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J. ROSICKÝ

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Our aim is to axiomatize properties of the tangent functor $T : \mathcal{M} \to \mathcal{M}$ on the category of smooth manifolds. The resulting abstract tangent functor $T : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ has the property that there is a well-behaved bracket operation of its sections $A \to TA$ on any object $A \in \mathcal{C}$. Representable abstract tangent functors are closely connected with rings of line type in the sense of Kock - Lawvere.

1. Natural group bundles

Let $\mathcal{C}$ be a category, $F,G : \mathcal{C} \to \mathcal{C}$ functors and $p : G \to F$ a natural transformation. Let $\oplus$ denote products in the category of functors over F. It means that

$$\begin{array}{ccc} G \oplus G & \xrightarrow{p_1} & G \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow[p]{} & F \end{array}$$

is a pullback. We say that (G,p) is a natural group bundle over F if it is a group in the category of functors over F. It means that G is equipped with natural transformations

$$+ : G \oplus G \to G\ , \quad - : G \to G\ , \quad O : F \to G$$

which are over F (i.e. $p.+ = p.p_1$, $p.- = p$ and $p.O = 1$) and satisfy the group axioms. A natural group bundle is a natural group bundle over the identity functor on $\mathcal{C}$.

Let (G,p) be a natural group bundle over F and (H,q) a natural group bundle over E. A homomorphism $(f,g):(G,p) \to (H,q)$ is a couple of natural transformations $f : G \to H$ and $g : F \to E$ such that the following diagrams commute

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow[g]{} & E \end{array} \qquad\qquad \begin{array}{ccc} G\oplus G & \xrightarrow{f\oplus f} & H\oplus H \\ {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\ G & \xrightarrow[f]{} & H \end{array}$$

Let (G,p) be a natural group bundle. If the pullbacks $G\oplus G$ and $G\oplus G\oplus G$ are pointwise then (G^2,pG) is a natural group bundle over G with respect to $+G$, $-G$, $0G$ and $(Gp,p) : (G^2,pG) \to (G,p)$ is a homomorphism. If G preserves $G\oplus G$ and $G\oplus G\oplus G$ then (G^2,Gp) is a natural group bundle over G and $(pG,p) : (G^2,Gp) \to (G,p)$ is a homomorphism.

2. Tangent functors

We say that $T : \mathcal{C} \to \mathcal{C}$ is a tangent functor if there are p, i and m such that

(T1) (T,p) is a natural commutative group bundle such that pullbacks $T\oplus T$ and $T\oplus T\oplus T$ are pointwise and preserved by T.

(T2) $(i,1):(T^2,pT) \to (T^2,Tp)$ is a homomorphism and $i^2 = 1$.

(T3) $(m,0):(T,p) \to (T^2,pT)$ is a homomorphism such that $i.m=m$ and the following diagram is a pointwise equalizer

$$T \xrightarrow{m} T^2 \overset{pT}{\underset{0.p.pT}{\overset{Tp}{\rightrightarrows}}} T$$

(T4) The following diagrams commute

$$\begin{array}{ccc} T & \xrightarrow{m} & T^2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mT} \\ T^2 & \xrightarrow[Tm]{} & T^3 \end{array} \qquad \begin{array}{ccccc} T^3 & \xrightarrow{iT} & T^3 & \xrightarrow{Ti} & T^3 \\ {\scriptstyle Ti}\downarrow & & & & \downarrow{\scriptstyle iT} \\ T^3 & \xrightarrow[iT]{} & T^3 & \xrightarrow[Ti]{} & T^3 \end{array}$$

$$\begin{array}{ccccc} T^2 & \xrightarrow{Tm} & T^3 & \xrightarrow{iT} & T^3 \\ {\scriptstyle i}\downarrow & & & & \downarrow{\scriptstyle Ti} \\ T^2 & & \xrightarrow[mT]{} & & T^3 \end{array}$$

These axioms are satisfied by the tangent functor $T : M \to M$ on the category of smooth manifolds. Since $T(\mathbb{R}^n) = \mathbb{R}^{2n}$, the following example also indicates the description of i and m in local coordinates.

<u>Example 1</u> : Let Ab be the category of abelian groups and $T = (-)^2$. Then $p_A(a,b) = a$, $(a,b) +_A (a,c) = (a,b+c)$, $-_A(a,b) = (a,-b)$, $0_A(a) = (a,0)$, $i_A(a,b,c,d) = (a,c,b,d)$ and $m_A(a,b) = (a,0,0,b)$ make from T a tangent functor. We indicate that $T \oplus T = (-)^3$ and $(a,b,c,d) +_{TA} (a,b,c',d') = (a,b,c+c',d+d')$ and $(a,b,c,d) \;{}_T{+}_A (a,b',c,d') = (a,b+b',c,d+d')$. Here $+_{TA}$ and ${}_T{+}_A$ denote the addition in group bundles (T^2A, p_{TA}) and $(T^2A, T(p)_A)$ over TA.

<u>Example 2</u> : Let R be the category of commutative rings. Let $TA = A[x] \setminus x^2$. Then TA consists of polynomials $a + bx$, $(T \oplus T)A = A[x,y] \setminus x^2, y^2, xy$ of polynomials $a+bx+cy$ and $T^2A = A[x,y] \setminus x^2, y^2$ of polynomials $a+bx+cy+dxy$. Then T is a tangent functor and its structure is given by the same formulas as in the preceding example.

Example 3 : Let R be a ring of the line type in a cartesian closed category E and $D = \{d \in R \setminus d^2 = 0\}$ (see Kock [2]). Let C be a full subcategory of E which consists of all infinitesimaly linear objects having the property W. Then $T = (-)^D$ is a tangent functor on C. Here $T \oplus T = (-)^{D(2)}$ where $D(2) = \{(d_1, d_2) \in D \times D \setminus d_1.d_2 = 0\}$, $+ : T \oplus T \to T$ is represented by the diagonal $D \to D(2)$, $T^2 = (-)^{D \times D}$, i is represented by the symmetry $s : D \times D \to D \times D$ and m by the multiplication $. : D \times D \to D$.

A concrete example is the category E of functors from the category R_o of finitely presented commutative rings to the category of sets. R is $R_o(Z,-)$ and D is $R_o(Z[x] \setminus x^2,-)$. We took R_o to avoid set theoretical difficulties with functor categories. It is evident that Example 2 works for R_o , too. Hence the tangent functor $(-)^D$ on $C \subset E$ is derived from the tangent functor on R_o. It is a general phenomenon.

Proposition 1 : Let C be a small category and $T : C \to C$ a tangent functor on C. Let B be the full subcategory of the functor category Set^C consisting of all functors which preserve pullbacks $T \oplus T$, $T \oplus T \oplus T$ and the equalizer from (T3). Then

$$T^*(V) = V.T \quad , \quad T^*(\alpha) = \alpha T$$

yields the tangent functor $T^* : B \to B$.

Another general construction of new tangent functors is the following one. Let T be a tangent functor on a category C. Consider the comma category $B = C \setminus A$ where $A \in C$. Then

$$\hat{T}(X,f) = (TX, f.p_X) \quad , \quad \hat{T}(h) = T(h)$$

yields the tangent functor $\hat{T} : B \to B$. Let

$$\bar{T}X \xrightarrow{v_X} TX \underset{O_A.p_A.T(f)}{\overset{T(f)}{\rightrightarrows}} TA$$

be an equalizer. If T preserves this equalizer then

$$\bar{T}(X,f) = (\bar{T}X, f.p_X.v_X)$$

provides the tangent functor $\bar{T} : \mathcal{B} \to \mathcal{B}$.

If $T : \mathcal{M} \to \mathcal{M}$ is the tangent functor on the category of smooth manifolds then $\bar{T}$ gives the vertical bundle on the category of fibered manifolds.

The following property of tangent functors is very important.

<u>Lemma 1</u> : Let T be a tangent functor and consider the composition

$$e : T\oplus T \xrightarrow{\langle m.p_2, T(O).p_1\rangle} (T\oplus T)T \xrightarrow{+T} T^2$$

Then the diagram

$$T\oplus T \xrightarrow{e} T^2 \underset{O.p.pT}{\overset{pT}{\rightrightarrows}} T$$

is an equalizer.

Since $T(p).e = p_1$, it says that if $f : S \to T^2$ is a natural transformation equalizing pT and $O.p.pT$ then there is a unique natural transformation $g : S \to T$ such that

(1) $f = T(p.O).f +_T m.g.$

3. <u>Bracket operation</u>

Let T be a tangent functor on a category $\mathcal{C}$ and $A \in \mathcal{C}$. A morphism $r : A \to TA$ is called a <u>section</u> of T if $p_A.r = 1$. Hence r is a T-coalgebra in the terminology of Kelly [1].

If $T : \mathcal{M} \to \mathcal{M}$ is the usual tangent functor, the sections are vector fields. We want to define the bracket $[r,s]$ of sections $r,s : A \to TA$ of any tangent functor. For this purpose, we would need the description of the bracket of vector fields in terms of T only and not using functions on manifolds. This description was given by Kolář [3] and we will follow it. Very similar description is stated in White [8].

Let $r,s : A \to TA$ be sections. Since $T(p)_A.T(s).r = r = T(p)_A.i_A.T(r).s$, it is defined the difference

$$v = (T(s).r)_{T}\,{-}_{A}\,(i_A.T(i).s) .$$

It is easy to see that $p_{TA}.v = 0_A$. Following lemma 1, there is a unique morphism $\overline{v} : A \to (T\oplus T)A$ such that $e_A.\overline{v} = v$. Put

$$[r,s] : A \xrightarrow{\overline{v}} (T\oplus T)A \xrightarrow{(p_2)_A} TA .$$

In example 1, sections $r : A \to A^2$ correspond to endomorphisms $r : A \to A$. The bracket is the usual bracket of endomorphisms

$$[r,s] = s.r - r.s .$$

In example 2, sections $r : A \to A[x]\setminus x^2$ coincide with derivations $r : A \to A$ and the bracket is the usual bracket of derivations. In example 3, the bracket is the bracket from the synthetic differential geometry (see [2]).It follows from the fact that

$$m_A.[r,s]=(+.k)_{TA}.T^2(+.k)_A.T(i)_{TA}.-_{T^3A}.T(-)_{T^2A}.T^3(s).T^2(r).T(s).r$$

and that this formula corresponds to the commutator of infinitesimal transformations. Here $k : T^2 \to T\oplus T$ is given by $p_1.k = pT$ and $p_2.k = Tp$.

<u>Theorem 1</u> : Let T be a tangent functor. Then the bracket operation has properties

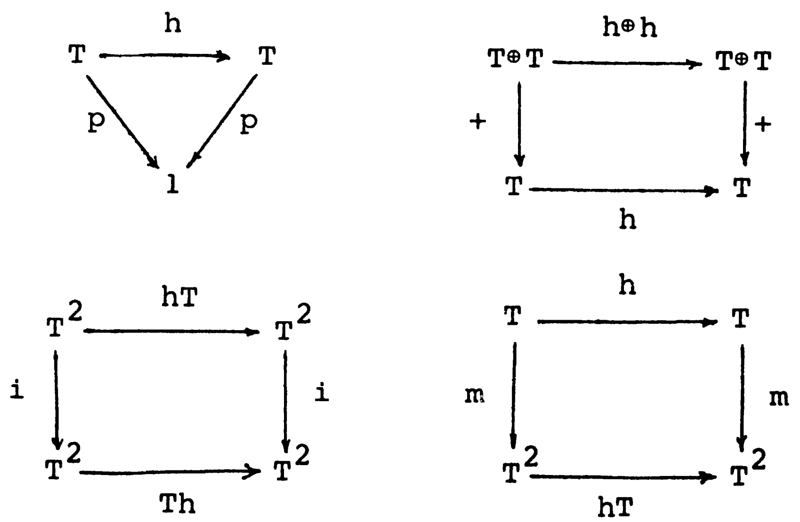
(B1) $[r+s,t] = [r,t] + [s,t]$

(B2) $[s,r] = -[r,s]$

(B3) $[r,[s,t]] + [s,[t,r]] + [t,[r,s]] = 0$

It is not difficult to calculate (B1) - (B3) for manifolds without using functions (see Vanžurová [7]). In the general case, we are facing the coherence problem for tangent functors. Remark that (B1) and (B2) hold on the basis of axioms (T1) - (T3) only. The proofs from synthetic differential geometry (see Reyes, Wraith [6], Lavendhomme [5] and Kock [2]) do not work in the general case. Our proof is "additive" and consists in calculations with natural group bundles T, T^2 and T^3 over 1, T and T^2 given by p, pT, Tp, pT^2, pTp and T^2p.

We did not need the $\mathbb{R}$-linear structure on $T : \mathcal{M} \to \mathcal{M}$. However, it is present in the general case, too. We say that $h : T \to T$ is an <u>endomorphism</u> of a tangent functor $T : \mathcal{C} \to \mathcal{C}$ if the following diagrams commute



It means that h is an endomorphism of the natural group bundle

T and preserves the tangent structure given by i and m. The set R of all endomorphisms of T is a ring with the composition as the multiplication and the addition

$$g + h : T \xrightarrow{\langle g,h\rangle} T \oplus T \xrightarrow{+} T .$$

The morphisms $h_A : TA \to TA$ put an R-module structure on TA and T becomes a natural R-module bundle. If $h \in R$ and $r,s : A \to TA$ are sections of T then it holds

$$[h_A . r,s] = h_A . [r,s] .$$

Kolář [4] proved that if $T : \mathcal{M} \to \mathcal{M}$ then natural transformations $h : T \to T$ with $p.h = p$ are precisely homotheties given by multiplying with $x \in \mathbf{R}$. Any homothety is an endomorphism of T and therefore $R = \mathbf{R}$ in this case. $R = \mathbf{Z}$ in ex. 1 and 2.

In synthetic differential geometry, any morphism $h : D \to D$ with $h(0) = 0$ gives an endomorphism $(-)^h : (-)^D \to (-)^D$. The ring of endomorphism of $(-)^D$ is the ring of 0 preserving morphisms $D \to D$. However, it is the starting ring R of the line type. To see it one has to realize that the line type property implies that 0 preserving morphism $h : D \to D$ coincide with morphisms $-.x$, $x \in R$. Hence the line R is determined by its infinitesimal segment D.

4. Representable tangent functors

Let $\mathcal{C}$ be a cartesian closed category and $D \in \mathcal{C}$ such that $T = (-)^D : \mathcal{C} \to \mathcal{C}$ is a tangent functor. Let p,+,-,0,i and m be represented by $0 : \mathbf{1} \to D$, $\delta : D \to D * D$, $- : D \to D$, $D \to \mathbf{1}$, $\iota : D^2 \to D^2$ and $. : D^2 \to D$.

Here,

$$\begin{array}{ccc} 1 & \xrightarrow{\;0\;} & D \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ D & \xrightarrow[\pi_2]{} & D*D \end{array}$$

is a pushout. There is a unique morphism $+ : D*D \to D$ such that $+ . \pi_1 = + . \pi_2 = 1$.

<u>Proposition 2</u> : $(D,.)$ is a semigroup with the zero 0 such that $d^2 = 0$ for any $d \in D$.

Categorical logic justifies the set theoretical terminology. The last assertion means that

$$\begin{array}{ccc} D & \xrightarrow{\;\Delta\;} & D^2 \\ \downarrow & & \downarrow{\scriptstyle \cdot} \\ 1 & \xrightarrow[0]{} & D \end{array}$$

commutes where Δ is the diagonal. It follows from the fact that Δ is the composition

$$D \xrightarrow{\;\delta\;} D*D \xrightarrow{\;\varkappa\;} D^2$$

and that

$$\begin{array}{ccc} D*D & \xrightarrow{\;\varkappa\;} & D^2 \\ \downarrow & & \downarrow{\scriptstyle \cdot} \\ 1 & \xrightarrow[0]{} & D \end{array}$$

commutes. The morphism $\varkappa$ represents $k : T^2 \to T \oplus T$, i.e. $\varkappa . \pi_1(d) = (0,d)$ and $\varkappa . \pi_2(d) = (d,0)$ for any $d \in D$. Hence $\varkappa(D*D) \subseteq D(2) = \{(d_1,d_2) \in D^2 \setminus d_1 . d_2 = 0\}$.

From now on, assume that T is represented in such a way that ι is the symmetry s. Then $(D,.)$ is commutative because $i.m = m$.

Let an endomorphism of T be represented by a morphism $h : D \to D$. Then $h(0) = 0$, $\delta.h = (h*h).\delta$ and

$$h(d_1.d_2) = h(d_1).d_2$$

for any $d_1, d_2 \in D$. Let $\bar{R}$ be the ring of these endomorphisms $h : D \to D$. The exponential transpose $\wedge : D \to D^D$ of $. : D^2 \to D$ yields a morphism $j : D \to \bar{R}$.

Proposition 3 : For any morphism $f : D \to \bar{R}$ there is a unique element $h \in \bar{R}$ such that

$$f(d) = f(0) + h.j(d)$$

holds for any $d \in D$.

It is the translation of formula (1). Hence $\bar{R}^D \cong \bar{R}^2$ and $\bar{R}$ thus has some properties of a ring of a line type. We are missing the commutativity of $\bar{R}$ and the fact that

$$D = \{h \in \bar{R} \setminus h^2 = 0\}.$$

Since $\mathbf{0} \times D \cong \mathbf{0}$ where $\mathbf{0}$ is an initial object, the tangent functors from examples 1 and 2 are not representable. The tangent functor from example 1 is not a restriction of a representable tangent functor on a subcategory closed with respect to a terminal object. It follows from the fact that 0 is not a unique natural transformation $1 \to T$. On the other hand, there are very convenient extensions of $T : \mathcal{M} \to \mathcal{M}$ to a representable tangent functor (see Kock [2]).

This paper had a rather slow development due to difficulties with the proof of theorem 1. The other material was completed

in 1982. Proofs will appear elsewhere. I profited from discussions with G. Wraith and A. Kock. However, I am especially indebted to I. Kolář who patiently introduced me into basic differential geometry.

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Department of Mathematics
Purkyně University
Brno, Czechoslovakia