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Théories des bornes

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THEORIES DES BORNES.

Chapitre 3: An algebraic description of ordinals.

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The class ON of all ordinals will be described as a certain free algebra in the category of classes. It gives a way to consider ordinals in more general situations than in sets.

We will work in a given model of the Gödel-Bernays set theory. Denote by $\mathcal{Cl}$ the category of all classes in this model with mappings which are classes as morphisms (of course, $\mathcal{Cl}$ is not a category in our model). Having a class A , $P(A)$ will denote the class of all non-empty subsets of A . Then $P: \mathcal{Cl} \longrightarrow \mathcal{Cl}$ is a functor where $P(f)$, for a mapping $f: A \longrightarrow B$, is given by the direct images (so, we have $P(f)(X) = \{ f(x) \ / \ x \in X \}$, for any $X \in P(A)$).

Under a complete semilattice in $\mathcal{Cl}$ we will mean a partially ordered class such that each non empty subset has a supremum. It is well known that complete semilattices are precisely couples (A,h) where $A \in \mathcal{Cl}$ and $h: P(A) \longrightarrow A$ is a mapping such that $h(\{x\}) = x$, for any $x \in A$, and $h(\bigcup \mathfrak{X}) = h(\{ h(X) \ / \ X \in \mathfrak{X} \})$, for any non-empty subset $\mathfrak{X} \subseteq P(A)$.

A triple (A,f,h) will be called an o-<u>algebra</u> if $A \in \mathcal{CL}$, (A,h) is a complete semilattice and $f: A \longrightarrow A$ a mapping such that $h(\{x,f(x)\}) = f(x)$ (i. e. $x \leqslant f(x)$), for any $x \in A$. A mapping $t: A \longrightarrow A'$ is a <u>homomorphism of</u> o-<u>algebras</u> (A,f,h) and (A',f',h') if $t.f = f'$ and $h'.P(t) = t.h$.

We have an o-algebra (ON,u,s) where $u(x) = x + 1$, for any $x \in ON$, and $s(X) = \sup X$, for any $X \in P(ON)$.

<u>THEOREM</u>. (ON,u,s) <u>is a free</u> o<u>-algebra over one generator</u>.

<u>Proof</u>. We will prove that 0 freely generates ON as an o-algebra. Let (A,f,h) be an o-algebra and $a \in A$. We have to show that there is a unique homomorphism $t: ON \longrightarrow A$ of o-algebras such that $t(0) = a$.
Define t by a transfinite recursion as follows: $t(0) = a$, $t(x+1) = f.t(x)$, for any $x \in ON$, and $t(x) = h.P(t)(x)$ (here x is understood as a set of all smaller ordinals), for any limit $x \in ON$. If $x \in ON$, then t_x will denote the restriction of t on $x+1 \subseteq ON$. To see that t is a homomorphism of o-algebras it suffices to verify that each $t_x: x+1 \longrightarrow A$ is a homomorphism of partial o-algebras (i. e. preserves all existing operations). At first, we show that each t_x preserves $\leqq$. Indeed, if t_x does it then t_{x+1} as well because $t(x+1) = f.t(x) \geq t(x)$. If x is limit, the induction step is evident.
It is clear that if t_x is a homomorphism of partial o-algebras then t_{x+1} as well. Assume that x is limit and t_y a homomorphism for any $y < x$. Then $h.P(t)(X) = t.s(X)$ for any $X \in P(x+1)$ such that $s(X) < x$. If $s(X) = x$, then either x is the greatest element of X or X is cofinal with x . Since t_x preserves $\leqslant$, the same alternative holds for $t(x)$ and $P(t)(X)$. In the first case, $h.P(t)(X) = t(x)$ is immediate and in the second one $h.P(t)(X) = h.P(t)(x) = t(x)$ by the

construction of t . Since $t(x) = t.s(X)$, t_x is proved to be a homomorphism.

The uniqueness of t is evident.

Remark. The just identified ordinal object ON , which is completely determined by "the calculus of the functor P " , may be compared with the natural number object in toposes (which is the free unary algebra over one generator).

(à suivre)