DIAGRAMMES

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B-catégories and gamuts

B-CATEGORIES AND GAMUTS

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Introduction.

In [7] R. Street defined "gamuts" as the V-counterpart of cofibrations in V-Cat in order to recover modules as codiscrete cofibrations. In [5] we observed that automata seen as enriched categories ([2]) were an example of gamuts and that the behaviour-realization *local* adjunction could be proved for a general *biclosed* category V , and for arbitrary modules rather than just *endobimodules of the trivial* V-*catégorie* 1 .

We made however a further step towards generality in [3], where we described tree automata as categories enriched over a *bicategory* B , or rather as gamuts in B-mod. The behaviour module *is not* any longer an endobimodule, and a behaviour-realization theorem holds, very similar to the classical one. Thus, we deem that an exposition of the B-categorical machinery necessary to get to this generalization is useful. This is the content of §1, of a somewhat expository nature, and of §2, where the main results are given.

In §3 we illustrate examples; in particular, in example 3.2 we give a brief account of our categorical treatment of tree automata, following the line of [3] and providing motivation for our general result. Thence, we think that a very short and informal recall of the classical notion of tree automata has an appropriate place in this Introduction.

The classical theory of tree automata (see e. g. [10]) relies on the notion of Σ-algebra, where Σ is a set of operations symbols that, together with an "arity" function from Σ to the non negative integers, constitutes a *similarity type*; a

Σ-algebra is then a set A (the *carrier* of the algebra) together
with a map α assigning to each symbol h of arity (say) n in
Σ an actual operation $\alpha_h: A^n \to A$. Nullary operations are called
constants.

A *deterministic* Σ-*automaton* is essentially a Σ-algebra (the
set of states of the automaton is given by the carrier and for
each *input symbol* h∈Σ the corresponding *transition map* is the
operation α_h) plus a subset F (the *final states*) of A .The
α-images of the constants are the *initial states*. The inputs of
the automaton are given by the set T_Σ of *terms* (or *trees*) of
the type Σ , that is the smallest subset of the free monoïd on
Σ such that:

  a) the constants are terms,
  b) if f is an n-ary operation symbol and $t_1, \ldots, t_n \in T_\Sigma$, then
$f(t_1, \ldots, t_n) \in T_\Sigma$.
Thus a term corresponds to a computation involving several
operations of various arities, the sequence of which is often
represented as a graph, namely a tree, where n-ary branchings
visualize n-ary operations: the nodes can be either constants or
subtrees, that is, according to the inductive definition above,
again terms.

The notion of *non deterministic automaton* is obtained in a
similar way from the notion of *relational algebra*, where
transition maps are substituted by *transition relations*
$\rho_h: A^n \to A$, relative to the input h .

We call *reachable states* (or definable elements) of a
deterministic (resp. non deterministic) automaton the α-images
(resp. ρ-images) of the terms. We say an automaton is *reachable*
when all the states are reachable. The *behaviour* βA of a non
deterministic automaton   A = (A,ρ,F)   is the set of trees
recognized by A, i. e. $\beta A = \{t \in T_\Sigma \ / \ \rho(t) \cap F \neq \emptyset\}$.

Of course, since the "dynamics" of an automaton (i. e.
forgetting terminal states) is just an algebra, all the notions
above can be rephrased in terms of Lawvere's functorial
semantics.

A *theory* T is a category whose objects are finite sets
[n] = {1,...,n} , n = 0,1,2, ... , and which admits the category
of finite sets as a subcategory. The initial object is [0] = ∅
and [1] = {1} is the terminal one. Notice further that [m] is
the m-fold coproduct of [1] , so that an arrow a:[m]→[n] is
equivalent to an m-tuple of "injections" $a_i:[1] \to [n]$.
A T-algebra is then a product preserving functor A:T^op→Set   and
a morphism of T-algebras is a natural transformation.

With this said, it is obvious to define a *deterministic automaton*
as a pair given by a T-algebra and a set of final states   F
which is a subset of A[1] , where A[1] is again the carrier of
the algebra.

Non deterministic automata correspond then analogously to
relationnal T-algebra, i. e. *lax functors* A:T^{op}→Rel(S) (Rel(S)
is the bicategory of sets and relations) sending coproducts in T
to quasi-products in   Rel(S) .We will describe in 3.2 a
bicategory B(T) constructed out of the theory T and such that
T-algebras corresponds to B(T)-categories, while automata
correspond to certain diagrams, to take into account final states
and behaviour.

In fact, the behaviour itself is a functor and, as a consequence
of our general result, it is possible to construct a
"realization" functor which assigns to each behaviour a canonical
automaton with a "quasi-universal" property which exhibits the
realization as a very weak adjoint to the behaviour. This is the
"best" analogue to the classical Goguen's behaviour-realization
adjunction in this non deterministic context.

Finally, let us observe further that gamuts and cofibrations,
crucial in the work of Rosebrugh and Wood ([6]), are even in this
case an instance of gamuts in a bicategory of the form B-mod (see
the remark at the end of §1).

1. Gamuts and B-categories.

Let   B   be a locally complete and cocomplete *biclosed*
bicategory, i. e. with right liftings and right extensions.

Namely, given 1-cells with common domain f:a→c and g:a→d ,
there is a 1-cell from c to d satisfying the universal
property of a right Kan extension. The extension will be denoted
by [f,g] .

As for the lifting of 1-cells with common codomain, it is
defined to be an extension in B^{coop} (the bicategory obtained
from B by reversing both the 1-cells and the 2-cells). The
lifting will be denoted by {f,g} .

We recall here briefly the basic notions of B-category
theory, referring the reader to (e. g.) [9] and [1] for further

details and applications. Our general reference for bicategories is [1].

A B-*category* X is a set X together with a function $e:X\to ObjB$ and a function $X(-,-):XxX\to MorphB$ such that:

(i) $X(x_1,x_2):e(x_1)\to e(x_2)$,

(ii) $\delta:1_{e(x)}\to X(x,x)$,

(iii) $\mu:X(x_2,x_3).X(x_1,x_2)\to X(x_1,x_3)$,

satisfying the obvious axioms of left and right identities and associativities.

If x is an object of a B-category X , e(x) is called the "underlying object" of x in B or also the "extent" of x . The 2-cells δ and μ are the identity and composition 2-cells. Notice that if we consider a monoïdal (biclosed) category V as a one object bicategory whose 1-cells are the "objects" of V and 2-cells are the "arrows" of V , then the definition above specializes to the classical notion of V-category.

Example ([11]). If L is a locale, the bicategory of relations Rel(L) is defined as follows:

- objects of Rel(L): opens u in L ,
- arrows from u to v : elements w≤u v ,
- 2-cells: order in L ,
- composition of arrows: intersection.

In [11] Walters showed that the category Sheaves(L) is equivalent to a suitable subcategory of Cauchy-complete (see below for a definition) Rel(L)-categories.

A B-*functor* F from X to a B-category Y is a function $F:X\to Y$ such that:

(i) $e(F(x)) = e(x)$,

(ii) $F_{x,x'}:X(x,x')\to Y(Fx,Fx')$.

The 2-cell $F_{x,x'}$ is called the *strength* of the functor; notice again that the definition above generalizes the notion of V-functor.

We call B-cat the bicategory of B-categories and B-functors.

If X and Y are B-categories a *bimodule* $\phi:X-+->Y$ assigns:

- to each pair of objects x in X and y in Y over u, v respectively, a 1-cell $\phi(x,y):u\to v$ in B,
- to objects x, x' in X and y in Y , a 2-cell (right action) $\rho:\phi(x',y)X(x,x')\to\phi(x,y)$,
- to objects x in X and y, y' in Y , a 2-cell (left action) $\lambda:Y(y,y')\phi(x,y)\to\phi(x,y')$,

satisfying the axions expressing the compatibilities of the left and right actions.

Again this definition generalizes the notions of V-module and distributor. Further, as in the V-categorical case, a B-functor $F:X{\to}Y$ yields a pair of bimodules $F^*:X{-}{+}{-}{>}Y$ and $F_*:Y{-}{+}{-}{>}X$ given by:

- $F^*(x,y) = Y(Fx,y)$,
- $F_*(y,x) = Y(y,Fx)$.

Given bimodules $\phi,\psi:A{-}{+}{-}{>}C$, a 2-cell $\theta:\phi{\Rightarrow}\psi$ is a family of 2-cells $\theta_{a,c}:\phi(a,c){\Rightarrow}\psi(a,c)$ in B compatible with the actions.

The composite $\psi\phi:A{-}{+}{-}{>}X$ of bimodules $\phi:A{-}{+}{-}{>}C$ and $\psi:C{-}{+}{-}{>}X$ is defined as follows:

- $\psi\phi(a,x)$ is the colimit in $B(ea,ex)$ of the diagram

$$\psi(c,x)\phi(a,c){<}{-}{-}{-}{-}{-}{-}{-}{-}(c',x)C(c,c')(a,c){-}{-}{-}{-}{-}{-}{-}{-}{>}\psi(c',x)\phi(a,c')$$
$$\rho.\phi(a,c) \qquad\qquad \psi(c',x)$$

where c, c' vary over all objects of C . The actions are the obvious ones induced by the actions of ϕ and ψ . Notice that the existence of the colimit is guaranteed by the assumption of the local cocompleteness of B . We write this colimit with the coend notation:

$$\int^c \psi(c,x).\phi(a,c).$$

Thus we can define the bicategory B-mod of B-categories and bimodules.

It is also possible to speak of *adjoint* bimodules.

An important example is provided by the bimodules F^* and F_* induced by a B-functor F : we have the adjonction $F_* \dashv F^*$.

A B-category is *Cauchy-complete* if every adjoint pair of bimodules $\phi,\psi:X{\rightleftharpoons}Y$ arises from a functor from X to Y.

To test Cauchy-completeness is sufficient to consider bimodules to/from *trivial* B-categories, that is one object B-categories (say $u^\wedge$) over an object u , with $u^\wedge(*,*) = 1_u$.

Notice that in our assumptions, B-mod is again a biclosed bicategory. Indeed in the diagramm:

A ————————|————→ C
          ⌀

ψ ⟋   ⇐   ⟍ [⌀,ψ]

      ↘   ↙
        D
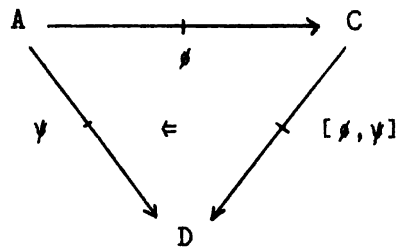
the components [⌀,ψ](c,d) of the bimodule [⌀,ψ] are given by
the equalizer of the following diagram:

$$\underset{a}{\Pi}\ [⌀(a,c),ψ(a,d)] \rightrightarrows \Pi\ [⌀(a,c).A(a',a),ψ(a',d)]$$

and are 1-cells e(c)→e(d).
We will use for this equalizer the end notation:

$$[⌀,ψ](c,d) = \int_{a}\ [⌀(a,c),ψ(a,d)].$$

Notice also that the various forms of Yoneda lemma, involving
integral notation, hold.

In [7] Ross Street defined gamuts in   V-mod , which are
actually defineable in any bicategory, being just diagrams
defined by lax functors from the bicategory 3 .
So, a *gamut* in B-mod (or a B-gamut) from A to D is a diagram
(C,⌀,ψ,θ,λ):

            C
          ⟋ ↗ ⟍
      ⌀ ⟋   ⇓λ   ⟍ ψ
       ⟋   θ   ↘↘
      A ————|————→ D
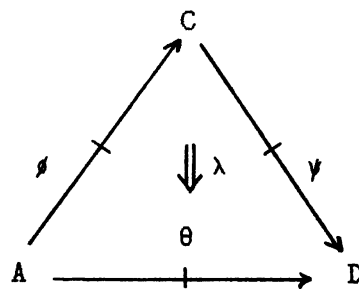
A morphism of gamuts (C,⌀,ψ,θ,λ)→(C',⌀',ψ',θ',λ') is then given
by (S,α,β,γ) where S is a bifunctor C→C' and α, β, γ are
2-cells α:⌀⇉S*.⌀' , β:ψ.S*⇉ψ' , γ:θ⇉θ' such that:
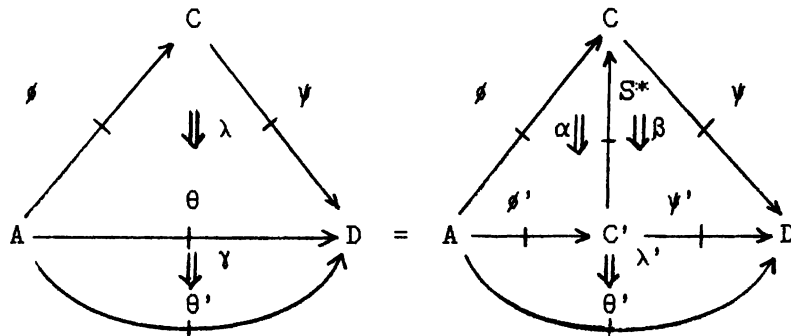
where   S*   denotes   the   right   adjoint   in   the   pair   of   modules
induced   by   the   functor   S .

As   an   aside,   we   mention   the   fact   that   the   result   of   Street
([7])   concerning   the   (bi)equivalence   between   the   (bi)category   of
V-gamuts   from   A   to   D   and   the   (bi)category   of   cofibrations
from   A   to   D   can   be   extended   to   the   B-categorical   context.
However   this   is   beyond   the   scope   of   the   present   paper.

Gamuts   have   been   also   considered   by   Rosebrugh   and   Wood   in
the   context   of   their   "proarrow   equipments"   ([6])   Top→LTop ,
where   Top   is   the   bicategory   of   topoï   and   geometric   morphisms
and   LTop   is   the   bicategory   of   topoï   and   left   exact   functors.

What   is   interesting   for   us   is   that   this   is   one   example   of
gamuts   in   bicategories   *of   the   form*   B-mod .   In   fact,   by   Barr's
theorem   (see   [4],   p.   252)   BLTop   (boolean   topoï)   is   a   *dense*
subbicategory   of   LTop ,   and   we   have   LTop $\simeq$ BLTop-mod$_{fin}$,   where
"$_{fin}$"   means   that   we   are   considering   just   *finite*   BLTop-
categories.

2.   The canonical decomposition.

For   each   small   category   A ,   there   exists   a   B-category   PA
which   represents   modules,   in   the   sense   that,   for   modules
θ:A-+->C ,   we   have:

(1)
$$\frac{\theta : A\text{-+->}C}{T : C \to PA}$$

Notice that   T(c)   is   θ(-,c) , which is a module from   A   to
e^(c) .

The B-category   PA   is defined as follows (see also [9]):

- objects are modules ϕ:A-+->u^ ,   u   ranging in   B , and for
ψ:A-+->v^   we set, using the biclosed structure of   B-mod,

$$PA(\phi,\psi) = [\phi,\psi] = \int_a [\phi_a,\psi_a] .$$

Since we will consider also gamuts with a large "top" category
(in fact   PA), we have to suppose   B   embedded in a   B'
corresponding to a higher universe   Set' , in such a way that the
inclusion of   B   in   B'   preserves local limits and   Set'-small
local colimits.

Let us recall now some classical propositions of V-category
theory (see [5]) which extend easily to the B-categorical case.

Proposition 2.1. *Let   F:A→C   be a fully faithful functor between
two small B-categories and let ϕ be a module from   A   to   D .
Then we have   F*[F*,ϕ] ≃ ϕ .*

Proposition 2.2. *Given a module   θ:A-+->C , the gamut below has
an invertible 2-cell:*
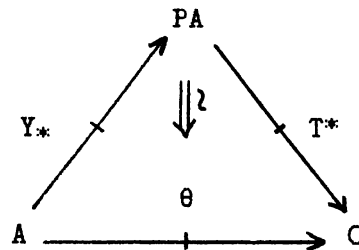
*(here   T   is the functor corresponding to   θ   in (1) and   Y   is
the Yoneda embedding).*

Notice that the proofs rely essentialy on the validity of the
B-categorical forms of Yoneda lemma.

The gamut of Proposition 2.2 is called *canonical decomposition
of the module* θ .
Obviously, to any gamut from   A   to   C   is possible to assign
canonically a module, the bottom one. This process amounts to a
"forgetful" homomorphism:

$$F : B\text{-gam}(A,C) \to B\text{-mod}(A,C)$$

We define also a homomorphism:

$$R : B\text{-mod}(A,C) \to B\text{-gam}(A,C)$$

sending a module θ to its canonical decomposition and a 2-cell $\gamma : \theta \Rightarrow \theta'$ to the morphism of gamuts $R\gamma = (1_{FA}, \text{id}, \gamma^*, \gamma)$, where $\gamma^* : T^* \Rightarrow T'^*$.
Observe that $B\text{-mod}(A,C)$ is a discrete bicategory.

Recalling now (see [2] for the details) that a homomorphism of bicategories $\Omega : W \to D$ has a right *local adjoint* $\Delta : D \to W$ if for each v in W and d in D there is an adjunction $\Delta_{vd} \dashv \Omega_{vd}$:

$$D(\Omega v, d) \underset{\Omega_{vd}}{\overset{\Delta_{vd}}{\rightleftharpoons}} W(v, \Delta d)$$

with appropriate naturality conditions, we have the following:

Theorem 2.3. *The homomorphism* $F : B\text{-gam}(A,C) \to B\text{-mod}(A,C)$ *has a right local adjoint* $R : B\text{-mod}(A,C) \to B\text{-gam}(A,C)$.

Proof. We have to prove that, for arbitrary θ in $B\text{-mod}(A,C)$ and $X = (D, \phi, \psi, \theta', \lambda)$ in $B\text{-gam}(A,C)$, it holds:

$$B\text{-mod}(FX, \theta) \underset{R_{X\theta}}{\overset{F_{X\theta}}{\rightleftharpoons}} B\text{-gam}(X, R\theta)$$

with $R_{X\theta} \dashv F_{X\theta}$.
Let $\gamma : \theta' \Rightarrow \theta$ in $B\text{-mod}(FX, \theta)$. Notice that since the latter is just a set, if $(S, \alpha, \beta, \lambda) : X \to R\theta$ is a morphism of gamuts, to say that $R_{X\theta}$ is left adjoint to $F_{X\theta}$ amounts to the following:

$$\frac{R_{X\theta}(\bar{\gamma}) \Rightarrow (S, \alpha, \beta, \gamma)}{\bar{\gamma} \Rightarrow \gamma}$$

where $\bar{\gamma} \Rightarrow \gamma$ is in a *discrete* category, so either $\bar{\gamma} = \gamma$ or there is no 2-cell. So there is exactly one 2-cell $R_{X\theta}(\bar{\gamma}) \Rightarrow (S, \alpha, \beta, \gamma)$ iff $\bar{\gamma} = \gamma$. This is to say that for each $\gamma : \theta' \Rightarrow \theta$, the full subcategory of $B\text{-gam}(X, R)$ given by the $(S, \alpha, \beta, \gamma) : X \to R\theta$ with *this* γ , has an initial object. It is easily checked that (just

like in [5], theorem 9) this is obtained observing that to give a morphism of gamuts $(S,\alpha,\beta,\gamma):X\rightarrow R\theta$ is to give the module $\sigma:A\text{-+-}>D$ corresponding to S under the bijection (1) , plus 2-cells $\alpha:\emptyset\Rightarrow\sigma$, $\epsilon:\gamma\sigma\Rightarrow\theta$ and $\gamma:\theta'\Rightarrow\theta$.
We have the initial object by putting $\sigma = \emptyset$, $\alpha = id$, $\epsilon = \gamma\lambda$.

3. Examples.

3.1. The one object categories A in B are to be thought of as *monads* (v,m) with $m = A(a,a)$ on $v = e(a)$ in B .
Then the fibre $(PA)_u$ is the $B(u,u)$-category of *m-opalgebras with codomain* u . So PA is the (large) B-category of opalgebras with *all* possible codomains.
We can define a morphism of opalgebras with *different* codomains $t:v\rightarrow u$ and $s:v\rightarrow w$ as $[t,s]$ in B. This is the hom in PA .
Now $Y_*(a,t) = PA(Ya,t) = [m,t]$.
Given a module $\sigma:A\text{-+-}>C$, where $C = (u,n)$, the corresponding functor S: $C\rightarrow PA$ is the 1-cell $\sigma:v\rightarrow u$.
So $S^*(t,c) = PA(t,\sigma) = [t,\sigma]$ and

$$Y_*S^*(a,c) = \int^t PA(m,t)\cdot PA(t,\sigma) \cong PA(m,\sigma) = [m,\sigma]$$

and, by Proposition 2.2, $[m,\sigma] \simeq \sigma$.

3.2. In [3] tree automata are described as categories enriched on a bicategory. We already recall in the Introduction that a (non deterministic) tree automaton is a (relational) T-algebra (plus the initial and terminal states), where T is a Lawvere's theory, i. e. a category whose objects are finite sets and which admits the category of finite sets as a subcategory.
Starting from the theory T , a bicategory $B(T)$ is defined as follows:
- $B(T)$ has the same objects as T ,
- the 1-cells from u to v in $B(T)$ are the subsets of $T(u,v)$ and 2-cells are inclusions,
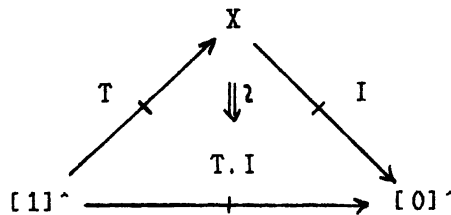- composition of 1-cells and identities are the obvious one.

B(T)   satisfies the assumptions of §1, in particular it is
biclosed, with

$$
\begin{array}{c}
\text{u} \xleftarrow{} \text{V} \\
R \quad \Leftarrow \quad \in [R,S] \\
S \quad \downarrow \\
\text{W}
\end{array}
$$
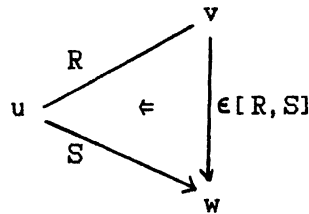
given by [R,S] = {k:v→w in T such that kR⊆S}.
In [3] it is shown that the category of "reachable" T-algebras is
isomorphic to the category of "reachable" B(T)-categories.
A tree automaton is defined as a gamut:

$$
\begin{array}{ccc}
& & X \\
& T \nearrow \quad \Downarrow 2 \quad \searrow I & \\
[1]^\wedge & \xrightarrow{\quad T.I \quad} & [0]^\wedge
\end{array}
$$

where X is a B(T)-category (i. e. an algebra) and [n]^
denotes the trivial category over [n] . If b = (b₁,...,bₙ) is
an X-object over [n] , then I(b) = X(b,x₀) , i. e. it provides
n-tuples of trees (x₀ denotes here the *unique* X-object over 0),
while T(b) = {g∈T([1],[n]) / ∃ a∈F⊆X[1] and g∈X(a,b)} , where
F is the set of final states, so that T determines sets of
operations which are succesful if performed on those trees (see
[3] for all the details). The composite module T.I is now the
behaviour of the automaton (the set of trees computed by the
operations of the automaton which are recognizable, i. e. which
belong to the set of final states).
We can apply here our theorem on local adjoints, where the right
local adjoint to the "behaviour" (i. e. the forgetful functor) is
the *realization* functor, which associates to a behaviour
θ:[1]^-+->[0]^ the gamut:

$$P[1]^\wedge$$
$$Y_* \qquad T^*$$
$$\theta$$
$$[1]^\wedge \longrightarrow [0]^\wedge$$
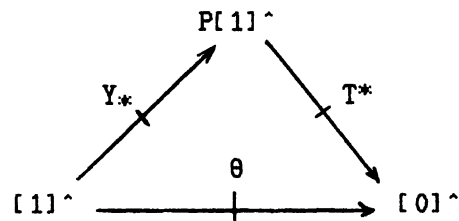
This result is a generalization of both [2] and [5], and provides furthermore a motivation for the "abstract" theorem of [5], where modules were taken between two *different* V-categories, although in the leading example of ordinary automata they both coïncide with the trivial category [1]^. In fact, in the case of tree automata the "behaviour" module is *necessary* defined between two different categories, namely [1]^ and [0]^ .

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