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CATEGORIES AS MONOIDS IN *Span*, *Rel* AND *Sup*

by Toby KENNEY and Robert PARE

Résumé. Nous étudions les représentations de petites catégories comme les monoïdes dans trois bicatégoies monoïdales, étroitement liées. Les catégories peuvent être exprimées comme certains types de monoïdes dans la catégorie *Span*. En fait, ces monoïdes sont aussi dans *Rel*. Il y a une équivalence bien connue, entre *Rel* et une sous-catégorie pleine de la catégorie des treillis complets et des morphismes qui préservent les sups. Cela nous permet de représenter une catégorie comme un monoïde dans *Sup*. Les monoïdes dans *Sup* s'appellent des quantales, et sont intéressants dans plusieurs domaines. Nous étudions aussi dans ce contexte la représentation d'autres structures catégoriques, par exemple, les foncteurs, les transformations naturelles, et les profoncteurs.

Abstract. We study the representation of small categories as monoids in three closely related monoidal bicategories. Categories can be expressed as special types of monoids in the category *Span*. In fact, these monoids also live in *Rel*. There is a well-known equivalence between *Rel*, and a full subcategory of the category *Sup*, of complete lattices and sup-preserving morphisms. This allows us to represent categories as a special kind of monoid in *Sup*. Monoids in *Sup* are called quantales, and are of interest in a number of different areas. We will also study the appropriate ways to express other categorical structures such as functors, natural transformations and profunctors in these categories.

Keywords. Category, Monoid, Span, Quantale

Mathematics Subject Classification (2010). 18B10, 18D35, 06F07

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1. Introduction

This research was originally conceived as an attempt to understand the following natural construction:

From a category C, we can form a quantale QC as follows:

- Elements of QC are sets of morphisms in C.
- The product of elements A and B in QC is the set $\{ab|a \in A, b \in B\}$ of composites.
- Join is union.

Examples 1.1.

1. When C is the indiscrete category on a set X, this is the quantale of all relations on X.
2. When C is a group, G, viewed as a 1-element category, this quantale is the quantale of all subsets of G.

These two examples are of interest because they give a deeper understanding of the well-known connection between equivalence relations on a set, and subgroups of a group. In both cases, these can be viewed as symmetric idempotent elements above 1 in their respective quantales.

This construction has also been studied in more detail in the case of étale groupoids by Resende [6]. In this case, instead of all sets of morphisms, he takes only open sets. Because he is considering only groupoids, the quantale is in addition involutive.

The question arises: which quantales occur in this way? We will answer this question indirectly by firstly producing a correspondance between categories and certain monoids in $\mathcal{R}el$. Using this, we will be able to describe which quantales correspond to categories, using a well-known equivalence:

Proposition 1.2. *The category of sets and relations is equivalent to the category of complete atomic Boolean algebras and sup-morphisms.*

Proof. On objects, there is a well-known correspondance between power sets and complete atomic Boolean algebras. We need to show that the direct image of a relation is a sup-morphism, and that every sup-morphism is the direct image of a relation. This is straightforward to check □

The correspondance with certain monoids in $\mathcal{Rel}$, or in $\mathcal{Span}$, has the additional advantage of holding for internal categories in other categories. However, there is not such a correspondence between internal relations in a category and quantales in that category, so for example, describing topological categories as quantales would need a different approach.

2. Monoids in *Span* and *Rel*

We begin by listing some basic properties of general monoids in $\mathcal{Span}$ and in $\mathcal{Rel}$, and the relation between the two. These properties will be of interest later when we are studying the particular monoids in $\mathcal{Span}$ and $\mathcal{Rel}$ that correspond to categories.

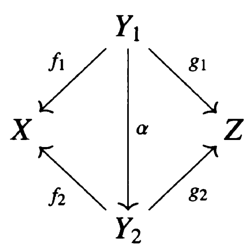
2.1 Preliminaries

To start with, we will clarify exactly what we mean by monoids in $\mathcal{Span}$, since this could be interpreted in several different ways. Firstly, we will view $\mathcal{Span}$ as a bicategory in the following way:

Objects Sets X

Morphisms Spans $X \xleftarrow{f} Y \xrightarrow{g} Z$ in **Set**.

2-cells Commutative diagrams:



in **Set**.

This is furthermore, a monoidal bicategory, with tensor product $\otimes$ given by cartesian product of sets, and the obvious tensor products of morphisms.

Of course, we can extend all of this to spans in an arbitrary category C, with pullbacks and products, and all the results we present are equally valid for this context. However, the equivalence between $\mathcal{Rel}$ and $\mathbf{CABA}_{sup}$ is specific to **Set**, so the results about quantales cannot be applied to internal categories in a category C.

Now, by a monoid in $\mathcal{S}pan$, we really mean a pseudomonoid with respect to the tensor product (rather than the categorical product, which is disjoint union in $\mathcal{S}pan$) – i.e. a diagram $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ of morphisms in $\mathcal{S}pan$, with the associativity and unit laws for a monoid commuting only up to isomorphism, with these isomorphisms satisfying the usual coherence axioms. The first reference for a monoid with respect to the tensor product of a tensor category appears to be [1].

Besides being the natural choice for the definition of monoid in $\mathcal{S}pan$, this definition also makes sense when we pass to the category of relations, because when we view relations as jointly monic spans, if two relations are isomorphic as spans, then the isomorphism is unique – indeed if two spans are isomorphic, and one is a relation, then the other is also a relation, and the isomorphism is unique. It will therefore be clear for the monoids which correspond to categories, that the isomorphisms present in the monoid axioms are unique, and therefore satisfy coherence conditions.

To save rewriting the same thing many times, we will begin by fixing our usual notation for monoids, in $\mathcal{S}pan$, $\mathcal{R}el$, or $\mathbf{CABA}_{sup}$. We will then use this notation without restating it each time.

We will denote monoids in these categories by $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ and $D \otimes D \xrightarrow{N} D \xleftarrow{J} 1$. In the case of $\mathcal{S}pan$, we will furthermore use the name of a span to denote the set that is the domain of both morphisms of the span. For instance the span $C \otimes C \xrightarrow{M} C$ will denote the span $C \times C \xleftarrow{m} M \xrightarrow{m'} C$.

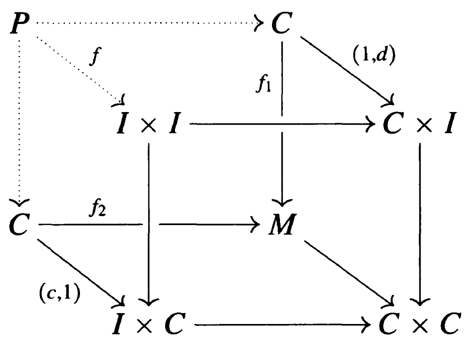
2.2 Monoids in $\mathcal{S}pan$

Proposition 2.1. *In any monoid in Span, the opposite of the unit is a partial function.*

Proof. By the unit laws, we get that there are pullbacks:

$$\begin{array}{ccc} C & \xrightarrow{f_1} & M \\ {\scriptstyle (1,d)}\downarrow & & \downarrow \\ C \times I & \longrightarrow & C \times C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{f_2} & M \\ {\scriptstyle (c,1)}\downarrow & & \downarrow \\ I \times C & \longrightarrow & C \times C \end{array}$$

for some choices of functions, $C \xrightarrow{d} I$, $C \xrightarrow{f_1} M$, $C \xrightarrow{c} I$ and $C \xrightarrow{f_2} M$ satisfying $C \xrightarrow{f_i} M \longrightarrow C$ is the identity for $i = 1, 2$. From this, in the following diagram, where the back square is a pullback, and the morphism f is the unique factorisation through the front pullback:



The front is clearly a pullback and the right and bottom squares are pullbacks by the unit laws. Therefore, by a standard argument, the top and left-hand squares are also pullbacks. We start by showing that P is isomorphic to I. In the following diagram:

$$\begin{array}{ccccc} P & \xrightarrow{f} & I \times I & \xrightarrow{\pi_1} & I \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & C \times I & \xrightarrow[\pi_1]{} & C \end{array}$$

where the left-hand square is a pullback, we know that the right-hand square is a pullback, and the bottom composite is the identity, so the whole rectangle is a pullback, and the top composite is an isomorphism, and so $P \cong I$. Thus, for the morphism f in the above cube, $\pi_1 f$ must be an isomorphism.

This means that the induced morphism $C \xrightarrow{d} I$ is a splitting (up to isomorphism) of the morphism $I \longrightarrow C$, which is therefore monic. $\square$

If (C, M, I) is a monoid in $\mathcal{S}pan$, then we have functions $M \xrightarrow{m} C \times C$ and $M \xrightarrow{m'} C$. Using these three functions from M to C, we partition M as a disjoint union of sets M_c^{ab} for $a, b, c \in C$, where

$$M_c^{ab} = \{x \in M | m(x) = (a, b), m'(x) = c\}$$

We can think of the elements of M_c^{ab} as witnesses to the fact that $ab = c$. We can then extend this to witnesses of more complicated expressions, so we can allow $M_d^{(ab)c}$ to be the set of witnesses to the fact that $(ab)c = d$. By composition in *Span*, we know that $M_d^{(ab)c} = \sum_{v \in C} M_v^{ab} \times M_d^{vc}$. We can let M' be the union of all these $M_d^{(ab)c}$. Now M' is given by the pullback

$$\begin{array}{ccc} M' & \xrightarrow{l} & M \\ {\scriptstyle k}\downarrow & & \downarrow \\ M \times C & \longrightarrow & C \times C \end{array}$$

in **Set**. By associativity, M' is also given by the following pullback in **Set**:

$$\begin{array}{ccc} M' & \xrightarrow{r} & M \\ {\scriptstyle s}\downarrow & & \downarrow \\ C \times M & \longrightarrow & C \times C \end{array}$$

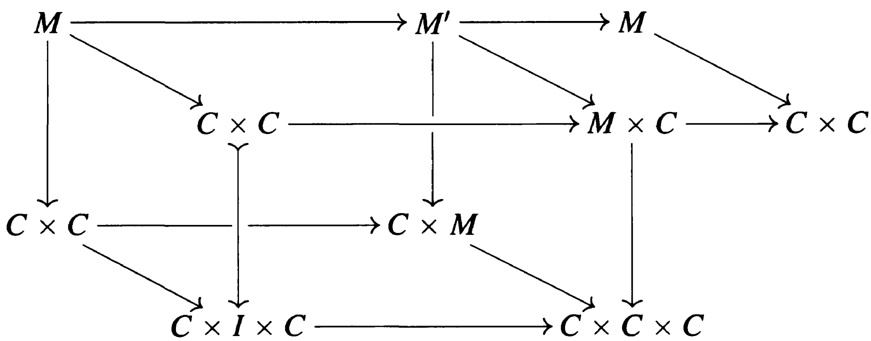
When we are trying to consider categories as monoids in *Span*, we will want the condition that $(ab)c$ (or equivalently $a(bc)$) should exist if and only if both the composites ab and bc exist, or equivalently, the following diagram in **Set**:

$$\begin{array}{ccc} M' & \xrightarrow{k} & M \times C \\ {\scriptstyle s}\downarrow & & \downarrow \\ C \times M & \longrightarrow & C \times C \times C \end{array}$$

is also a pullback. We will call a monoid in *Span* that satisfies this condition *categorical.*

Proposition 2.2. *The multiplication of any categorical monoid in Span is a partial morphism.*

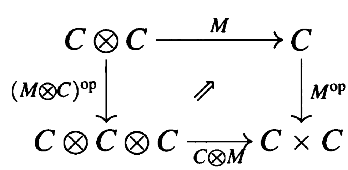
Proof. Consider the pullback squares:



The front and bottom squares are pullbacks by the unit laws. The right square in the cube is the pullback in the definition of a categorical monoid. The top right square is the pullback in the definition of M'. Also by the unit law, the top front composite $C \times C \longrightarrow M \times C \longrightarrow C \times C$ is the identity, so we know that the top left arrow is isomorphic to the morphism $M \longrightarrow C \times C$. However, the front left morphism is monic, since it is split by the projection. Therefore $M \longrightarrow C \times C$ is also monic. □

Proposition 2.3. *If M is a partial function, then the following are equivalent:*

1. *The monoid is categorical*
2. *There is a (necessarily unique) 2-cell:*



in Span.

Proof. Firstly suppose the monoid is categorical. Now in the diagram in **Set**:

$$\begin{array}{ccccc}
C \times C \times C & \longleftarrow & C \times M & \longrightarrow & C \times C \\
\uparrow & & \uparrow & & \uparrow \\
M \times C & \longleftarrow & M' & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
C \times C & \longleftarrow & M & \longrightarrow & C
\end{array}$$

all except the lower right square are pullbacks. Because the lower right square commutes, it must factor through the pullback:

$$\begin{array}{ccc}
N & \longrightarrow & M \\
\downarrow & & \downarrow \\
M & \longrightarrow & C
\end{array}$$

This factorisation is exactly the 2-cell we require.

Conversely, suppose that the 2-cell exists. Now in the diagram:

$$\begin{array}{ccccc}
C \times C \times C & \longleftarrow & C \times M & \longrightarrow & C \times C \\
\uparrow & & \uparrow{\scriptstyle a} & & \uparrow \\
M \times C & \overset{}{\underset{d}{\longleftarrow}} & M' & \overset{b}{\longrightarrow} & M \\
\downarrow & & \downarrow{\scriptstyle c} & & \downarrow \\
C \times C & \longleftarrow & M & \longrightarrow & C
\end{array}$$

The top right and bottom left squares are pullbacks by associativity, and the top left square must factor through the pullback:

$$\begin{array}{ccc}
N & \overset{w}{\longrightarrow} & C \times M \\
{\scriptstyle z}\downarrow & & \downarrow \\
M \times C & \longrightarrow & C \times C \times C
\end{array}$$

We will denote this factorisation $M' \overset{f}{\longrightarrow} N$.

On the other hand, because of the 2-cell, we get a commutative diagram:

$$\begin{array}{ccccc}
C \times C \times C & \longleftarrow & C \times M & \longrightarrow & C \times C \\
\uparrow & & \uparrow{\scriptstyle w} & & \uparrow \\
M \times C & \xleftarrow[z]{} & N & \xrightarrow{x} & M \\
\downarrow & & \downarrow{\scriptstyle y} & & \downarrow \\
C \times C & \longleftarrow & M & \longrightarrow & C
\end{array}$$

The top right and bottom left squares of this diagram have M' as the pullback. We will denote the factorisation of the bottom left square through the pullback by $N \xrightarrow{g} M'$. Now we have that $dgf = zf = d$. Since $M \longrightarrow C \times C$ is monic, the pullback d is also monic, so we have that $gf = 1_{M'}$. On the other hand, we also have that $zfg = dg = z$, and z is a pullback of $C \times M \longrightarrow C \times C \times C$, which is monic. Therefore, z is also monic, showing that f and g are inverses, yielding an isomorphism between N and M'. It is straightforward to check that this extends to an isomorphism between the labelled morphisms in the diagrams. □

2.3 Premonoidal Structures

The study of monoids in *Span* is of some interest as they correspond to Day's premonoidal structures on discrete categories [2]. As such, they correspond to monoidal closed structures on products of the category of sets

$$\prod_{C} \mathbf{Set} = \mathbf{Set}^{\mathbf{C}} \simeq \mathbf{Set}/\mathbf{C}$$

If C is a set, then a premonoidal structure on C, considered as a discrete category with values in **Set**, consists of:

(1) a triplely indexed family of sets $\langle M_c^{ab} \rangle_{a,b,c \in C}$;

(2) a singlely indexed family of sets $\langle I_a \rangle_{a \in C}$;

(3) isomorphisms $\alpha_d^{abc} : \sum_{x \in C} M_x^{ab} \times M_d^{xc} \to \sum_{x \in C} M_x^{bc} \times M_d^{ax}$;

(4) isomorphisms $\lambda_b^a : \sum_{x \in C} I_x \times M_b^{xa} \cong \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

(5) isomorphisms $\rho_b^a : \sum_{x \in C} I_x \times M_b^{ax} \cong \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

satisfying the well-known coherence conditions.

If we combine all the M_c^{ab} into a single set $M = \sum_{a,b,c} M_c^{ab}$ together with indexing functions $C \times C \leftarrow M \rightarrow C$, we have a morphism $M : C \times C \rightarrow C$ in $\mathcal{S}pan$. Similarly $I = \sum_a I_a$ with $I \rightarrow C$ gives a morphism $I : 1 \rightarrow C$ in $\mathcal{S}pan$. The isomorphisms α_d^{abc}, λ_b^a, ρ_b^a fit together into isomorphisms α, λ, ρ expressing the fact that (C, M, I) is a monoid in $\mathcal{S}pan$. Thus we have an equivalence between monoids in $\mathcal{S}pan$ and discrete premonoidal structures.

We can recast the proof of Proposition 2.1 in this setting. The conclusion says that for every $a \in C$, $I_a = 0$ or 1. Suppose then that some I_c has more than one element. The isomorphisms λ_b^a imply that $M_b^{ca} = 0$ for all a and b. Then $\rho_c^c : \sum I_x \times M_c^{cx} \rightarrow 1$ is not an isomorphism as the domain is 0.

On the other hand, Day's theory of convolution products tells us that premonoidal structures on C correspond to monoidal closed structures on $\mathbf{Set}^{\mathbf{C}}$. Given two families $\langle X_a \rangle$, $\langle Y_a \rangle$ in $\mathbf{Set}^{\mathbf{C}}$,

$$\langle X_a \rangle_a \otimes \langle Y_a \rangle_a = \langle \sum_{x,y \in C} X_x \times Y_y \times M_a^{xy} \rangle_a$$

is a monoidal closed structure if and only if M_c^{ab} is part of a premonoidal structure on C. Furthermore, all monoidal closed structures on $\mathbf{Set}^{\mathbf{C}}$ arise this way.

2.4 Example – Monoid Structures on 0, 1 and 2 in *Span*

Using the results from the previous sections, we list all the monoid structures on sets with at most 2 elements, in $\mathcal{S}pan$. When looking at examples, the notation from the previous section will be useful i.e., we express M as the disjoint union of M_c^{ab}, for elements a, b, c in C, and I as a disjoint union of I_a for elements a in C. By Proposition 2.1, we know that each I_a has at most one element.

Lemma 2.4. *If all $I_a = 1$ (i.e. if $I = C$), then*

$$M_c^{ab} = \begin{cases} 1 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

The corresponding monoidal structure in $\mathbf{Set}^{\mathbf{C}}$ *is the cartesian one.*

Proof. If $b \neq c$, $\lambda_c^b : \sum_x M_c^{xb} \cong 0$, so $M_c^{ab} = 0$, for all a. If $a \neq c$, then $\rho_c^a : \sum M_c^{ax} \cong 0$ so $M_c^{ab} = 0$ for all b. Thus the only non zero Ms are the M_a^{aa} and so λ_a^a gives that $M_a^{aa} = 1$.

The claim that the monoidal structure on $\mathbf{Set}^{\mathbf{C}}$ is cartesian is clear from the definition of the corresponding monoidal structure. □

We will call the monoid in the above lemma, the *discrete* monoid on C. It is the monoid which comes from a discrete category, when applying the construction of Section 3. On the other hand:

Lemma 2.5. *If $I = 0$, then also $C = 0$.*

Proof. If I were 0, then the composite $M(I \otimes C)$ would also be 0. However, by the unit law, it is isomorphic to the identity span on C. □

Example 2.6. The empty set admits a unique monoid structure, given by $I = M = 0$, since the empty set is strict initial, it is clear that this is the only possible monoid structure. It is easy to check that it is indeed a monoid structure. This monoid structure corresponds to the unique monoidal closed structure on the category $\mathbf{Set}^{\mathbf{0}} \simeq \mathbf{1}$.

Example 2.7. The set $1 = \{0\}$ admits a unique monoid structure. Indeed the unique I_0 can't be 0 so $I_0 = 1$, so Lemma 2.4 applies, and the monoid structure has M as the diagonal subset of 1×1.

Example 2.8. For the two-element set $\{0, 1\}$, there are multiple possible monoid structures in *Span*. By Proposition 2.1, we know that I must be a subset of $\{0, 1\}$. It is therefore, up to isomorphism, either a singleton (w.l.o.g. $\{0\}$) or the whole of $\{0, 1\}$ (it can't be empty, by Lemma 2.5). In the latter case, Lemma 2.4 tells us exactly what the monoid has to be.

Now suppose that the unit is the singleton $\{0\}$. By the unit laws, we know that:

$$\begin{array}{lll} M_0^{00} = 1 & M_0^{01} = 0 & M_0^{10} = 0 \\ M_1^{00} = 0 & M_1^{01} = 1 & M_1^{10} = 1 \end{array}$$

So the only choices we have are M_0^{11} and M_1^{11}. We will show that

Proposition 2.9. *For any two sets A and B, setting $M_0^{11} = A$ and $M_1^{11} = B$ gives a monoid.*

Proof. The unit laws are easily checked – they clearly don't involve M_0^{11} and M_1^{11}, so the fact that $A = 1, B = 0$ gives a monoid (in **Set**, and therefore also in *Span*) means that they hold.

For associativity, the composite $M(M \otimes 1)$ is given by the pullback:

$$\begin{array}{ccc} M' & \longrightarrow & M \times C \\ \downarrow & & \downarrow{\scriptstyle c \times 1} \\ M & \longrightarrow & C \times C \end{array}$$

We can partition M' into 16 sets $M_l^{(ij)k}$ for $i, j, k, l \in \{0, 1\}$. We can explicitly calculate the $M_l^{(ij)k}$ as follows:

(i,j) / (k,l)	00	01	10	11
00	1	0	0	A
01	0	1	1	B
10	0	A	A	AB
11	1	B	B	$A + B^2$

It is easy to check that this is also isomorphic to the composite $M(1 \otimes M)$, so that the monoid is indeed associative for any A and B. Furthermore, the isomorphism is canonical – we can just take the identity function on each (i, j, k, l). It is clear that this satisfies the required coherence conditions. □

The induced monoidal structure on **Set** × **Set** is the following:

$$(X_0, X_1) \otimes (Y_0, Y_1) = (X_0Y_0 + AX_1Y_1, X_0Y_1 + X_1Y_0 + BX_1Y_1)$$

(where we have used juxtaposition to denote product). The internal hom for the tensor product on **Set** × **Set** is given by

$$(Y_0, Y_1)^{(X_0, X_1)} = ({Y_0}^{X_0}{Y_1}^{X_1}, {Y_0}^{AX_1}{Y_1}^{X_0}{Y_1}^{BX_1})$$

Remark 2.10. Note that we have determined all the monoid structures on 2 in $\mathcal{S}pan$, as far as the identity and multiplication. However, we have not shown that the isomorphisms are unique. If there are any non obvious ways of defining the α, they will give a non-symmetric tensor product on **Set** $\times$ **Set**.

For our treatment of categories as monoids in $\mathcal{S}pan$, the choice of isomorphism for the pseudomonoids will be unique, because the multiplication and unit of monoids corresponding to categories are relations, so there will be a unique isomorphism, and so it will obviously satisfy the coherence conditions. Therefore, we will not have to worry about coherence conditions in that section.

Remark 2.11. In the introduction, we said that we are most interested in the monoids in $\mathcal{S}pan$ that come from categories, using the construction we will give in Section 3. There are three categories with exactly two morphisms – the discrete two-object category, and two monoid structures. We already said that the discrete category corresponds to the discrete monoid structure on 2. The two monoids correspond to the cases $A = 0, B = 1$, and $A = 1, B = 0$, above. This is not surprising, because in these cases, we see that the multiplication for the monoid in $\mathcal{S}pan$ actually becomes a function, and the unit is already a function, because I is a one-element set, so these monoids in $\mathcal{S}pan$ actually live in the subcategory **Set**.

2.5 Monoids in $\mathcal{R}el$

There is a morphism of monoidal bicategories from $\mathcal{S}pan$ to $\mathcal{R}el$, sending sets to themselves, and sending a span $A \xleftarrow{l} S \xrightarrow{r} B$ to it's underlying relation – i.e. the relation that relates an element a of A to an element b of B if and only if there is at least one element $s_{a,b}$ of S satisfying $l(s_{a,b}) = a$ and $r(s_{a,b}) = b$. This functor preserves monoids, so from a monoid in $\mathcal{S}pan$, we get a monoid in $\mathcal{R}el$.

In the other direction, we can view a relation as a jointly monic span. However, this is merely an oplax morphism, because the composite of two jointly monic spans need not necessarily be jointly monic. Therefore, not all monoids in $\mathcal{R}el$ are monoids in $\mathcal{S}pan$. Being a monoid in $\mathcal{S}pan$ imposes additional equations on a monoid in $\mathcal{R}el$. We will call a monoid in $\mathcal{R}el$ which can be viewed as a monoid in $\mathcal{S}pan$, by sending the multiplication and unit to the corresponding jointly monic spans, *spanish*.

Example 2.12. There is a monoid in $\mathcal{R}el$, on the 4-element set $\{e, x, y, z\}$, where e is the unique identity, and multiplication is given by the following table. (The sets in the table are the collection of all elements related to the pair given by the row and the column.)

	e	x	y	z
e	$\{e\}$	$\{x\}$	$\{y\}$	$\{z\}$
x	$\{x\}$	$\{y,z\}$	$\{x,z\}$	$\{x,y\}$
y	$\{y\}$	$\{x,z\}$	$\{x,y\}$	$\{y,z\}$
z	$\{z\}$	$\{x,y\}$	$\{y,z\}$	$\{x,y,z\}$

It is straightforward to check that this is indeed associative in $\mathcal{R}el$, and so a monoid. However, it is not a monoid in $\mathcal{S}pan$, since for example,

$$M_y^{(xy)z} = \sum_{w\in\{e,x,y,z\}} M_w^{xy} \times M_y^{wz} = 2$$

by taking the values $w = x$ and $w = z$. However, on the other hand,

$$M_y^{x(yz)} = \sum_{v\in\{e,x,y,z\}} M_y^{xv} \times M_v^{yz} = 1$$

with the only non-zero value when $v = z$.

In this section, we will show that monoids in $\mathcal{R}el$ that satisfy that the multiplication is a partial morphism are spanish.

Lemma 2.13. *If* $1 \xrightarrow{I} C \xleftarrow{M} C \otimes C$ *is a monoid in* $\mathcal{R}el$*, then*

$$\begin{array}{ccc} I\otimes I & \rightarrowtail & C\otimes C \\ {\scriptstyle \Delta^{op}}\downarrow & & \downarrow{\scriptstyle M} \\ I & \rightarrowtail & C \end{array}$$

commutes in $\mathcal{R}el$*, where* I *represents the subset of all elements in* C *that are related to the unique element of* 1*, and* $I \xrightarrow{\Delta} I \times I$ *is the diagonal function, viewed as a relation.*

Proof. From the unit law:

$$\begin{array}{ccccc} C & \longleftarrow & C \otimes I & \longrightarrow & C \otimes C \\ & \searrow^{1_C} & & & \downarrow M \\ & & & & C \end{array}$$

we see that the relational composite $C \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to the first projection $C \otimes I \xrightarrow{\pi_1} C$. Similarly, from the other unit law, we see that the composite $I \otimes C \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to the second projection $I \otimes C \xrightarrow{\pi_2} C$. Restricting to the subset $I \otimes I$, we get that the composite $I \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is less than or equal to both projections $I \otimes I \xrightarrow{\pi_1} I \longrightarrow C$ and $I \otimes I \xrightarrow{\pi_2} I \longrightarrow C$. The intersection of these projections is $I \otimes I \xrightarrow{\Delta^{\mathrm{op}}} I \longrightarrow C$, so one inclusion in the square is proved.

Since $I \longrightarrow C$ is a function, the inequality we have proved means that we have a commutative square

$$\begin{array}{ccc} I \otimes I & \longrightarrow & C \otimes C \\ f \downarrow & & \downarrow M \\ I & \longrightarrow & C \end{array}$$

for some relation $f \leqslant \Delta^{\mathrm{op}}$. We want to show that $f = \Delta^{\mathrm{op}}$. By the unit law, the composite $I \xrightarrow{{\pi_1}^{\mathrm{op}}} I \otimes I \longrightarrow C \otimes C \xrightarrow{M} C$ is the inclusion $I \longrightarrow C$. Therefore, we know that

$$I \xrightarrow{{\pi_1}^{\mathrm{op}}} I \otimes I \xrightarrow{f} I \longrightarrow C = I \xrightarrow{{\pi_1}^{\mathrm{op}}} I \otimes I \xrightarrow{\Delta^{\mathrm{op}}} I \longrightarrow C$$

Since $I \longrightarrow C$ is monic, this gives that $f{\pi_1}^{\mathrm{op}} = 1_I$, and $f \leqslant \Delta^{\mathrm{op}}$. It is easy to see that the only solution to this is $f = \Delta^{\mathrm{op}}$, giving the required commutativity. □

In the case where the multiplication M is a partial function, the commutative diagram in the above proposition lives entirely within the bicategory

Part, of sets, partial functions, and inclusions of graphs of partial functions. However, *Part* is a subcategory of both *Rel* and *Span*, so we see that the lifting of the above diagram to *Span* also commutes.

When the multiplication is a partial function, then the associativity square lives in the subcategory *Part*, so it lifts to a commutative square in *Span*. Therefore, to show that a monoid in *Rel*, with a partial multiplication is spanish, it is sufficient to show that the unit laws lift to commutative diagrams in *Span*, or equivalently that there are morphisms $C \xrightarrow{d} I$ and $C \xrightarrow{c} I$ such that

$$\begin{array}{ccc} C & \xrightarrow{(1,d)} & C \times I \\ \downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{(c,1)} & I \times C \\ \downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array}$$

are pullbacks in **Set**.

Proposition 2.14. *If a monoid in Rel has a partial multiplication, then it is spanish.*

Proof. Let

$$\begin{array}{ccc} P & \overset{(i,f)}{\rightarrowtail} & I \times C \\ {\scriptstyle g}\downarrow & & \downarrow \\ M & \rightarrowtail & C \times C \end{array}$$

be a pullback in **Set**. By the unit law, we know that $P \xrightarrow{g} M \longrightarrow C$ is equal to f, and is an extremal epimorphism. Now let

$$\begin{array}{ccc} P_3 & \xrightarrow{a} & P \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow[f]{} & C \end{array}$$

be a pullback in **Set**. We will show that $a = b$, so that f is mono. Hence we will have that f is an isomorphism, and so the unit law lifts to *Span*. A similar argument will show the same for the other unit law, giving that the monoid is spanish. We consider the diagram of pullbacks:

$$\begin{array}{ccccc} P_3 & \rightarrowtail & P' & \longrightarrow & P \\ \downarrow & \searrow^{(j,k,f)} & \downarrow & & \downarrow{\scriptstyle (i,f)} \\ P'' & \rightarrowtail & I \times I \times C & \longrightarrow & I \times C \\ \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow[(i,f)]{} & I \times C & \longrightarrow & C \end{array}$$

We see that we can describe P_3 entirely by the morphisms j, k, and f. We will show that $j = k$. However, $j = ia$ and $k = ib$. Furthermore, we already know that $fa = fb$, and that (i, f) is a monomorphism. Thus, we can deduce that $a = b$, and so f is a monomorphism.

To show that $j = k$, we consider the commutative diagram in $\mathcal{R}el$:

$$\begin{array}{ccccc} I \times I \times C & \rightarrowtail & C \times C \times C & \xrightarrow{1 \times M} & C \times C \\ {\scriptstyle \Delta^{op} \times 1_C}\downarrow & & {\scriptstyle M \times 1}\downarrow & & \downarrow{\scriptstyle M} \\ I \times C & \rightarrowtail & C \times C & \xrightarrow[M]{} & C \end{array}$$

The right-hand square is associativity, while the left hand square is from Lemma 2.13. Since all morphisms are partial functions, the diagram lifts to $\mathcal{S}pan$. In $\mathcal{S}pan$, the top-right composite is

$$I \times I \times C \xleftarrow{(j,k,f)} P_3 \longrightarrow P \longrightarrow M \longrightarrow C$$

Since the diagram commutes in $\mathcal{S}pan$, the left-hand leg of this span must factor through the diagonal $I \xrightarrow{\Delta} I \times I$, and so $j = k$ as required.

Using the other unit law in a similar way, we can deduce that it also lifts to a commutative diagram in $\mathcal{S}pan$, so that the monoid is spanish. □

3. Categories, Functors, Profunctors and Natural Transformations in *Span*

3.1 Categories

In this section, we will establish a bijective correspondence between categories and certain types of monoid in *Span*, or equivalently in *Rel*. We will fix some notation. For a category $\mathcal{C}$, the corresponding monoid will be $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$. For a category $\mathcal{D}$, the corresponding monoid will be $D \otimes D \xrightarrow{N} D \xleftarrow{J} 1$.

Given a small category $\mathcal{C}$, we can form a monoid in *Span* as follows: The underlying set is the set C of morphisms of $\mathcal{C}$. Composition gives a partial function from $C \times C$ to C, defined on composable pairs, i.e. pairs (f, g) such that $\operatorname{dom} f = \operatorname{cod} g$. The identity is the opposite to the partial function from C to 1 that is defined only on identity morphisms. It is easy to check that this is indeed a monoid in *Span*. This also works for internal categories in any category with all finite limits, and the following theorems also all apply in this case, with the exception of Proposition 3.2.

Theorem 3.1. *A monoid in Span can be expressed as the result of the above construction for a category if and only if the multiplication is a partial morphism and there is a (necessarily unique) 2-cell*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{M} & C \\ {\scriptstyle (M\otimes C)^{\mathrm{op}}}\downarrow & \nearrow & \downarrow{\scriptstyle M^{\mathrm{op}}} \\ C \otimes C \otimes C & \xrightarrow[C\otimes M]{} & C \otimes C \end{array}$$

in Span.

Proof. It is easy to see that the monoid we obtain from a category using the above construction, has a partial function for its multiplication, and also has the unique 2-cell in the above theorem.

Conversely, given a monoid $C \otimes C \xrightarrow{M} C \xleftarrow{I} 1$ in *Span*, where M is a partial function and I^{op} is also a partial function, C will be the object of morphisms. The domain of I^{op} will be the object I of objects. The left identity law for the monoid:

$$C \xrightarrow{I \otimes 1_C} C \otimes C \xrightarrow{M} C = 1_C$$

says that for every element f of C, there is exactly one element j of I such that the composite $jf = f$, and no other composites are possible. We will call this unique j the *codomain* of f. Similarly, there is exactly one element j of I such that $fj = f$. We will call this the *domain* of f. These give the functions $C \xrightarrow{\text{dom}} I$ and $C \xrightarrow{\text{cod}} I$, needed for a category. Also, M must be the object of composable pairs in the category. We need to show that it is the object of pairs (f, g) such that $\text{dom}(f) = \text{cod}(g)$, as is required for a category.

The 2-cell shows that if the composites fg and gh both exist, then the composite $(fg)h$ also exists. Associativity then gives us that $f(gh)$ also exists, and is equal to $(fg)h$. Also, associativity gives us that if either $f(gh)$ or $(fg)h$ exists, then the other also exists, and they are equal. This means that in this case, both fg and gh must exist.

Finally, from the case where g is an identity, we know that the composite fh exists if and only if the domain of f is equal to the codomain of h. This is exactly what we need for a category. □

We note that since the multiplication is a partial morphism, and the unit is the opposite of a partial morphism, they are both relations. From Lemma 2.14, we see that a monoid in $\mathcal{R}el$ comes from a category if and only if the multiplication is a partial morphism, and the same 2-cell exists.

In the particular case of **Set**, it is possible to write the third condition in a different way. This will be useful when we discuss categories as quantales.

Proposition 3.2. *A monoid in $\mathcal{R}el$ can be expressed as the result of the above construction for a category if and only if the multiplication is a partial morphism and whenever the products xy and yz are both defined, then so is xyz.*

Proof. We need to show that for a monoid in $\mathcal{R}el$, whose multiplication is a partial morphism, the condition about products being defined is equivalent to the condition in Theorem 3.1.

We know that if the products xy and yz both exist, then the composite $(C \times M)(M \times C)^{\text{op}}$ relates (xy, z) to (x, yz). Therefore, by the condition in

Theorem 3.1, the composite $M^{\mathrm{op}}M$ must also relate them. For this to happen, $M(xy, z) = (xy)z$ must be defined.

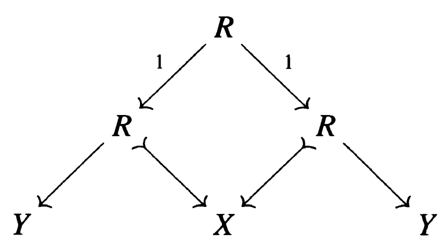
Conversely, suppose we have that whenever xy and yz are both defined, so is xyz. Now suppose that $(C \times M)(M \times C)^{\mathrm{op}}$ relates (a, b) to (c, d). This means that there is a triple (c, x, b) which is related to (a, b) by $M \times C$, and to (c, d) by $C \times M$. Thus, $cx = a$ and $xb = d$ are both defined, so the products $(cx)b$ and $c(xb)$ are both defined (and equal by associativity). Now M relates both (a, b) and (c, d) to cxb, so the composite $M^{\mathrm{op}}M$ relates (a, b) to (c, d). Since (a, b) and (c, d) were an arbitrary pair related by $(C \times M)(M \times C)^{\mathrm{op}}$, this means that $(C \times M)(M \times C)^{\mathrm{op}} \leqslant M^{\mathrm{op}}M$ in $\mathcal{R}el$. □

To make the description from Theorem 3.1 internal in $\mathcal{S}pan$, or $\mathcal{R}el$, we need to give a way of identifying which spans are partial functions.

Proposition 3.3. *A span is a relation if and only if it is a subterminal object in its hom-category in the bicategory $\mathcal{S}pan$.*

Proposition 3.4. *A relation $X \xrightarrow{R} Y$ is a partial function if and only if there is a 2-cell $RR^{\mathrm{op}} \Longrightarrow 1_Y$ in $\mathcal{R}el$, or equivalently in $\mathcal{S}pan$.*

Proof. If R is a partial function, then in $\mathcal{S}pan$, the composite RR^{op} is given by the pullback:



The function from R to Y then gives the required 2-cell in $\mathcal{S}pan$, and in $\mathcal{R}el$.

Conversely, suppose R has the required 2-cell, then the composite RR^{op} is a span in which both functions are the same. Because the composite is the pullback square, and the morphisms from R to X and to Y are jointly monic, this means that the two functions f and g of the pullback square:

$$\begin{array}{ccc} P & \xrightarrow{f} & R \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \longrightarrow & X \end{array}$$

are equal. This can only happen if $R \longrightarrow X$ is a monomorphism, so R is a partial function. □

3.2 Functors

Proposition 3.5. *If C and $\mathcal{D}$ are categories, corresponding to the monoids (C, M, I) and (D, N, J) in Span, then functors $C \xrightarrow{F} \mathcal{D}$ correspond bijectively to lax monoid homomorphisms from C to D in Span, which are also functions.*

Proof. Given a lax monoid homomorphism $C \xrightarrow{f} D$ in *Span*, where f is a function, one lax monoid homomorphism condition says that fm' admits a 2-cell to $n'(f \times f)$. The composite $n'(f \times f)$ is the pullback:

$$\begin{array}{ccccc} P & \longrightarrow & N & \xrightarrow{n'} & D \\ \downarrow & & \downarrow & & \\ C \times C & \xrightarrow[f \times f]{} & D \times D & & \end{array}$$

The 2-cell therefore says that there is a morphism $M \rightarrowtail P$. This means that any morphisms that compose in C are sent to morphisms that compose in $\mathcal{D}$. We get a commutative square in **Set**:

$$\begin{array}{ccc} M & \xrightarrow{f \times f} & N \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle n'} \\ C & \xrightarrow[f]{} & D \end{array}$$

This is exactly the functoriality condition involving composition. Similarly, the other lax monoid homomorphism condition for f gives the square

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle I}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle J} \\ C & \xrightarrow[f]{} & D \end{array}$$

in $\mathcal{S}pan$, with a 2-cell from fI to J. This gives a morphism from I to J, sending an identity 1_X to $F(1_X)$, which this 2-cell shows is an identity. Since f preserves composition, this must be 1_{FX}.

Conversely, suppose $C \xrightarrow{F} \mathcal{D}$ is a functor. Then its action on morphisms is a function $C \xrightarrow{f} D$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \xrightarrow{m'} & C & \xleftarrow{i} & I \\ {\scriptstyle f\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow \\ N & \xrightarrow[n']{} & D & \xleftarrow[j]{} & J \end{array}$$

This induces a morphism from M to the pullback of $N \rightarrowtail D \times D$ along $f \times f$, and a morphism from I to the pullback of J along f. These give the 2-cells required to make f into a lax monoid homomorphism in $\mathcal{S}pan$. □

Composition of functors is the obvious composition of functions. We can identify morphisms as the spans with right adjoints. The situation is identical in $\mathcal{R}el$ – lax monoid homomorphisms in $\mathcal{R}el$ remain lax homomorphisms in $\mathcal{S}pan$, and functions are relations with a right adjoint.

Remark 3.6. The reader may find it strange that categories correspond to pseudomonoids, and yet functors only correspond to lax monoid homomorphisms. This leads us to consider what lax monoids correspond to. There is a correspondance between certain kinds of protocategories [3], and certain lax monoids (unbiassed in Leinster's [4] terminology). Given a protocategory C, with at most one composite for each pair of protomorphisms, we form a lax monoid in $\mathcal{S}pan$ as follows: C is the set of protomorphisms; M is the set of composable pairs of protomorphisms; I is the set of objects.

It turns out that this is a lax monoid. However, we can view a category as a protocategory in which every morphism has exactly one source and target. From this point of view, any function between categories that preserves the protocategory structure (i.e. preserves identities and composition) is a functor. These are lax monoid homomorphisms.

Strict monoid homomorphisms are functors that are injective on objects, since for the 2-cell to be an isomorphism would require that any pair of morphisms whose images are composable in $\mathcal{D}$ must also be composable in $\mathcal{C}$.

3.3 Natural Transformations

Proposition 3.7. *Given functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$, corresponding to lax monoid morphisms $C \xrightarrow{f} D$ and $C \xrightarrow{g} D$ in $\mathcal{S}pan$, respectively, natural transformations correspond to functions $C \xrightarrow{a} D$ such that we have (necessarily unique) 2-cells:*

$$\begin{array}{ccc} C\otimes C & \xrightarrow{(a\otimes g)} & D\otimes D \\ {\scriptstyle M}\downarrow & \nearrow & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C\otimes C & \xrightarrow{(f\otimes a)} & D\otimes D \\ {\scriptstyle M}\downarrow & \nearrow & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array}$$

Proof. Given a natural transformation α, the function $C \xrightarrow{a} D$ sends the morphism $X \xrightarrow{h} Y$ in $\mathcal{C}$, to the morphism $FX \xrightarrow{\alpha_X} GX \xrightarrow{Gh} GY$, or equivalently $FX \xrightarrow{Fh} FY \xrightarrow{\alpha_Y} GY$. It is straightforward to check that the 2-cells above do indeed exist.

Conversely, given a morphism a such that the above 2-cells exist, in $\mathcal{S}pan$, we can form a natural transformation α by $\alpha_X = a(1_X)$. If we apply the left-hand 2-cell to $(\mathrm{cod}(h), h)$, the lower-left way around sends it to $a(h)$, while the upper-right way sends it to $a(\mathrm{cod}(h))F(h)$. We deduce that these are equal. On the other hand, if we apply the right-hand 2-cell to $(h, \mathrm{dom}(h))$, we get that $a(h) = G(h)a(\mathrm{dom}(h))$. The equality of these two is exactly the commutativity of the naturality square. □

Again, the existence of these 2-cells does not depend whether we are in $\mathcal{S}pan$ or $\mathcal{R}el$.

For composition of natural transformations, there are two types to consider. The easier type is horizontal composition. It is easy to see that this is just composition of the morphisms corresponding to the natural transformation. For vertical composition, suppose we have categories C and $\mathcal{D}$; functors F, G and H, all from C to $\mathcal{D}$; and natural transformations $F \overset{\alpha}{\Longrightarrow} G$ and $G \overset{\beta}{\Longrightarrow} H$. Let C and D be the monoids in *Span* obtained from C and $\mathcal{D}$ respectively; let f, g, and h be the lax monoid homomorphisms corresponding to the functors F, G and H respectively; and let a and b be the functions from C to D corresponding to α and β respectively. Let $C \xrightarrow{c} D$ be the function corresponding to the composite $\beta\alpha$. For a composable pair $(x, y) \in M$, We know that $c(xy) = b(x)a(y)$. The naturality gives that the value of this composite does not change if we choose a different factorisation of the composite xy. Since every morphism has a factorisation – for example, the factorisation through an identity – we can construct c as the following composite in *Span* or *Rel*:

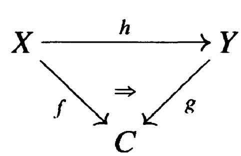
$$C \xrightarrow{M^{\mathrm{op}}} C \otimes C \xrightarrow{a \otimes b} D \otimes D \xrightarrow{N} D$$

From the above characterisation of natural transformations, it looks like they should be thought of as a kind of bimodule. To find the appropriate context in which they are bimodules, we will need the following well-known fact.

Proposition 3.8. *If $(C, \otimes, I)$ is a monoidal category, and (C, M, I') is a monoid in C, then C/C is a monoidal category.*

Proof. Given $X \xrightarrow{f} C$ and $Y \xrightarrow{g} C$, their tensor product is the composite $X \otimes Y \xrightarrow{f \otimes g} C \otimes C \xrightarrow{M} C$. The unit is $I \xrightarrow{I'} C$. It is straightforward to check that this satisfies all the required axioms. □

In fact, we will need to modify this for a monoidal bicategory: we define a bicategory $C /\!/ C$ to have the same objects as C/C, but the morphisms are now lax triangles:



and 2-cells are just 2-cells between the top morphisms in these triangles, subject to the obvious compatibility conditions with the 2-cells in the triangles. It is straightforward to see that the same argument as above makes this into a monoidal bicategory. In this monoidal bicategory, for another monoid D in C, a lax monoid homomorphism $D \xrightarrow{f} C$ becomes a monoid in this slice category $C \,/\!\!/\, C$.

We see that when we view a category C as a monoid, C, in $\mathcal{S}pan$, we can view a functor with codomain C as a certain monoid in the slice $\mathcal{S}pan \,/\!\!/\, C$. Now for two functors from $\mathcal{D}$ to C, we have the corresponding monoids in the slice category $\mathcal{S}pan \,/\!\!/\, C$. Now a natural transformation is a kind of bimodule between these monoids, in this slice category.

3.4 Profunctors

In this context, the best way to view a profunctor $P : C^{\text{op}} \times \mathcal{D} \longrightarrow \textbf{Set}$, is through the collection of elements, i.e. $\sum_{A \in \text{ob}(C), B \in \text{ob}(\mathcal{D})} P(A, B)$. This collection admits a sort of left action by C, and a right action by $\mathcal{D}$. When we look at the corresponding monoids C and D in $\mathcal{S}pan$, these actions are partial functions. We therefore see that a profunctor is a special kind of bimodule.

Proposition 3.9. *Given categories C and $\mathcal{D}$, and corresponding monoids C and D in $\mathcal{S}pan$, a bimodule E (a left C, right D module with the obvious coherence conditions between the actions) comes from a profunctor from C to $\mathcal{D}$ if and only if it satisfies the following conditions:*

1. *The actions $C \otimes E \xrightarrow{a} E$ and $E \otimes D \xrightarrow{b} E$ are partial functions.*
2. *There are (necessarily unique) 2-cells:*

$$\begin{array}{ccc} C \otimes E & \xrightarrow{C \otimes a^{\text{op}}} & C \otimes C \otimes E \\ {\scriptstyle a}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle M \otimes E} \\ E & \xrightarrow[a^{\text{op}}]{} & C \otimes E \end{array} \quad \text{and} \quad \begin{array}{ccc} E \otimes D & \xrightarrow{b^{\text{op}} \otimes D} & E \otimes D \otimes D \\ {\scriptstyle b}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle E \otimes N} \\ E & \xrightarrow[b^{\text{op}}]{} & E \otimes D \end{array}$$

Proof. We will denote the actions $a(f, e)$ by $f.e$ and $b(e, g)$ by $e * g$. It is obvious that for a profunctor, the actions a and b are partial functions. Now we consider the first 2-cell in condition 2: The top-right composite is a span that above any pairs of elements (h, e) and (f, e') of $C \otimes E$, has the set of triples $(f, g, e) \in C \otimes C \otimes E$, such that $fg = h$ and $g.e = e'$. Therefore, the 2-cell in question sends all triples (f, g, e) such that fg and $g.e$ both exist, to an element e'' of E, satisfying both $f.(g.e) = e''$ and $(fg).e = e''$. Thus, the existence of this 2-cell simply indicates that if (fg) and $g.e$ both exist, then $(fg).e$ also exists (the fact that it is equal to $f.(g.e)$ is automatic by the associativity conditions required for a bimodule). Also, by the associativity conditions, we know that if $(fg).e$ exists, then both (fg) and $g.e$ must also exist.

Furthermore, by the unit laws for a bimodule, for any $e \in E$, there is a unique identity $i \in C$ such that $i.e$ exists, and for this i, we have that $i.e = e$. We will call this i, the codomain of e. Now if we substitute this codomain of e for g in the above observation, we see that $f.e$ exists if and only if $f\,\mathrm{cod}(e)$ exists, or equivalently, if and only if $\mathrm{cod}(e) = \mathrm{dom}(f)$.

By a similar argument, we see that $e * d$ exists if and only if $\mathrm{dom}(e) = \mathrm{cod}(d)$. From these conditions, it is clear that E comes from a profunctor. □

For composition of profunctors, let C, $\mathcal{D}$ and $\mathcal{E}$ be categories, and let $C \xrightarrow{P} \mathcal{D}$ and $\mathcal{D} \xrightarrow{Q} \mathcal{E}$ be profunctors. Let the corresponding sets in *Span* be C, D, E, P and Q respectively. We know that the composite profunctor has as elements, equivalence classes of "composable" pairs $(p \in P, q \in Q)$ under the equivalence relation that relates two pairs (p, q) and (p', q') if there is $f \in D$ such that $p = fp'$ and $q' = qf$. This is the product of P and Q over D, as bimodules in *Span*, or $\mathcal{R}el$.

4. Quantales

Just as monoids in *Span* corresponded to premonoidal structures on discrete categories and consequently monoidal closed structures on powers of **Set**, monoids in $\mathcal{R}el$ correspond to **2**-enriched premonoidal structures on discrete sets and thus monoidal closed structures on powers of **2**. These are quantale structures on power sets, ordered by inclusion. In this way a small category gives a quantale. In this section, we study the interplay between categorical constructions and quantale ones. Niefield considers the closely related questions of the quantale of subsets of a monoid and quantales of subobjects of the unit object in certain closed categories (see [5] and the references cited there).

4.1 The Quantale of a Category

To determine which quantales occur as the quantale from a category, we just need to translate our characterisation of categories as monoids in $\mathcal{R}el$ through the equivalence between categories $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. Rosenthal [7] calls a quantale whose underlying lattice is a power set, a *power quantale*.

The most direct translation is just in terms of atoms (or equivalently join-irreducible elements). A morphism of powersets corresponding to a relation, corresponds to a partial morphism if and only if it sends atoms to either atoms or the empty set. The final condition for a monoid in $\mathcal{R}el$ to be categorical says that if the composites fg and gh both exist, then so does the triple composite fgh. (By associativity, it doesn't matter which way we express the triple composite.) We can express this for a quantale by using the contrapositive – if the triple composite fgh of three atoms is 0 (i.e. undefined) then either $fg = 0$ or $gh = 0$.

However, conditions involving atoms are not natural conditions on quantales, except in the case where the lattices are CABAs. We therefore seek to rephrase these conditions in a way that looks more natural for all quantales. We hope that these conditions might give a better guide for how we might be able to generalise our results to internal categories, for example in $\mathcal{O}rd$ or $\mathcal{L}oc$. However, such a generalisation would still require significant further work, and we would expect further conditions to be necessary. In such cases, we may find that we no longer get a strict quantale, but a lax quantale.

Lemma 4.1. *A sup-morphism between CABAs corresponds to a partial function if and only if its right adjoint preserves all non-empty sups.*

Proof. Let f be a relation. The right adjoint to its direct image is just its inverse image – i.e. it sends a subset A to the set $\{x|f(x)\subseteq A\}$, where $f(x)$ represents the set of things to which x is related. To say that this preserves non-empty sups says that if the image of a point x is contained in $\bigcup A_i$, then it is contained in one of the A_i. Since we can express any set as the union of its points, this means that the image of x is either empty or a singleton, i.e. f is a partial function. □

Remark 4.2. This lemma also applies in a constructive context if we replace CABAs by powersets, and non-empty by inhabited. However, some of the results later in this section will require complements, so the classification of which internal quantales come from internal categories in an arbitrary topos will require further work.

Remark 4.3. It is sufficient for the right adjoint to preserve binary sups, because the key point of the proof is the fact that the image of any x is join irreducible. However, in a powerset lattice, the elements that are irreducible as binary joins are exactly the singletons and the empty set. These are exactly the elements that are irreducible as non-empty joins.

An equivalent way to express this is to say that the sup-morphism sends $\bigvee$-irreducible elements to either $\bigvee$-irreducible elements or the bottom element. This will be a more natural condition for our purposes, since in the powerset, the $\bigvee$-irreducible elements are singleton sets. When we are looking at the quantale C from a category, singleton sets of C will be morphisms of the category, while singleton sets of $C \otimes C$ will be pairs of morphisms. Saying that the multiplication sends irreducible elements to irreducible elements or the bottom element therefore means that a pair of morphisms has at most one composite.

To express the final condition, we need to find an expression for the sup-morphism corresponding to the opposite of a relation. In fact, it is sufficient for our purposes to only find an expression for the opposite of a partial function.

Lemma 4.4. *If* $PA \xrightarrow{f} PB$ *is the sup-morphism corresponding to a partial function, then the sup-morphism corresponding to its opposite is given by* $f^{\mathrm{op}}(x) = f^*(x)\backslash f^*(\bot)$, *where* f^* *is the right adjoint to* f, *and* $\backslash$ *is the relative complementation that exists in the power set.*

Proof. The opposite of the sup-morphism $PA \xrightarrow{R} PB$ corresponding to a relation is given by $R^{\mathrm{op}}(B') = \{a \in A | (\exists b \in B')(b \in R(\{a\}))\}$, or equivalently, $R^{\mathrm{op}}(B') = \{a \in A | R(\{a\}) \cap B' \neq \varnothing\}$. In the case where the relation is a partial morphism, we can express the condition $f(\{a\}) \cap B' \neq \varnothing$ simply as $(f(\{a\}) \subseteq B') \wedge (f(\{a\}) \neq \varnothing)$. Furthermore, this can be expressed as $(\{a\} \subseteq f^*(B')) \wedge (\{a\} \nsubseteq f^*(\varnothing))$. This can be further simplified to $\{a\} \subseteq f^*(B')\backslash f^*(\varnothing)$. From this, we see that $f^{\mathrm{op}}(B') = f^*(B')\backslash f^*(\varnothing)$. □

Lemma 4.5. *The multiplication of a quantale whose underlying lattice is a CABA is the direct image of a partial function if and only if:*

- *For any $xy \leqslant \bigvee_{i \in I} a_i$, where I is non-empty, we can find a collection of pairs of elements $(x_j, y_j)_{j \in J}$ such that $x \otimes y \leqslant \bigvee_{j \in J} x_j \otimes y_j$, and for each $j \in J$, there is some $i \in I$ such that $x_j y_j \leqslant a_i$.*

Proof. We know that the condition that multiplication is a partial function in the monoid in $\mathcal{R}el$ corresponds to saying that the right adjoint to the quantale multiplication $Q \otimes Q \xrightarrow{m} Q$ preserves non-empty joins. The right adjoint is the function $Q \xrightarrow{m^*} Q \otimes Q$ given by $m^*(a) = \bigvee\{x \otimes y | xy \leqslant a\}$. To say that this preserves non-empty joins says that if $x \otimes y \leqslant m^*(\bigvee a_i)$, then $x \otimes y \leqslant \bigvee m^*(a_i)$, but in a tensor product, this means that $x \otimes y \leqslant \bigvee_{j \in J} x_j \otimes y_j$, where for every j there is an $i \in I$ such that $x_j \otimes y_j \leqslant m^*(a_i)$, or equivalently $m(x_j \otimes y_j) \leqslant a_i$. □

Definition 4.6. *For a quantale, Q, the* binary kernel, *Z_2, is the largest element of $Q \otimes Q$ whose product is 0. The* ternary kernel, *Z_3 is the largest element of $Q \otimes Q \otimes Q$ whose product is 0.*

Proposition 4.7. *A quantale Q is the quantale from a category if and only if it has the following properties:*

1. *The underlying lattice is a CABA.*
2. *If $xy \leqslant \bigvee_{i \in I} a_i$, for non-empty I, then we can find a collection of pairs of elements $(x_j, y_j)_{j \in J}$ such that $x \otimes y \leqslant \bigvee_{j \in J} x_j \otimes y_j$, and for each $j \in J$, there is some $i \in I$ such that $x_j y_j \leqslant a_i$.*
3. *$Z_3 = (Z_2 \otimes \top) \vee (\top \otimes Z_2)$.*

Proof. We just need to show that Conditions (2) and (3) are equivalent to the conditions on a monoid in $\mathcal{R}el$ from Proposition 3.2. We showed in Lemma 4.5 that the multiplication in $\mathcal{R}el$ is a partial map if and only if the powerset construction sends it to a quantale with the second property.

We want to show that Condition (3) is equivalent to the condition that for atoms x, y, and z, if the product $xyz = 0$, then either $xy = 0$ or $yz = 0$.

However, the condition $xyz = 0$ is equivalent to $x \otimes y \otimes z \leqslant Z_3$, so the condition for atoms says that if $x \otimes y \otimes z \leqslant Z_3$, then either $x \otimes y \leqslant Z_2$ or $y \otimes z \leqslant Z_2$. Since Z_3 is the join of all products of atoms $\{x \otimes y \otimes z | xyz = 0\}$, this gives that $Z_3 \leqslant (Z_2 \otimes \top) \vee (\top \otimes Z_2)$. The opposite inequality is trivial.

For the converse, we know that for atoms x, y and z, if $xyz = 0$, then $x \otimes y \otimes z \leqslant Z_3$, so by Condition (3), $x \otimes y \otimes z \leqslant (Z_2 \otimes \top) \vee (\top \otimes Z_2)$. Since $x \otimes y \otimes z$ is an atom, this means that either $x \otimes y \otimes z \leqslant Z_2 \otimes \top$ or $x \otimes y \otimes z \leqslant \top \otimes Z_2$, i.e. either $xy = 0$ or $yz = 0$. □

The condition on the binary and ternary kernel can be thought of as a variation of the condition for a ring to be an integral domain, namely, if $ab = 0$, then either $a = 0$ or $b = 0$. In a quantale, we can express the condition $a = 0$ or $b = 0$ as $a \otimes b \leqslant (0 \otimes \top) \vee (\top \otimes 0)$. (This can also be simplified to $a \otimes b = 0$, but the form we use makes the connection with our condition clearer.) Now we can take joins, to get the condition $Z_2 = (0 \otimes \top) \vee (\top \otimes 0)$. The condition from the theorem is now clearly a generalisation of this. It corresponds to the condition "if $abc = 0$ then either $ab = 0$ or $bc = 0$" in a ring. For unital rings, by substituting $b = 1$, we see that this is equivalent to the integral domain condition, but for quantales we can express b as a join of elements $b = \bigvee_{i \in i} b_i$, and we can have some of these elements satisfying $ab_i = 0$, and others satisfying $b_i c = 0$, so the condition is a weaker version of the integral domain condition. It may be of interest to study what the condition "if $abc = 0$, then either $ab = 0$ or $bc = 0$" means in the context of a non-unital ring.

In a similar manner to Theorem 4.7, we can express functors as certain kinds of morphisms between quantales.

Proposition 4.8. *For categories $\mathcal{C}$ and $\mathcal{D}$, and corresponding quantales $Q\mathcal{C}$ and $Q\mathcal{D}$, functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ correspond to lax quantale homomorphisms $Q\mathcal{C} \xrightarrow{f} Q\mathcal{D}$, where the right adjoint to f has a right adjoint of its own.*

Proof. This is automatic from the equivalence between $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. □

Proposition 4.9. *If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$ are functors, with corresponding lax quantale homomorphisms f and g, then a natural transformation between F and G corresponds to a sup-morphism $C \xrightarrow{a} D$, whose right adjoint preserves sups, such that there are 2-cells*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{(a \otimes g)} & D \otimes D \\ {\scriptstyle m}\downarrow & \leqslant & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} C \otimes C & \xrightarrow{(f \otimes a)} & D \otimes D \\ {\scriptstyle m}\downarrow & \leqslant & \downarrow{\scriptstyle N} \\ C & \xrightarrow[a]{} & D \end{array}$$

Proof. This is automatic from the equivalence between $\mathcal{R}el$ and $\mathbf{CABA}_{sup}$. □

4.2 Retrieving the Category

We have shown that categories correspond to certain types of quantale. We now consider the inverse part of this bijection – given one of these quantales, how can we recover the category we started with?

This can actually be done fairly easily – we know that a category can be described by the collection of functors from the cocategory object

$$1 \rightleftarrows 2 \rightrightarrows 3$$

in *Cat*. More explicitly, objects of the category correspond bijectively with functors from 1 to the category. Morphisms correspond to functors from 2 to the category, and the composite of a composeable pair is calculated by looking at the functor from 3 to the category. We can describe these functors explicitly for quantales. The quantale corresponding to the one morphism category is the two-element quantale. A functor from this to another quantale must send 0 to 0, so the only question is where it must send 1. It must send 1 to a join-irreducible element below 1, which is idempotent. The objects of the category therefore correspond to join-irreducible idempotent elements below 1 in the quantale. Similarly, morphisms correspond to triples (x, y, f), where x and y are objects, and f is an irreducible element such that $yfx = f$. Finally, the composite of two morphisms (x, y, f) and (y, z, g) is (x, z, gf).

If the quantale corresponds to a category, then this will give us the corresponding category.

However, it is worth observing that while the above is a cocategory object in *Cat*, it is not a cocategory object when we extend to the category of all quantales and lax quantale homomorphisms whose right adjoint is also a supmorphism. This means that for a general quantale, we do not get a category using this approach.

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