that $k(a^m, a^n)$ consist of all n-tuples of members of the carrier X_m of the sort m [along with the unique 'void' 0-tuple in case of $n = 0$]. Once the composite $g \circ f$ of $f = (f_0, \dots, f_{n-1}) \in k(a^m, a^n)$ and $g = (g_0, \dots, g_{p-1}) \in k(a^n, a^p)$ is defined as

$$g \circ f = (S^n_m(g_0; f_0, \dots, f_{n-1}), \dots, S^n_m(g_{p-1}; f_0, \dots, f_{n-1})),$$

an abstract clone arises because of (E1)–(E3).

The first order language of abstract clones (see [2] or [18], for instance) is then the first order language of the ω-sorted algebras just described. Let k be an abstract clone. For each $m \in \omega$, the symbols of the language are, first, countably many variables $f^{(m)}, g^{(m)}, \dots$ of the m-th sort (interpreted as members of $k(a^m, a)$), secondly, constants $\pi^{(m)}_j$ with $j \in m$ of the m-th sort (interpreted as the named projections in $k(a^m, a)$) and, finally, operation symbols S^n_m (interpreted as the composition in k in the manner already described). Terms of the m-th sort are then described in the usual way: all variables and constants of the m-th sort are terms of the m-th sort, and so is any expression $S^n_m(t^{(n)}; t^{(m)}_0, \dots, t^{(m)}_{n-1})$ in which $t^{(n)}$ is a term of the n-th sort and $t^{(m)}_0, \dots, t^{(m)}_{n-1}$ are terms of the m-th sort, and every term of the m-th sort is created through these rules. Equalities $t^{(n)} = u^{(n)}$ of terms of the same sort are the atomic formulas of the language. As is usual, any formula is obtained by means of logical connectives and quantifiers, and a sentence is a closed formula.

Informally, for any abstract clone k, this first order language recognizes all its named product projections, and allows only equalities of composites falling into the same $k(a^m, a)$ as its atomic formulas.

Clone or clone segments are elementarily equivalent if their corresponding ω-sorted algebras satisfy the same sentences of the first order language. It is clear that isomorphic clones are elementarily equivalent. In fact, since each first order sentence refers only to finitely many variables and hence to finitely many sorts,

(e) locally isomorphic clones are elementarily equivalent.

Examples 2.4 and 3.2 thus respectively exhibit pairs of algebras of the same similarity type whose polynomial or term clones are elementarily equivalent but not isomorphic. Algebras in $\mathrm{Alg}(1, 1)$ whose centralizer clones are elementarily equivalent but their n-segments are isomorphic exactly when $n \leq N$ for some given finite N were constructed already in [17].

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